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Construction of the Envelope of Holomorphy for a Domain

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Summary

When we move from the complex plane to several complex variables, several things changes. One of these things is that we can sometimes extend all holomorphic functions defined on a domain to a larger domain. This gives rise to a question: for any domain in $\mathbb{C}^n$, what is the largest domain all holomorphic functions defined on it can be extended to, and how do we find this domain? Several methods have been found, but in 2008 Burglind Jöricke published a paper called "Envelopes of holomorphy and holomorphic discs", that provided a way to construct this largest domain, called an envelope of holomorphy, in a far more effective way than previously. However, the proofs were only given in the case of $n = 2$, and a lot of new concepts made it at times difficult to understand. Therefore my Master's thesis consists of generalizing the proofs to all $n > 1$, and hopefully simplifying and changing some of the proofs.

There are three chapters: Chapter 1 is a short review of some notation and fundamental concepts and ideas. Chapter 2 proves some results needed for Jöricke's proofs. Finally, Chapter 3 is my version of the proofs in Jöricke's paper, which sometimes differ from the original.

Oppsummering

Når vi går fra det komplekse planet til flerdimensjonalt komplekst rom, er det flere ting som endrer seg. En av disse tingene er at vi noen ganger kan utvide alle holomorfe funksjoner definert på et domene til et større domene. Dette leder til et spørsmål: for et domene i $\mathbb{C}^n$, hva er det største domenet vi kan utvide alle holomorfe funksjoner på det opprinnelige domenet til, og finnes det en metode for å finne det? Flere metoder har blitt oppfunnet, men i 2008 publiserte Burglind Jöricke en artikkel kalt "Envelopes of holomorphy and holomorphic discs", som gav en veldig effektiv måte å konstruere den største utvidelsen, kalt holomorfienvelopen. Bevisene i den artikkelen var kun gitt i tilfellet hvor $n = 2$, og den introduserte mange ne konsepter, noe som gjorde den vanskelig å forstå. Derfor handler min masteroppgave om å generalisere bevisene til alle $n > 1$,

og forhåpentligvis endre og forenkle noen av bevisene.

Det er tre kapitler: Kapittel 1 introduserer noe notasjon og flere fundamentale konsepter og ideer. Kapittel 2 beviser noen resultater som trengs for Jöricke's bevis. Til slutt er Kapittel 3, som er min versjon av Jöricke's artikkel, noen ganger med endrede beviser.

Trondheim, May 30, 2016

Håkon Strand Bølviken

Contents

Chapter 1

Preliminaries

1.1 Notations and Early Notions

This chapter will define some of the most important concepts used in this Master's thesis, and provide examples that will motivate concepts in the later chapters. First, here is some notation regarding holomorphic functions.

Definition 1.1.1. *We will often call the set of all holomorphic functions on a domain D by $H(D)$.*

Definition 1.1.2. *Say that we have a domain D in $\mathbb{C}^n$ and a domain B for which D is a proper subset. Let f be a holomorphic function on D. We say that a holomorphic function F defined on B is an extension of f if $f(z)=F(z)$ for all $z \in D$. If such an F exists, we say that f can be extended to B.*

This Master's thesis will revolve around extending all holomorphic functions defined on a domain to a lager domain. The next theorem will show that this is not possible in one complex dimension, but we will later show that extending every single function on a domain to some larger domain may be possible in several complex

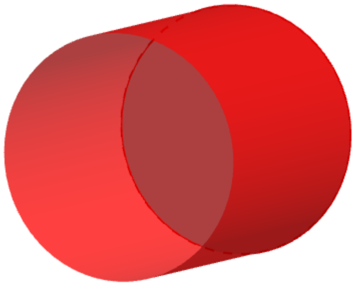


Figure 1.1: A cut of G when $b = 0$.

Figure 1.2: G when $b = 0$ and $y = 0$.

dimensions.

Theorem 1.1.1. *Let D be a connected domain in $\mathbb{C}$, and let B be a domain so that $D \subsetneq B$. Then there is at least one analytic function defined on D that does not extend to B.*

Proof: Let $a \in B\ D$. Then $f(z) = \dfrac{1}{z-a}$ is holomorphic everywhere except at a. That means that it is holomorphic on D, but not on B. It is also impossible that there is an extension of $f(z)|_D$ which differs from $f(z)$ on B, because extensions of analytic functions are unique, thus any extension must go to infinity when we approach a. □

Next we have an example showing that extending all analytic functions on a domain to a larger one is possible for at least one domain in $\mathbb{C}^2$.

Example 1.1.1. *Let G be a domain in $\mathbb{C}^2$. We write z=x+iy and w=a+ib. Let G be the* *union of the following domains:*

$$G_1 = \{(z,w) \in \mathbb{C}^2 : |z| < 2, a \in (-1, 0.5], b \in (-1,1)\}$$

$$G_2 = \{(z,w) \in \mathbb{C}^2 : \frac{7}{4} < |z| < 2, a \in (0,3), b \in (-1,1)\}$$

An idea about the shape of this domain can be gotten from Figures 1.1 and 1.2.

A holomorphic function f on this domain must be on the form

$$f(z,w) = \sum_{l=-\infty}^{\infty} a_n(z) w^n$$

where a_n is an analytic function. We can find a point b=(z,w) in the domain where w=0. In b a_n and all its derivatives must be 0 for all $n < 0$, otherwise f or one of its derivatives with respects to z would get a singularity in G. This means that $a_n = 0$ for $n < 0$, so

$$f(z,w) = \sum_{l=0}^{\infty} a_n(z) w^n$$

Let us now look at a point inside the "glass", that is to say a point

$$\xi_0 \in H = \{|z| \leq \frac{7}{4}, a \in [\frac{1}{10}, 3), b \in (-1,1)\}$$

Write $\xi_0 = (z_0, w_0)$. Now change the value of the imaginary part of w_0 until you get a point $\xi_1 = (z_0, w_1)$ inside G, so that $\mid w_0 \mid < \mid w_1 \mid$. At this point the power series of f converges absolutely. But by the direct comparison test for convergence of series, f must also converge absolutely in ξ_0, as $\mid a_n(z_0) w_0^n \mid < \mid a_n(z_0) w_1^n \mid$ for all n.So f converges for all points $\xi_0 \in H$, and so we can extend any $f \in H(G)$ to $G \cup H$.

A result we will use without proof is that on a pseudoconvex domain, one can not extend all analytic functions to a larger one. $G \cup H$ is convex, and thus pseudoconvex. Therefore $G \cup H$ is the largest domain we can extend all analytic functions to.

Example 1.1.1 shows us that unlike in 1 complex dimension, in several dimensions there are domains so that all holomorphic functions can be extended to a larger domain. From this concept it naturally follows to think about the largest possible domain so that all holomorphic functions on a domain can be extended to it. This gives rise to the idea of an envelope of holomorphy. In this chapter it will not be proven that the envelope of holomorphy of a domain exists, that will be proven in chapter 2.

Definition 1.1.3. *The envelope of holomorphy of a domain D is the largest domain containing domain containing D such that all holomorphic functions defined on D can be extended to the envelope of holomorphy. The envelope of holomorphy of D can be written as $\widetilde{D}$.*

Definition 1.1.4. *A domain of holomorphy D is a domain which is equal to its own envelope of holomorphy, that is to say that for any domain B for which D is a proper subset, at least one holomorphic function defined on D can not be extended to all of B.*

Example 1.1.1 shows a way to find the envelope in a simple case, as the extension found is convex, and thus pseudoconvex. However, there are problems that may arise for some domains, as the next example will show. This example will resemble the last, but with one part added that will drastically change the properties of the domain.

Example 1.1.2. *Let U be a domain, $U \subset\subset \mathbb{C}^2$. It will be the union of three parts, U_1, U_2 and U_3. We will write a point in $\mathbb{C}^2$ as (z, w), where $z = x + iy$ and $w = a + ib$.*

$$U_1 = \{(z, w) \in \mathbb{C}^2 : |z| < 2, -\frac{1}{2} < a < \frac{1}{10}, b \in (-1, 1)\}$$
$$U_2 = \{(z, w) \in \mathbb{C}^2 : \frac{7}{4} < |z| < 2, 0 < a < 3, b \in (-1, 1)\}$$

U_3 needs some more explanation. It will be a kind of tail going from the U_1, going on the outside of U_2 and then dropping down into the inner part of U_2. We start with defining a curve made of these parts:

The curve starts at $(0,0)$, then follows $(3+3e^{it}, 0), -\pi \le t \le \frac{\pi}{2}$ to $(3+3i, 0)$. Note that 3 is in the centre of this curve, that will be become important. Next, let the curve go straight

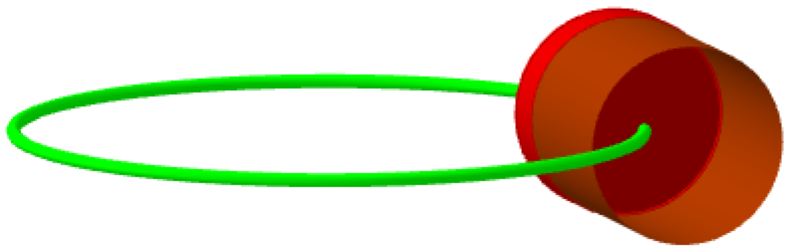


Figure 1.3: Glass with tail as seen when $b = 0$. The tail is not quite accurate, but gives the idea.

from $(3+3i,0)$ *to* $(3,4)$*, and then to* $(0,4)$*. Finally, we have the curve go to* $(0,2)$*. All of this, except for the first part, is done through straight lines.* U_3 *is gotten by slightly fattening the curve described, by taking the points that lie less than* $\frac{1}{100}$ *from it. Note that except at the beginning,* U_3 *does not intersect with* U_1 *or* U_2*, and that it ends inside of* U_2*.*

Now set $U = U_1 \cup U_2 \cup U_3$*. One can imagine this domain as a glass with a tail growing out of the base, where the tail curves around the glass and into the opening. 1.3 and 1.4 are two images of this domain, one cut in 3D and one in 2D that together gives some idea about U's appearance.*

A holomorphic function f *on this domain must be on the form*

$$f(z,w) = \sum_{l=-\infty}^{\infty} a_n(z) w^n$$

where a_n *is an analytic function. We can find a point b=(z,w) in the domain where w=0. In b* a_n *and all its derivatives must be 0 for all* $n < 0$*, otherwise* f *or one of its derivatives with respect to z would get a singularity in U. This means that* $a_n = 0$ *for* $n < 0$*, so*

$$f(z,w) = \sum_{l=0}^{\infty} a_n(z) w^n$$

Let us now look at a point inside the "glass", that is to say a point $\xi_0 \in V = \{(z,w) \in \mathbb{C}^2 : |z| \leq \frac{7}{4}, 0 \leq a < 3, b \in (-1,1)\}$

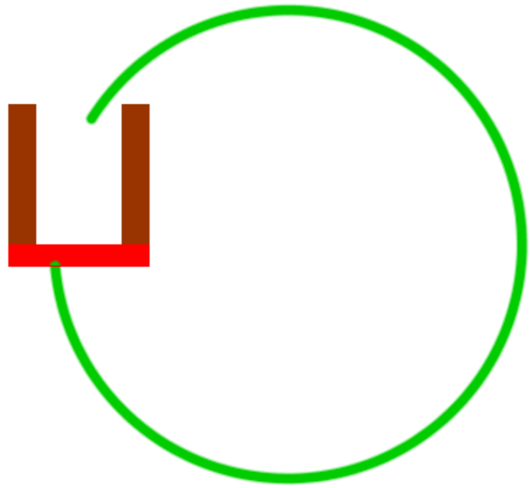


Figure 1.4: Glass with tail as seen when $w = 0$.

Write $\xi_0 = (z_0, w_0)$. Now change the value of the imaginary part of w_0 until you get a point $\xi_1 = (z_0, w_1)$ inside U, so that $| w_0 |<| w_1 |$. At this point the power series of f converges absolutely, but by the direct comparison test for convergence of series, f must also converge absolutely in ξ_0, as $| a_n(z_0) w_0^n |<| a_n(z_0) w_1^n |$.So f converges for all points $\xi_0 \in V$, and so we can extend any $f \in H(U)$ to $U \cup V$.

One might think that $U \cup V$ is the envelope of holomorphy for U (or at least a subset of it), but there is a problem, namely the "tail", U_3. This part goes in a loop around $z = 3$ and ends inside V. Now consider the function $g(z, w) = \sqrt{z-3}$, a holomorphic function on U. $\sqrt{z-3}$ has two branches, and as U loops around $z = 3$ the function must change branch. This means that for a point $v \in V \cap U_3$, $\sqrt{v-3}$ would have two different values, so the extension fails.

The solution is to, in any point where there is such a conflict of values, create two or more separate points, each said to be on a different "sheet", which each takes on one of the values the function could have. This means that we no longer have domains in $\mathbb{C}^n$, but rather manifolds of complex dimension n. More on this will come in chapter

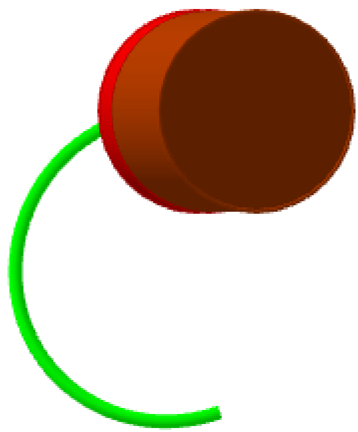


Figure 1.5: The envelope of holomorphy of U. The glass has been filled in and the tail is now on a different sheet.

2, for now I will refer to 1.5 which shows the extension of the domain mentioned in the example, the envelope of holomorphy of U.

To be able to work with these, more complicated, cases, we generalize the concept of a domain to one that is used in "Several Complex Variables" by Raghavan Narasimhan (Narasimhan (1971)).

We will when it is convenient define an unramified domain D, also known as a Riemann domain, as p: D→ $\mathbb{C}^n$. D is an analytic manifold, except that instead of charts there is a locally homeomorphic function p that projects D down to $\mathbb{C}^n$. The analytic structure on D is determined by p. A holomorphic function on D is defined in the same way as on a manifold.

This way of defining a domain gives rise to new definitions when it comes to extending domains and what the envelope of holomorphy is.

Definition 1.1.5. *Let p: $D \to \mathbb{C}^n$ be a connected, unramified domain and $f \in H(D)$. Let p_1: $X \to \mathbb{C}^n$ be a connected domain and g: $D \to X$ a continuous map with $p \circ g = p_1$. We say that $F \in H(X)$ is an extension on p_1: $X \to \mathbb{C}^n$ if $F \circ g = f$.*

Definition 1.1.6. *Let p: $D \to \mathbb{C}^n$ be a connected domain, and p_1: $X \to \mathbb{C}^n$ be its envelope of holomorphy. If the function p_1 is injective, we say that X is schlicht. This is equivalent with saying that the envelope of holomorphy of D is in $\mathbb{C}^n$, the same dimension D is in. In this case we can use the first definitions of domain of holomorphy.*

The construction of the envelope used in this Master's thesis will rely largely on the concept of analytic discs. To complete this preliminary chapter, we define an analytic disc as such:

Definition 1.1.7. *An analytic disc in $\mathbb{C}^n$ is the image of a holomorphic function $f : \Delta \to \mathbb{C}^n$. A closed analytic disc is the image of a function $g : \bar{\Delta} \to \mathbb{C}^n$ which is continuous on $\bar{\Delta}$ and holomorphic on Δ. Analytic disc can also refer to the function f or g. In the case of unramified domains, an analytic disc disc can also be a holomorphic function from the unit disc into the unramified domain.*

Chapter 2

Sheaves and analytic discs

This chapter will prove results that are be necessary for Jöricke's paper. Most of the results come from Narasimhan (1971) or Forstnerič and Globevnik (1992). At the end some ideas from Jöricke's paper will be introduced.

2.1 Sheaf Theory and Domains of Holomorphy

This section bases itself on Narasimhan (1971). It will use sheaf theory to define and prove the existence of an envelope of holomorphy for a general connected Riemann domain.

Why do we have to make this generalization? In the cases where an envelope of holomorphy is not schlicht, one needs to work with manifolds. For this purpose, we use the theory of sheaves and germs, which will be defined in the following paragraphs.

First, let U be an open set in $\mathbb{C}^n$ and $\{f_s\}$ be a family of analytic functions on U, where $s \in S$, S being an index. For a point $z \in \mathbb{C}^n$ and $(U, \{f_s\})$, $(V, \{g_s\})$ with $a \in U$, $a \in V$, we create an equivalence relation by saying that these two pairs are equivalent if there exists a neighbourhood $W \subset U \cap V$ such that, for all s, $f_s = g_s$ when restricted to W. It is easy to confirm that this is an equivalence relation, and one says that the quotient space we get from this relation is the S-germ of holomorphic functions at a, denoted by

$\vartheta_a(S)$. We now define the sheaf $\vartheta(S) = \bigcup_{a\in\mathbb{C}^n} \vartheta_a(S)$.

Further, define the projection $p_S : \vartheta(S) \to \mathbb{C}^n$, where for $g \in \vartheta_a$ $p_S(g) = a$. Usually it will be simply be known as p, when it is clear to what projection we are referring. As any element in the sheaf lies uniquely in one of the S-germs, this function is well-defined.

Next we define the basis of a topology on $\vartheta(S)$: let $g_a \in \vartheta_a$ and let U be a basis element of the topology on $\mathbb{C}^n$ containing a and f_s a family of holomorphic functions on U. Let g_b be the element of the S-germ at a point $b \in$ U defined with functions $\{f_s\}$. Let $N(U, \{f_s\}) = \bigcup_{b\in U} g_b$. The set of these elements create a basis for the topology on $\vartheta(S)$.

Theorem 2.1.1. *Under the topology defined previously the map p is continuous and a local homeomorphism. Also $\vartheta(S)$ is a Hausdorff space.*

Proof: Take an open set $U \subset \mathbb{C}^n$ and consider $p^{-1}(U)$, which contains all g_a where $a \in U$ and the indexed functions in g_a are either holomorphic on U or on some open subset of U. Either of these cases are open balls in $\vartheta(S)$ and that means that $p^{-1}(U)$ is the union of several open balls, which gives an open set. Thus, from the topological definition of continuity, p is continuous. To see that it is a local homeomorphism, take a basis element of the sheaf ad restrict p to it. Projecting it down, each element in the basis goes to a different element in $\mathbb{C}^n$, implying p is locally 1-1, and this projection forms precisely the open set U from which the basis element was constructed above. This will also hold for any basis element inside this one, it will go to a basis element in $\mathbb{C}^n$. This means that p restricted to the basis element is an open map, so its inverse is continuous. As such, p is a local homeomorphism.

As for why the sheaf is Hausdorff, first note that for two different points x,y in $\vartheta(S)$, either $p_S(x) \neq p_S(y)$ or some pair of functions in the indexed functions are not equal. In the first case we can find two open sets around $p_S(x)$ and $p_S(y)$. In this case we can create two open disjoint sets X,Y and then take $N(X, \{f_s\})$ and $N(Y, \{g_s\})$. These two must be disjoint. In the second case simply take $N(f_a, V), N(g_a, V)$ for some small

V, and this again gives disjointedness. This is because at least one pair of functions indexed by some s must differ on V. □

As mentioned before an unramified domain is a special type of a complex manifold. Normally the analytic structure on the manifold is defined by a collection of charts, but for an unramified domain there is instead a function $p : M \to \mathbb{C}^n$, where p is a local homeomorphism, and M is the manifold. p then defines the structure on M in the usual way on manifolds. M will always be assumed to be connected.

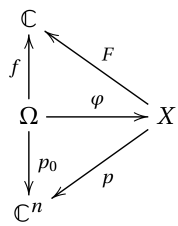
A small but important theorem concerning unramified domains is this:

Theorem 2.1.2. *Say $p : X \to \mathbb{C}^n$ and $p_1 : Y \to \mathbb{C}^n$ are two unramified domains, and let u be a continuous function between X and Y such that $p_1 \circ u = p$. Then u is a local analytic isomorphism, meaning that for a sufficiently small restriction of p to a neighbourhood U around any point $a \in X$ p creates a biholomorphism between U and $p(U)$, that u is holomorphic, and that u^{-1} is locally holomorphic.*

Proof: Both p and p_1 are local analytic isomorphisms, which easily follows from the fact that they are local homeomorphisms, and that they define the analytic structure on their respective manifold. u is analytic, as $p^{-1} \circ u \circ p_1 = id$ for a small enough restriction. For a small enough restriction, we also have $u = p_1^{-1} \circ p$, which is a homeomorphism on a small enough restriction on X, meaning u is a local homeomorphism. On such a restriction, $u^{-1} = p^{-1} \circ p_1$, and by the definition of an analytic function between manifolds this is analytic. Putting all this together proves the assertion. □

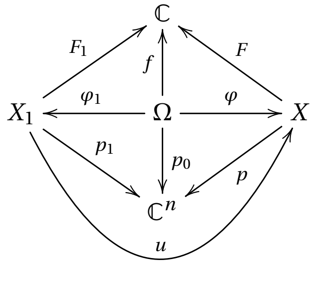
For an unramified domain we can define generalized versions of the concept of extension and envelope. All unramified domains will from this point on be assumed to be connected.

Definition 2.1.1. *Let $p_0 : \Omega \to \mathbb{C}^n$ be an unramified domain, and let $p : X \to \mathbb{C}^n$ be another unramified domain where we have a continuous function $\varphi : \Omega \to X$ such that $p_0 = p \circ \varphi$. Let $S \subset H(\Omega)$. Then we say that X is an S-extension of Ω if for each $f \in S$ there* *is an $F \in H(X)$ such that $f = F \circ \varphi$. Simply put, the diagram below should commute for every function $f \in S$.*



If $S = H(\Omega)$, p_0, p are just identity maps on $\mathbb{C}^n$ and φ is the standard inclusion then this notion is exactly the notion of extending all analytic functions on a domain to a larger domain. Next, we define the envelope:

Definition 2.1.2. *Let $p_0 : \Omega \to \mathbb{C}^n$ be an unramified domain and $S \subset H(\Omega)$. One says that $p : X \to \mathbb{C}^n$ is an S-envelope of Ω if X is an extension of Ω with function $\varphi : \Omega \to X$ such that for any other extension $p_1 : X_1 \to \mathbb{C}^n$, $\varphi_1 : \Omega \to X_1$ we have an analytic function $u : X_1 \to X$ so that $p_1 = p \circ u$, $\varphi = u \circ \varphi_1$ and $F_1 = F \circ u$, where F_1 and F are the extensions of the same function $f \in S$ on X_1 and X respectively. This corresponds to this commuting diagram for every $f \in S$:*



An easy consequence of this definition is that all S-envelopes of a domain $p_0 : \Omega$ are isomorphic to each other, so the envelope, if it exists, is unique up to isomorphism. To see this, say that there are two envelopes with the necessary functions: X, p, φ and $Y, p_1,_1$ where with analytic functions $u : X \to Y$ and $v : Y$ as described in the definition(the exists because X and Y are extensions). Then $\varphi = v \circ \varphi_1$ and $\varphi_1 = u \circ \varphi$. This gives us $u \circ v \circ \varphi_1 = u \circ \varphi = \varphi_1$, implying that $u \circ v$ is the identity on $\varphi_1(\Omega)$ and by uniqueness of analytic functions it is the identity on all of X_1. In the same way we find that $v \circ u$ is the identity on X, and so one finds an isomorphism.

The next theorem will justify that we can always talk about an envelope of holomorphy:

Theorem 2.1.3. *For an unramified domain $p_0 : \Omega \to \mathbb{C}^n$ the S-envelope of holomorphy exists.*

Proof: Let $p_0 : \Omega \to \mathbb{C}^n$ be an unramified domain, and S be a fixed, ordered subset of $H(\Omega)$. We want to construct a function φ from Ω to $\vartheta(S)$. For an $a \in \Omega$, let $a_0 = p_0(a)$. Take a small open neighbourhood U around a, so that $p_0|_U$ is a homeomorphism between U and its image. Say $U_0 = p_0(U)$. Now, for any function $s \in S$ let $f_s = s \circ (p_0|_U)^{-1}$. Then say that $g_a = [(U_0, f_s] \in \vartheta_{a_0}(S)$ is an equivalence class in the S-germ at a_0, and thus in the sheaf. Now put $\varphi(a) = g_a$.

It is easy to show that φ is continuous. Take a basis element in the topology in the sheaf. Notice that $p(\varphi(a)) = a_0$. So locally $\varphi = p^{-1} \circ p_0$. This is continuous on a small restriction, and so it is continuous on Ω and a local analytic isomorphism by previous theorem. We assume that Ω is connected, and so $\varphi(\Omega)$ is also that. Let X be the connected component of it in $\vartheta(S)$. The claim is that $p : X \to \mathbb{C}^n$ is then the envelope of holomorphy of Ω, with φ as the inclusion of Ω into X.

First we prove that X is an S-extension of Ω. We have the φ function needed, so we just need to find functions $F_s \in H(X)$. Take an element g in $\vartheta(S)$ and take a representative $(U, \{f_s\}) \in \vartheta_a(S)$ of the equivalence g represents. Then set $F_s(g) = f_s(a)$, that is to say, take the value of the s-th defining function at the point which g "belongs" to. To see that it is an analytic function, take a small basis element B in $\vartheta(S)$ which is homeomorphic to its projection. Then $F_s \circ (p|_B)^{-1}$ is holomorphic, as f_s is holomorphic on $p(B)$ and $F_s \circ (p|_B)^{-1}$ takes a point a to $f_s(a)$.

Next, to show that X is the envelope, take another extension $p_1 : X_1 \to \mathbb{C}^n$ with $\varphi_1 : \Omega \to X_1$. We need a function $u : X_1 \to X$, and it simply defined the same way that φ was using p_1 and the extensions G_s of each $s \in S$, by going down to $\mathbb{C}^n$, pulling the extensions down to some neighbourhood around the point and using that to find a point in the sheaf. Note that for a point $b \in X_1$ with $b = \varphi(a)$ for some a, we have $p_1(b) = a_0 = p_0(a)$ by definition, and that for any extended function G_s, being an extension of the function s, in some small ball N around g we have:

$$G_s \circ (p_1|_N)^{-1} = G_s \circ \varphi_1 \circ \varphi_1^{-1} \circ (p_1|_N)^{-1} = s \circ (p_0|_{\varphi_1^{-1}(N)}$$

This means that the extended functions when pulled down to $\mathbb{C}^n$ are similar to the original functions on some small neighbourhood. All this means that $u(\varphi_1(a)) = \varphi(a)$, as the value of both are defined by the point in $\mathbb{C}^n$ and the functions around the point. u is continuous, which one finds in the same way it was shown for φ, and obviously $p_1 = p \circ u$, so it is a local analytic isomorphism, and thus holomorphic. As it is continuous and X_1 is connected, $u(X_1)$ is connected, and as $u \circ \varphi_1 = \varphi$, u must go to the connected component known as X. It is obvious that $G_s = F_s \circ u$ and thus all the requirements for X being an envelope are fulfilled.

This means that the envelope always exists and is unique up to isomorphism.□

Let us now define a domain of holomorphy:

Definition 2.1.3. *A domain of holomorphy X is an unramified domain which is isomorphic to its own envelope of holomorphy, and where φ is the isomorphic function between them. Another way of saying this is that φ is bijective. The envelope of holomorphy is the S-envelope when $S = H(X)$.*

Theorem 2.1.4. *An envelope of holomorphy is a domain of holomorphy.*

Proof: Remembering how X was constructed in Theorem 2.1.3, and how φ was defined in the same Theorem, we get that φ is surjective when constructing the envelope of X, as Y must be the same connected component that X is. In fact, φ is easily shown to be the identity on X. □

Theorem 2.1.5. *Let $p_0 : \Omega \to \mathbb{C}^n$ be an unramified domain and $p : X \to \mathbb{C}^n$ be the envelope of holomorphy with φ as the function between them. Then, if φ is injective, the holomorphic functions on Ω separates the points of Ω.*

Proof:

One can write $p_0 = (p_1, p_2.....p_n)$, and it is easy to see that each p_j is holomorphic by using the definition of holomorphic functions on a manifold. Now, take two points $a, b \in \Omega, a \neq b$. Either $p_0(a) = p_0(b)$ or not. If they are different, then for some j, $p_j(a) \neq p_j(b)$ and the holomorphic functions separate a and b.

Next, assume $p_0(a) = p_0(b) = z$. As φ is injective $\varphi(a) \neq \varphi(b)$, and associating X with the connected component of the sheaf as in theorem 2.1.3 . Now, let U and U_1 be neighbourhoods around a and b that is mapped by p_0 unto the same neighbourhood P around z. φ being injective implies that there is some analytic function f defined on Ω such that

$$f \circ (p_0|_U)^{-1} \neq f \circ (p_0|_{U_1})^{-1}$$

This must be true because if no such function exists, then φ will map a and b to the same point, as the point they are mapped to in X is determined by their projection down to $\mathbb{C}^n$, which is equal, and the functions defined around it.

From this one gets that there must be some $\alpha \in \mathbb{N}^n$ so that

$$D^\alpha f \circ (p_0|_U)^{-1}(z) \neq D^\alpha f \circ (p_0|_{U_1})^{-1}(z)$$

and this proves the theorem. □

Closely related to unramified is the concept of sheets.

Definition 2.1.4. *An unramified domain $p_0 : \Omega \to \mathbb{C}^n$ is called n-sheeted if $max\{k \in \mathbb{N} | \exists a_1 ... a_k \in \Omega, p_0(a_1) = p_0(a_2) = ... p_0(a_k)\} = n$. If a domain is 1-sheeted p_0 is injective, and Ω can be embedded into $\mathbb{C}^n$*

Note that the case where Ω is 1-sheeted and its envelope is not is exactly the case where the domain is not schlicht.

2.2 Analytic Discs

Kontinuitätssatz, which we will often denote as "satisfying the disc property", is a property of a domain Ω first defined by by Hartog. We will in this section show that this property is equivalent with being pseudoconvex, and use this to find a way to construct the envelope of holomorphy of a domain. Here we will first work with the case where Ω is embedded in $\mathbb{C}^n$.

Definition 2.2.1. *A domain satisfies the disc property if for any $G(t,z) : [0,1] \times \bar{\Delta} \to \mathbb{C}^n$, with G continuous in the first coordinate and analytic in second coordinate on Δ, where $G(0, \bar{\Delta}) \subset \Omega$ and $G(t, \partial\Delta) \subset \Omega$ for all $t \in [0,1]$, the entire image of G is in Ω.*

This definition can easily be extended to general manifolds embedded into a larger manifold X, simply replace $\mathbb{C}^n$ with X.

This next theorem comes from Fornæss and Stensønes (1987).

Theorem 2.2.1. *A domain $\Omega \subset \mathbb{C}^n$ is pseudoconvex if and only if it satisfies the disc property.*

Proof: First, let us prove that Ω being pseudoconvex implies that the disc property holds. Assume the opposite, that Ω is pseudoconvex but that the disc property does not hold. Then there is a function $G(x,z)$ that gives a continuous family of analytic discs, where for some $x \in [0,1]$, we have that $G(x, \partial\Delta) \subset \Omega$ but that some part of $G(x, \Delta)$ is not in Ω.There is also a plurisubharmonic exhaustion function $\rho(z)$, $z \in \Omega$.

As there is some part of the image of $G(x,z)$ that is not in Ω we can find a sequence of points $z_x \in \Delta$ such that $G(x,z_x) \to \partial\Omega$ as we go from $x = 0$ as x tends to some value s(recall that for $x = 0$ the image lies in Ω). This implies that $\rho(G(x,z_x)) \to \infty$ as $x \nearrow s$. On the other hand, $\rho(G(x,\partial\Delta))$ is a compact set, meaning $\rho(G(x,\partial\Delta)) < C$ for some C. But for some point y we have that $\rho(G(y,z_y)) > C$, which gives us an analytic disc $G(y,\bar{\Delta})$ where the maximum principle does not hold. This is a contradiction.

For the second part of the proof, we want to show that the disc property being satisfied on $\Omega \subset \mathbb{C}^n$ implies that Ω is pseudoconvex. For this we prove the contrapositive. Say that Ω is not pseudoconvex and let $u(z)$ be a distance-function that for a point z in Ω measures that points distance to $\partial\Omega$. Our assumptions would imply that $\rho(z) = -log(u(z))$ fails to be a plurisubharmonic exhaustion function. That means that there must exist a set $L = \{aw + b : w \in \mathbb{C}\}$ partially in Ω such that $\rho\mid_{L\cap\Omega}$ is not subharmonic. This again gives us the existence of an open disc D around some part of L and a harmonic function h such that $-log(u(z)) \leq h(z), z \in \partial D$ and $-log(u(z_0)) > h(z_0)$ for some point $z_0 \in D$.

From these results we get $u(z) \geq e^{-h(z)}, z \in \partial D$ and $u(z_0) < e^{-h(z_0)}$. Define the analytic function g so that $Re(g) = h$. Also define D as being the image of $\gamma : \bar{\Delta} \to \mathbb{C}^n$, γ being holomorphic. Then define the function

$$H(x,z) : [0,1] \times \bar{\Delta} \to \mathbb{C}^n$$

where

$$H(x,z) = \gamma(z) + cxe^{-g(z)}$$

c is here a unit vector constructed so that $xe^{-g(z_0)}$ goes in the same direction as the vector going from z_0 to the closest point from it on the boundary. Note that for any fixed x, we get an analytic disc. The distance between z_0 and the boundary is less than $e^{-g(z_0)}$, meaning that $H(x,z_0)$ will go past the boundary of Ω as x increases, and so we get a continuous family of closed analytic discs where the first discs are contained in Ω

but the later have part of their interior outside Ω. But $H(x, \partial D)$ is contained entirely in Ω because the distance between w and the boundary is larger than $|e^{-g(w)}|$. This means that the disc property does not hold. Having proven that not pseudoconvex implies that we do not have the disc property, we know that the disc property implies pseudoconvex. □

Pseudoconvexity is also equivalent with the disc property for Riemann domains. This comes from Theorem 3 in Chapter N of Gunning (1990). No proof will be given here.

Next, here is a way of finding, for a domain Ω, a larger domain where we can extend all $H(\Omega)$.

Theorem 2.2.2. *Let Ω be a Riemann domain embedded into a another domain X and let G be a family of continuous discs such that $\{G(0, \bar{\Delta})\} \cup \{G(x, \partial\Delta), x \in [0,1]\}$ is in Ω, but where not all of the image of G is in Ω. Then any $f \in H(\Omega)$ can be extended to $\Omega \cup Image(G)$.*

Proof: Assume that this was not true. Then there would have to exist a family of continuous discs G with $\{G(0, \bar{\Delta})\} \cup \{G(x, \partial\Delta), x \in [0,1]\}$ in Ω but image not entirely contained in Ω, and some holomorphic function f defined on Ω which can not be extended to $\Omega \cup Image(G)$. That means that the image of G is not contained in the envelope of Ω. But as $\{G(0, \bar{\Delta})\} \cup \{G(x, \partial\Delta), x \in [0,1]\}$ is contained in the envelope, we get that the envelope does not satisfy the disc property, implying it is not pseudoconvex. The envelope must therefore contain the image of G to be pseudoconvex. □

Of course, the envelope might not be schlicht, in which case pushing the discs can cause problems with one point having two values as in example 1.1.2. In that case, for a general Riemann domain Ω, we use the inclusion of Ω into its envelope, and then use the method as described earlier. Next, a construction of the envelope of holomorphy.

Theorem 2.2.3. *Let $\Omega_0 = \phi(\Omega)$, Ω's inclusion into its envelope of holomorphy. Create a sequence of domains Ω_n, $n = 0,1,2...$ in the following way: for a Ω_n you construct Ω_{n+1} by taking all continuous families of discs with first disc and boundary of all the discs in Ω_n, and add the image of the entire family of discs to Ω_n. Then*

$$\widehat{\Omega} = \bigcup_n \Omega_n = \widetilde{\Omega}$$

That is to say, the union of these domains equals the envelope of holomorphy of Ω.

Proof: There are two parts to this proof: proving that any $f \in H(\Omega)$ can be extended to $\widehat{\Omega}$ and that $\widehat{\Omega}$ is pseudoconvex. For the first part: any point $z \in \widehat{\Omega}$ must be in some Ω_N. That means that it is in a domain made by extending Ω by repeated disc-pushing, and from our earlier theorem we know that that implies that f can be extended to z.

For the second part, take a family of continuous discs G where $\{G(0,\bar{\Delta})\} \cup \{G(x,\partial\Delta), x \in [0,1]\}$ lies in $\widehat{\Omega}$. $\{G(0,\bar{\Delta})\} \cup \{G(x,\partial\Delta), x \in [0,1]\}$ is compact, and Ω_n is an open covering of $\widehat{\Omega}$ and thus of $\{G(0,\bar{\Delta})\} \cup \{G(x,\partial\Delta), x \in [0,1]\}$. By the definition of compactness, a finite covering $\Omega_{n_1}....\Omega_{n_k}$ covers $\{G(0,\bar{\Delta})\} \cup \{G(x,\partial\Delta), x \in [0,1]\}$. But $\Omega_{n-1} \subset \Omega_n$, so $\{G(0,\bar{\Delta})\} \cup \{G(x,\partial\Delta), x \in [0,1]\}$ must be in Ω_M, where $M = max\{n_j\}$. But Ω_{M+1} is created by pushing discs every way we can, so the entire image of G lies in Ω_{M+1}, meaning it lies in $\widehat{\Omega}$ too.

Since this holds for any G the disc property is true for $\widehat{\Omega}$, and by the previous theorems that implies that it is pseudoconvex. So $\widehat{\Omega}$ is a pseudoconvex domain that all analytic functions defined on Ω can be extended to. Thus it is the envelope of holomorphy. □

An obvious question would be whether it really is necessary to create a sequence of domains Ω_n or if pushing discs all ways you can only once is enough. Jöricke (2009) is about exactly this question, and this question will be what Chapter 3 builds towards answering. For now, here is an example for which it is not obvious that you can do it in one go:

Example 2.2.1. *The domain in this example can be thought of as a cylinder partially inside of a glass, where the glass and cylinder are connected by a thin thread. Some intuition can be gotten from the images 2.1 and 2.2.*

This domain V lies in $\mathbb{C}^2$ and will be the union of four different domains V_i. We use coordinates $z = x + iy$ and $w = a + ib$, and the function $k(t) = (1 + \frac{3t}{4}, 2)$ will be used to define the domains.

$$V_1 = \{(z, w) \in \mathbb{C}^2 : |z| < 2, -\frac{1}{2} < a < 0, b \in (-1, 1)\}$$

$$V_2 = \{(z, w) \in \mathbb{C}^2 : \frac{7}{4} < |z| < 2, 0 \le a < 3, b(-1, 1)\}$$

$$V_3 = \{(z, w) \in \mathbb{C}^2 : \frac{3}{4} < |z| < 1, 1 < a < 6, b \in (-1, 1)$$

$$V_4 = \{(z, w) \in \mathbb{C}^2 : ||(z, w) - k(t)|| < \frac{1}{10}\}$$

Let V be the union of these four domain.

Let us try to use the disc property to find domains to extend all analytic function on V to. We see that we can fit the $\{G_t(0, \bar{\Delta})\} \cup \{G_t(x, \partial\Delta), x \in [0, 1]\}$ of a family of continuous discs G_t into the union of V_1 and V_2 for any $b \in (-1, 1)$, having one disc in V_2 and the boundary of the other discs in V_1 and V_2. One can for example set $G_s(x, z) = (2z, x - 0.1 + is), s \in (-1, 1)$. One can also extend slightly further in the a-direction by other families of continuous discs, so that any points "inside" V_2 can be put in such a disc. This means that all analytic functions can be extended to $V \cup B$, where

$$B = \{(z, w) : |z| \le \frac{7}{4}, 0 \le a < 3, b \in (-1, 1)\}$$

This domain can be seen in Figure 2.3.

There are no other obvious families of continuous discs G with $\{G(0, \bar{\Delta})\} \cup \{G(x, \partial\Delta), x \in [0, 1]\}$ we can place inside V, but for $V \cup Image(G_s)$ one can create a new family of an-

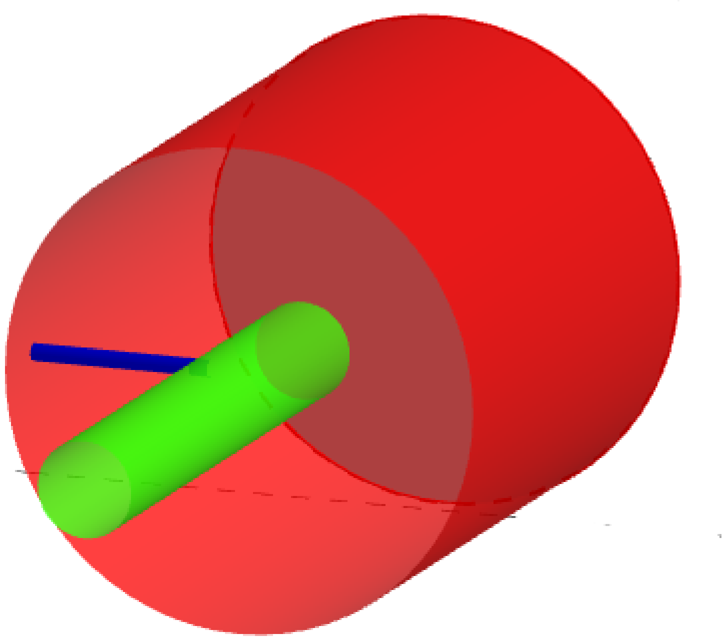


Figure 2.1: A cut of the domain when $b = 0$.

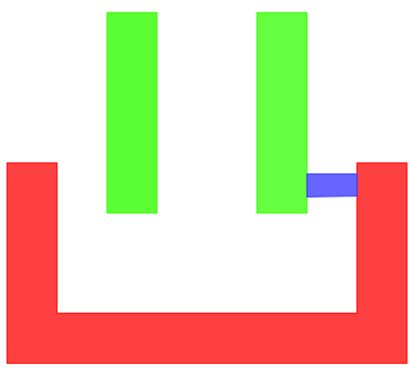


Figure 2.2: A cut of the domain when $w = 0$.

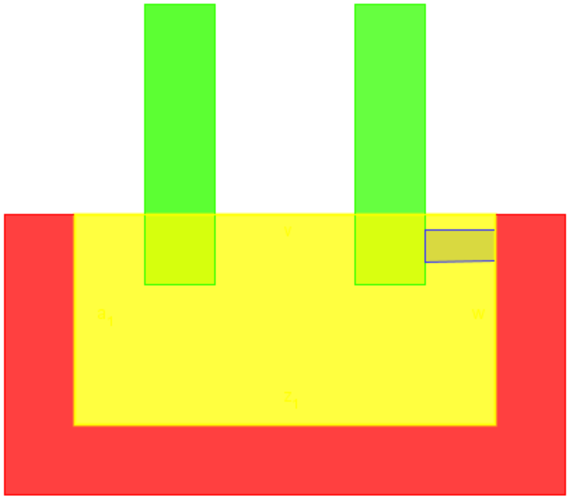


Figure 2.3: A 2D-cut of $V \cup B$.

alytic discs where $\{G_s(0,\bar{\Delta})\} \cup \{G_s(x,\partial\Delta), x \in [0,1]\}$ *is in* $Image(G_s)$ *and* V_3*, the first disc being in* $Image(G_s)$ *and the boundary of the others being in* $Image(G_s)$ *and* V_3*. For instance, families of continuous discs* H_t *might be defined as* $H_t(x,z) = (0.9z, 3x+2.9+it)$*. Like in the previous extension, one can also extend slightly further in the a-direction by other families of analytic discs. Thus one get an extension to*

$$D = \{(z,w) : |z| \le \frac{3}{4}, 3 \le a < 6, b \in (-1,1)\}$$

This domain, $V \cup B \cup D$*, can be seen in Figure 2.4.*

We can also show that the extension here is schlicht. Imagine if there were some collision of values, that would happen at the points lying in a disc with boundary in V_2*, and also in a disc with boundary in* V_3 *(or a point in* V_3*). Basically, the difference would come from the extension we get from extending the large glass not matching with the cylinder.*

It is the tread connecting the glass and the inner cylinder that makes it schlicht. Take a fattended line segment going through that thread. The extension of any analytic func-

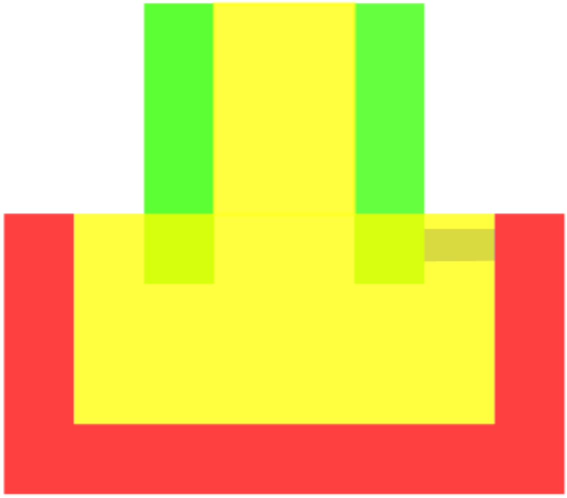


Figure 2.4: A 2D cut of a further extension of V.

tion defined on the glass, has a unique extension along this line, which must agree with how the function was already defined there. Thus the extension must match the original function in some small open ball in V_3*, and by unique extension the extension is equal to the original in the entirety of* V_3*. Finally, there can by the same arguments as before be only one possible extension to the inside of* V_3*. This means that the domain is schlicht.*

Note that the schlichtness in the example above was dependent on V_4 lying inside V_2. Should V_4 go on the outside, as in Figure 2.5, we would get a situation similar to Example 1.1.2, where we could construct a function that separates V_3 into a different sheet from the extension of V_1 and V_2.

From Theorem 2.2.3 we know that using families of discs to extend domains gives the envelope of holomorphy if one repeats the process an infinite number of times. The previous example showed a case where it is not obviously possible to do it in one go, that is to say, it is not obvious that Ω_1 equals the envelope for a domain Ω. Jöricke (2009) nevertheless states that this is possible, and some of the results from that paper will be investigated further down. The next example, courtesy of Professor John Erik

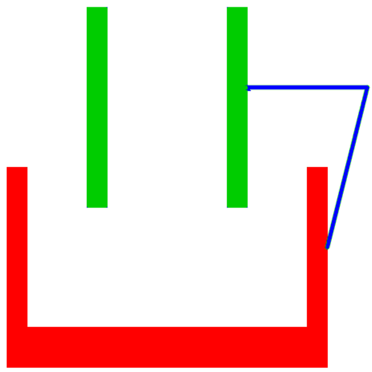


Figure 2.5: Different version of V with non-schlicht extension.

Fornæss, is an example of how one can get the points in D in the previous example to be in a family of discs with boundary in V and whose first disc is embedded in V.

Before the example, a lemma is needed, which is a consequence of Theorem 2 of Chapter 5 in Goluzin (1969).

Lemma 2.2.1. *Say that we have a continuous set of simply connected domains B in $\mathbb{C}$ whose boundary is a Jordan curve, where the domains are indexed by $t \in [0,1]$. We define a continuous set of domains as such: around for any domain indexed by k, where the boundary of the domain is parametrized by $S : S^1 \to \mathbb{C}$, we can for any $\epsilon > 0$ find a small interval around k, where the boundary of each domain indexed by one of the numbers in that interval has a parametrization $A : S^1 \to \mathbb{C}$, such that $|S(z) - A(z)| < \epsilon$ for all $z \in S^1$. This means that boundaries of discs close to each other have similar boundary.*

Here we say that the domains in B are either increasing or decreasing, so for $a, b \in [0,1]$, we either have $B_a \subset B_b$ for all $a < b$, or $B_b \subset B_a$. We also have that some point ξ is in all B_t. If we have this, then the Riemann mappings $C(t,z) : [0,1] \times \Delta \to \mathbb{C}$ from the unit disc to the various B_t which sends 0 to ξ and has positive derivative in 0 is continuous with respect to t.

Proof: Take any converging sequence $x_1, x_2 ... \in [0,1]$, where $x_n \to x$. It is sufficient

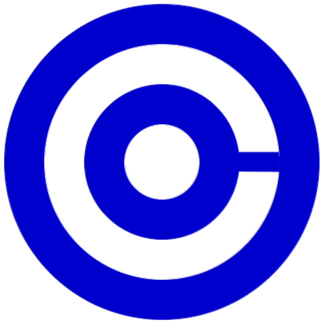


Figure 2.6: Cut of V at $w = 2$.

to show that $C(x_n, z) \to C(x, z)$ for any z. But that follows from Theorem 2 of Chapter 5 in Goluzin (1969), when you take A_n to be the unit disc, B_n to be the domains B_{x_n} and f_n to be the maps of the Riemann mapping theorem. With the assumptions above, where either $B_{x_1} \subset B_{x_2}$... or the other way around, the conditions of the theorem are obviously fulfilled, and so $C(t, z)$ is continuous in the first coordinate. □

Example 2.2.2. *Let V be the domain defined in Example 2.1. In the images used here all parts of V will be coloured blue for the sake of clarity. The task is to construct a family of discs indexed by $t \in [0,1]$ such that the first one is embedded in V, all have their boundary in V and any point on the "inside" of V_3 can be made the centre of the last disc. In this example the family will be constructed so that the point $(0,5)$ lies inside in the family. For all other points inside the cylinder V_3 the process is pretty much the same.*

Start with the disc $f(z) = (1.9z, -0.25)$. This is embedded into V. Then create a family given by $F(t, z) = (1.9z, -0.25 + 2.25t)$. To picture the disc one ends up with, look at Figures 2.6 and 2.7. These are cuts taken along $w = 2$.

Next, continually remove a part of the disc, so that one gets a "valley" and so that the boundary lies partially inside V_4. One step of this process is shown in Figure 2.8. This can be done so that each step is simply connected, and as every disc can be considered to lie in the z-plane, as its coordinate in the w-plane is constant, it is an analytic disc by

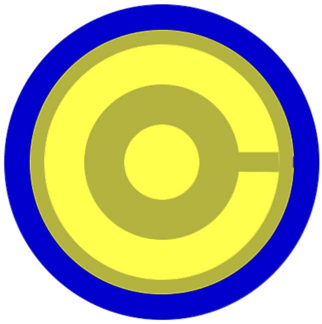


Figure 2.7: Cut of V at $w = 2$, analytic disc is yellow.

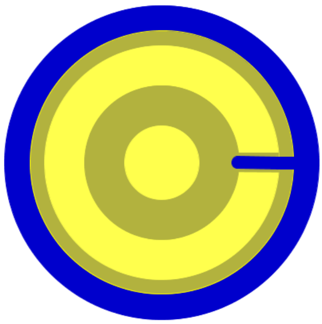


Figure 2.8: Cut of V at $w = 2$, analytic disc is yellow.

the Riemann mapping theorem. Carathéodory's theorem tells us that one can extend the function to the closure of the disc, and it is known from Lemma 2.2.1 that the Riemann mappings can be made continuously.

Using the same theorems as above, one can continually change the analytic disc to create two curves going almost around the cylinder, leaving only a very small gap, as can be seen in Figure 2.9. The exact size of the gap will be determined later. Call this disc $d : d(z) = (d_1(z), 2)$, d_1 *being the Riemann mapping.*

Take a compact set K in the complex plane that looks almost like d, except that the the small gap is just a line segment l. Let f be a function defined on K, where f is 0 on the outer almost-annulus, 1 on the disc inside of the annulus, and a linear function going

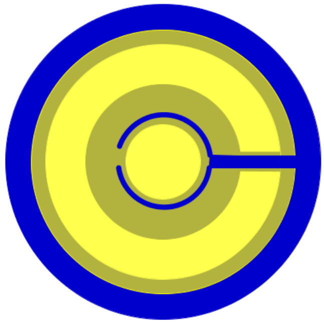


Figure 2.9: Cut of V at $w = 2$, analytic disc is yellow.

from 0 to 1 on the line segment, so that f is continuous on K, and analytic on the interior of K.

Note that K is compact and simply connected and as the w coordinate is constant one can take it to lie in the complex plane. By Mergelyan's theorem f can be approximated by a complex polynomial P, so that it approximates better than some small ϵ. Extend it in a small closed neighbourhood of the line segment l, so that the values P takes in the neighbourhood is only ϵ from the values it takes on l.

Let d be so that it matches K together with the neighbourhood. Create a new family of analytic discs $D(t,z) = (d_1(z), 2+3tP(d_1(z)))$. At $t = 0$ this is the disc d. At $t = 1$ the outer half-moon of d has not moved more than 3ϵ, which for small epsilon is hardly anything, but the disc in the center now has w coordinates close to 5. The line segment creates a line between the two parts. It is obvious that for ϵ small enough the boundary lies inside V, and by linearly shifting the disc a maximum of 5ϵ in one direction(which is again possible if ϵ is small enough), it contains the point $(0,5)$. The disc is pictured in Figure 2.10. Taking all the families of discs used here together and re-indexing, one gets the family of discs needed.

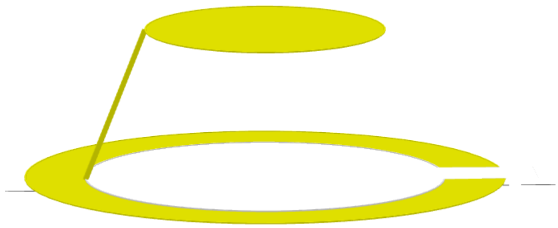


Figure 2.10: The final analytic disc.

2.3 Pulling a Disc

The following theorem will be very important. The theorem and its proof comes from Forstnerič and Globevnik (1992).

Theorem 2.3.1. *Say that we have a point p*$\in \mathbb{C}^n$ *and a closed analytic disc defined by H:*$\bar{\Delta} \to \mathbb{C}^n$ *such that H(0)=p. Further, let us say that we have a continuous function* $G(e^{i\theta},\zeta) : [0,2\pi]\times\bar{\Delta} \to \mathbb{C}^n$ *that is analytic in the second coordinate, so that for any* $G(e^{i\theta},\cdot)$ *we get a closed analytic disc. For now we shall also assume that* $G(\cdot,\zeta)$ *is continuous for all* ζ*. We also say that* $G(e^{i\theta},0) = H(e^{i\theta})$ *for all* θ*. Then we can for any* $\epsilon,\delta > 0$ *and* $r, 0 < r < 1$*, find a function Q(z):*$\bar{\Delta} \to \mathbb{C}^n$*, so that* $\| Q(z) \| < \delta$ *for any* $| z | < r$*, Q(0)+H(0)=p and* $min \| Q(z) + H(z) - G(z,\partial\Delta) \| < \epsilon$ *for* $z \in \partial\Delta$.

What does this mean intuitively? We have a closed analytic disc H with the point p contained in its interior. G is here a function that for each point q in the boundary of H($\bar{\Delta}$) gives us a closed analytic disc that contains q in its interior. What we want to show is that there is that there exists another analytic disc, Q(z)+H(z) which is almost like H for most of the interior, contains p in its interior and where any point on the boundary of the disc Q(z)+H(z) is arbitrarily close to the boundary of one of the discs G generates. There are three figures here to help with understanding what we are trying to accomplish. Figures 2.11 and 2.12 shows the starting disc and what it ends up as, while Figure 2.3.2 gives a 2D image containing H, G and the final disc.

Proof: We say that G=($g_1, g_2.....g_n$) and H=($h_1....h_n$). We then create a function v=($v_1,....v_n$), where $v_j(e^{i\theta},\zeta) = g_j(e^{i\theta},\zeta) - h_j(e^{i\theta})$. Note that this means that $v_j(e^{i\theta},0) =$

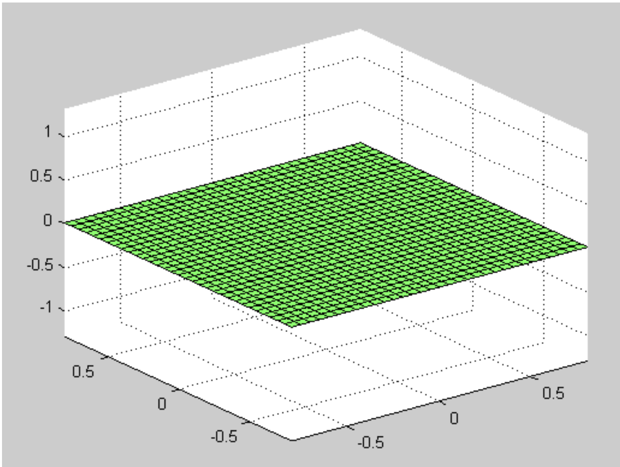


Figure 2.11: Let the green surface represent H.

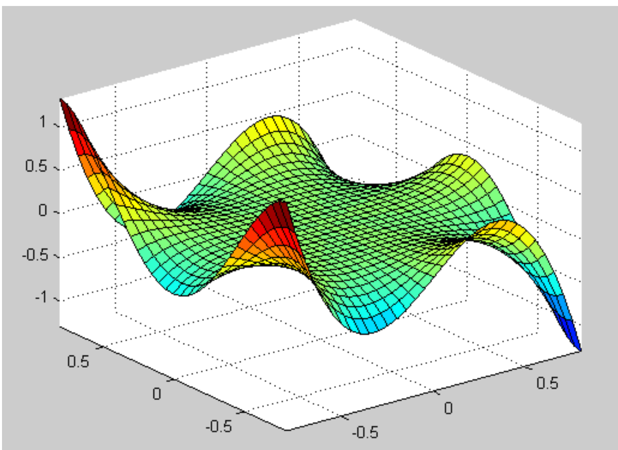


Figure 2.12: Q+H, which keeps most of H the same, but varies wildly towards the edges.

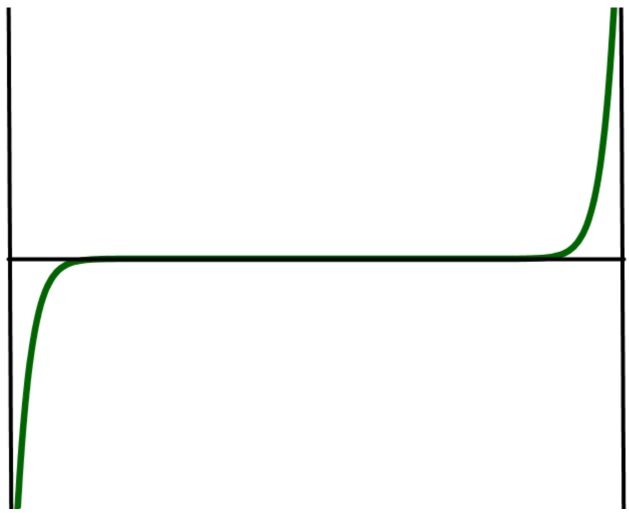


Figure 2.13: Here H is represented by the horizontal line, and G are the two vertical lines at each end of H(we represent the entire boundary of H by the two end points of the horizontal line). The green line shows the disc we end up with, which most of the time is like H, but towards the boundary it approximates the boundary of the discs in G.

0.

v_j is continuous in the first coordinate and analytic in the second, so we can expand to

$$v_j(e^{i\theta},\zeta) = \sum_{l=1}^{\infty} a_{j,l}(e^{i\theta})\zeta^l$$

Here $a_{j,l}$ is a continuous function on the unit circle. The sum starts at l=1 because we know that there is no constant term.

We now remove all terms in the sum above a certain limit, and thus get a finite sum

$$\bar{v}_j(e^{i\theta},\zeta) = \sum_{l=1}^{N} a_{j,l}(e^{i\theta})\zeta^l$$

We adjust N so that $\| v_j - \bar{v}_j \| < \frac{\epsilon}{2n}$.

Next, by using the complex version of the Stone-Weierstrass theorem we get that any continuous function on the unit circle can be approximated uniformly by a sum of the type $\sum_{n,m=0}^{\infty} a_{n,m} z^n \bar{z}^m$, where $a_{n,m}$ is a constant. This is because the identity function I(z)=z separates points, so one can construct the necessary *-algebra from that.

For each $a_{j,l}$ we can find a function $A_{j,l}$ where $| a_{j,l} - A_{j,l} | < \frac{\epsilon}{4nN}$ and where

$$A_{j,l}(e^{i\theta}) = \sum_{t,s=0}^{\infty} b_{j,l,t,s} e^{it\theta} e^{\bar{is\theta}}$$

However, as we are on the unit circle, $e^{\bar{i\theta}} = e^{-i\theta}$. Also, we can approximate $A_{j,l}$ by a finite sum

$$B_{j,l}(e^{i\theta}) = \sum_{t,s=0}^{M} b_{j,l,t,s} e^{it\theta} e^{-is\theta}$$

where we cut off t and s at M, where M is so large that $| A_{j,l} - B_{j,l} | < \frac{\epsilon}{4nN}$ for all l. As there are only N functions $A_{j,l}$ we only take the maximum of a finite set, so we can find such a maximum. By the triangle inequality we get that $| a_{j,l} - B_{j,l} | < \frac{\epsilon}{2nN}$.

One can now replace each instance of $a_{j,l}$ in the formula for $\bar{v_j}$ with $B_{j,l}$. Naming this new series C_j, we get

$$C_j(e^{i\theta}, \zeta) = \sum_{l=1}^{N} B_{j,l}(e^{i\theta}) \zeta^l$$

Putting in the definition of $B_{j,l}$ gives:

$$C_j(e^{i\theta}, \zeta) = \sum_{l=1}^{N} (\sum_{t,s=0}^{M} b_{j,l,t,s} e^{it\theta} e^{-is\theta}) \zeta^l$$

Using the triangle inequality repeatedly we get that $\| C_j - v_j \| < \frac{\epsilon}{n}$. Next, we set $\zeta = \eta^{kM}$, where k is an integer larger than or equal to 2. We put $e^{i\theta} = z$ for some $z \in \partial\Delta$, and finally we also extend C_j a little so that the function is defined on a half-open annulus R where one part of R's boundary is the unit circle, and this part of the boundary is in R, and the other part is a circle of radius r < 1. If you could not do this, then there would have to be singularities of C_j arbitrarily close to the circle, meaning that the absolute value of C_j would have to be arbitrarily big at some point on the circle. But C_j must be bounded on the circle, so that is impossible.

We now have an analytic function C_j: $R \times \Delta \to \mathbb{C}$. Next define a function q_j:

$$q_j : R \to \mathbb{C} = C_j(z, z)$$

This gives us

$$q_j = \sum_{l=1}^{N} (\sum_{t,s=0}^{M} b_{j,l,t,s} z^t z^{-s}) z^{kMl}$$

$$= q_j = \sum_{l=1}^{N} (\sum_{t,s=0}^{M} b_{j,l,t,s} z^t z^{M-s}) z^{(k-1)Ml}$$

The last rewriting of q_j makes it apparent that q_j can be extended to the entire closed disc, as none of the powers of z are negative, and q_j is a finite sum. Now we can define Q(z)=($q_1, q_2 ... q_n$). This is the function we wanted. First, for any disc with center in 0 and radius r less than 1, we can get $\mid q_j \mid < \frac{\epsilon}{n}$ by increasing the constant k. As the absolute value of z is less than r, increasing k by 1 will mean that the absolute value of q_j will lower by at least $\mid r^M \mid$. This can be done until the absolute value is low enough.

It is obvious that Q(0)=(0,0...0), so Q(0)+H(0)=p. Finally it needs to be shown that

$min \parallel Q(z) + H(z) - G(z, \partial\Delta) \parallel < \epsilon$ for any ϵ >0 and z$\in \partial\Delta$. Remember that $\mid q_j(z) - v_j(z, z^{kM}) \mid < \frac{\epsilon}{n}$ when $z \in \partial\Delta$. Using the triangle inequality and recalling that $v_j(e^{i\theta}, \zeta) = g_j(e^{i\theta}, \zeta) - h_j(e^{i\theta})$ we get the correct result. □

2.3.1 Taking a Disc from Ω_n to Ω_{n-1}

We here assume that Ω is an open domain embedded in $\mathbb{C}^n$.

The following lemma will motivate a lot of the new concepts in Chapter 3. The eventual goal of chapter 3 is to show that if we have a continuous family of analytic discs where the first disc and the boundary of all the discs lie in Ω_1, then we can create a family of analytic discs with first disc and boundary of all the discs in Ω, but where the centres of the discs in the first family are the centres of discs in the second. This would show that those points are actually in Ω_1, thus showing that $\Omega_1 = \Omega_2$.

These next lemmas will prove something much smaller. In the case where we have an analytic disc d with centre p and boundary in Ω_n, we want to find a continuous family of discs with centres along the boundary of d and who themselves have boundary in Ω^{n-1}. Then we can use Theorem 2.3.1 to get a disc with boundary in Ω_{n-1}. First, let us assume that around the boundary of d we have a continuous family of discs $G(e^{i\theta}, z) : [0, 2\pi] \times \bar{\Delta} \to \mathbb{C}^n$, where the centres of the discs follow d's boundary, and the discs have boundary in Ω_{n-1}. Then we can construct an analytic disc with centre p and boundary in Ω_{n-1}.

Note that by the way Ω_n is constructed, any point in it is the centre of an analytic disc with boundary in Ω_{n-1}.

Lemma 2.3.1. *Say that we have an analytic disc $d : \bar{\Delta} \to \mathbb{C}^n$ with centre p and boundary in Ω_{n-1}. Also say that there is a continuous family of analytic discs $G(e^{i\theta}, z) : [0, 2\pi] \times \bar{\Delta} \to \mathbb{C}^n$, where $G(e^{i\theta}, 0) = d(e^{i\theta})$ and $G(e^{i\theta}, \partial\Delta) \in \Omega_{n-1}$. Then there exists an analytic disc e with centre p and boundary in Ω_{n-1}.*

Proof: This is simply an application of Theorem 2.3.1. First, by a compactness-argument find an ϵ so small that if any point lies less than ϵ from the boundary of any of the discs in G, then it lies in Ω_{n-1}. Now simply use Theorem 2.3.1 to get a disc e with centre p and whose boundary lies a distance less than ϵ from the boundary of the discs in G □

This shows that for any element p in Ω_{n+1}, if we can find a d and G as above where p is the centre, we can find an analytic disc with boundary in Ω_{n-1} that p is the centre of. This does not prove that p is in Ω_n, but the lemma above is the starting point for theorems that will work basically the same as the lemma, but allow us to make sure that p is in fact a member of Ω_n. This would show that the process of constructing the envelope which has been discussed before, can be halted after a finite number of steps.

It might not be possible to find a continuous set of analytic discs around the boundary of d. But let us say that it was possible to choose the discs so that there were only a finite number of discontinuities. Having only a finite number of discontinuities is justified, as for any analytic disc we can always shift them a little bit. This means that if a disc has a centre on some point of the boundary, the same disc shifted continuously have centres at some interval of the boundary around that point. A compactness argument gives us a finite number of discontinuities.

Next, imagine that we could at any discontinuity, change the disc d a bit. More specifically, say that we could create a new disc, d_1, which also has centre p. d_1 would be virtually identical to d, except at the discontinuity, where it would, like in Figure 2.15, have an added curve that would allow us to bridge the discontinuity of the discs around the boundary. We basically add a curve to the boundary in a way that would give us a new analytic disc, and with continuous discs along the boundary.

The first, simple case, is where we can find a curve going from the discontinuity into Ω_n, and where the discs at each side of the discontinuity could be continually along the curve, until we at the end had a disc embedded in Ω. It is fairly obvious that any two discs in Ω with the same centre can be continually changed into each other, by shrinking and rotating the discs, so in this case we would have a continuous family of discs.

We do need to make sure that what we create an analytic disc. We can approximate the line in the following way: add to the disc d the curve previously discussed in $\mathbb{C}^n$. Attach a line segment to the closed unit disc in $\mathbb{C}$, and we can use Mergelyan's theorem to find a function that approximates d on the unit disc, and the added line on the segment added to the unit disc. This is because if the function f approximates d on the unit disc, and the curve on the added line, then all requirements for using Mergeylan's theorem are fulfilled. Mergelyan approximates function in one complex dimension, but it is obvious that it can be used to approximate functions from one to several complex dimensions.

After making the approximation we can slightly fatten out the line to on the closed unit disc and extend the approximation function to that new set, and in this way get something similar to the curve. This also means that we get something that is biholomorphic to the closed unit disc by the Riemann mapping theorem. We can also slightly shift the discs along the boundary so that they are continuous along the boundary. Then we can use Lemma 2.3.1 and get the result.

We could imagine that we can not get continuous discs along a curve, that there is another discontinuity along the curve. Then we might repeat the process, adding a new curve to the first one, and repeat the process. Also, it might be that we can not get a curve that ties the discontinuity together, but that we can use two curves to tie both sides to some third disc at the discontinuity. Then we can add two curves to get continuous discs along the boundary. This almost gives a notion of transitivity of discs with the property that we can use curves to bridge the discontinuity. One here sees the notion of an equivalence relation of discs showing up. This notion will be defined properly in Chapter 3. Next follows a result that further motivates the concept of using curves to remove discontinuities.

In the next lemma we will assume that the discs along the boundary of d were chosen so that,at the points of discontinuities, we could place trees along the boundary. Along the trees there should be a set of continuous discs which starts with the first disc in the discontinuity, and ends with the last, thus creating a continuous family of discs with boundary in Ω_{n-1} around a disc similar to d but with trees added to it, which we can call d_1. A picture of a disc with trees added can be found as Figure 2.15, and an example of two discs being connected by a tree can be seen in Example 3.1.1. It is not obvious that we can choose the discs this way, that will be covered in Chapter 3.

A quick word on what trees are. In the complex plane a tree is a finite union of line segments of finite length, so that the union of the segments is simply connected. We call one of the endpoints of one of the line segments the root, this one should not be meet any other line segments on the tree. In several complex dimensions, any continuous function from a tree in the complex plane into $\mathbb{C}^n$ is a tree. The root is the image of the root of the tree in the plane.

Theorem 2.3.2. *Assume that Ω is a domain in $\mathbb{C}^n$ with schlicht envelope. Let p be a point* *in Ω_{n+1}, and $d : \overline{\Delta} \to \mathbb{C}^n$ be a closed analytic disc which contains p (one can assume that $p = d(0)$) and which has boundary in Ω_n. If we can choose discs around the boundary of d in a way that there is only a finite number of discontinuities, and those discontinuities can be bridged by attaching trees along which we get a homotopy of analytic discs with boundary in Ω_{n-1}, then p is contained in a disc with boundary in Ω_{n-1}.*

Proof: The way to do this is to find a continuous family of discs with boundary in Ω_{n-1}, where all the discs have centres along the boundary of the analytic disc d. Then we use Lemma 2.3.1. What we will do, however, is to replace d with another analytic disc d_1 with boundary in Ω_n and $d_1(0) = p$, and find a family of continuous discs around d_1's boundary instead. The next paragraphs will construct d_1. It is essential to find a continuous set of analytic discs around d_1, we will denote those discs by $G(e^{i\theta}, z)$.

From the assumptions one can construct a piece-wise continuous family $G_0(e^{i\theta}, z) : [0, 2\pi] \times \bar{\Delta} \to \mathbb{C}^n$ (piecewise continuous in the first argument, holomorphic in the second) of discs along the boundary of d. The trouble is now to make them continuous.

For any discontinuity point q we can, by assumption, find a tree T in $\mathbb{C}^n$ so hat we can find a continuous set of discs along them bridging the discontinuity. Now, one can take a tree T_1 in the plane so that there is a continuous bijection from T_1 unto T. This is because T consists of line segments that overlap only at end points, so make the tree out of equally many line segments put together in the same way, and the construction of the function is obvious.

Take the union of the tree T_1 and the closed unit disc, so that the tree and the disc only intersect at the point q, at the root of the tree, as can be seen in figure 2.14. Taking the analytic function d and extending it along the tree using the bijection mentioned above, one gets a continuous function $(d_1, d_2 .... d_n) : \bar{\Delta} \cup T \to \mathbb{C}^n$ defined on a simply connected compact set which is holomorphic on the interior of the set. Thus, Mergelyan's theorem gives us a polynomial P approximating the polynomial as closely as one wishes by approximating each variable d_i by a polynomial P_1 and letting $P = (P_1 ... P_n)$. If the approximation is close enough, one can then shift it by a small constant

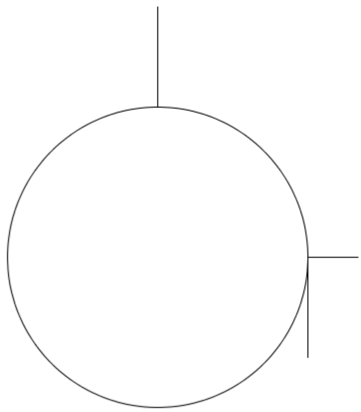


Figure 2.14: The unit disc with trees glued on at the discontinuity points of the family of discs.

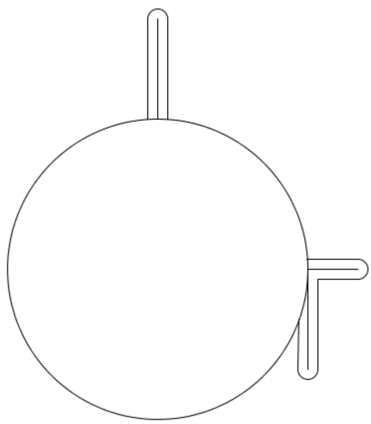


Figure 2.15: The unit disc with trees, now "fattened" so that one gets a domain biholomorphic to the unit disc.

so that $P(0) = p$ without the boundary going out of Ω_n. By assumption there is a homotopy of discs going along the tree that creates a continuous family of discs between the two discs.

Next one "fattens" the tree T_1 to a set T_2, by taking a very small constant δ and taking any taking any point with distance less than δ from the tree. Some requirements for how small δ must be are necessary.

As P is a polynomial it can be extended to the union of the disc and T_2, and we will simply call the extension P. It is required that δ is so small that that the values P takes in T_2 are close to the values it takes along T_1, particularly so that the boundary is still in Ω, but also so that any of the discs in the homotopy of discs from Lemma 3.1.2 on the tree can be shifted by a constant to create a disc on the new boundary. δ must be so small that the fattened lines in $\mathbb{C}$ does not intersect with itself in a way that would create a set that is not simply connected in the plane, furthermore one wants it so that the boundary of the domain is a Jordan curve. δ must also be so small that all other discontinuity points are further away from q than δ.

We also require that the boundary of the fattened tree meets the disc, and that the discs along the boundary of the tree and the discs along the closed unit discs meet continuously. This is always possible, but requires some work which will be detailed further down.

To remove the discontinuity one needs to make sure that the discs are continuous at the point where the tree meets the boundary of the disc. Look at the discs with centres on the unit disc closer than the δ one has gotten in the previous steps to q on either side. On each side, these discs create a continuous family $J(\gamma(t), z) : [0,1] \times \bar{\Delta} \to \mathbb{C}^n$(if one includes the end discs), where you get one of the original pair of discs at the point q when $t = 1$. Here γ is a parametrization of the closed interval from q to the point of the boundary of the unit disc δ away in either direction.

By the same arguments used earlier one can shift the entire family a value σ away.

Now, make a tiny change in index by using kt (the same k on both sides) instead of t with some k slightly more than one, and so that no disc is shifted more than σ. This means that the discs are pulled slightly back. The final requirement on δ is that it is smaller than the distance from q to $\gamma(k)$, which is the same on both sides by construction. In the point where the tree and the unit disc meets the discs from both sides is simply one of the discs in q shifted by a consonant to that point. As such there is a continuous family of discs and the discontinuity disappears.

We do this with all the discontinuities at once, making sure that nothing intersects. The final remaining problem is that we no longer have a disc, but a disc with trees glued to it. But we know that this set is simply connected, and so the Riemann mapping theorem gives us a biholomorphism unto the domain, and the discs associated to the boundary of the disc with trees can now be associated to the disc. Since the boundary of the disc with trees is a Jordan curve, the biholomorphism extends to the boundary, by Carathèodory's theorem. As such there is an analytic disc with boundary in Ω_n containing p with a continuous family of discs along the boundary, each of which have boundary in Ω_{n-1}. Theorem 2.3.1 gives us the result, when we choose ϵ to be so small that all of the boundary lies in Ω^{n-1}. □

The concept of the discs at the discontinuity being connected discs along a tree might seem arbitrary, but in Chapter 3 we will create an equivalence between discs based on trees, and this will lead to a way of constructing the envelope of holomorphy for a domain in $\mathbb{C}^n$. The concept of discs along trees will then prove quite useful.

2.4 Pulling a Family of Discs

Having shown how to pull one disc, this section will show that this can be done for a family of discs, that is to say that for a continuous family of discs $H(t,z)$, with a disc $G(t,e^{i\theta},z)$ going through every boundary point of every disc, and where the discs along the boundary change in a continuous manner, one can create a new family of discs that is mostly close H, but whose boundary lies close to the boundary of the discs $G(t,e^{i\theta},z)$.

The proof will for the most part be exactly the same as the proof for Theorem 2.3.1, just with an added variable.

Theorem 2.4.1. *Say that we have a curve $p(t) \in \mathbb{C}^n$ and a family of closed analytic discs defined by $H : [0,1] \times \bar{\Delta} \to \mathbb{C}^n$ such that $H(t,0) = p$. Further, let us say that we have a continuous function $G(t, e^{i\theta}, \zeta) : [0,1] \times [0,2\pi] \times \bar{\Delta} \to \mathbb{C}^n$ that is analytic in the third coordinate, so that for any $G(t, e^{i\theta}, \cdot)$ we get a closed analytic disc. We also say that $G(t, e^{i\theta}, 0) = H(t, e^{i\theta})$ for all θ. Then we can for any $\epsilon, \delta > 0$ and $0 < r < 1$ find a function $Q(t,z) : [0,1] \times \bar{\Delta} \to \mathbb{C}^n$, so that $\parallel Q(t,z) \parallel < \delta$ when $\mid z \mid < r$, $Q(0) + H(0) = p$ and $min \parallel Q(t,z) + H(t,z) - G(t,z,\partial\Delta) \parallel < \epsilon$ for $z \in \partial\Delta$.*

Proof: We say that $G = (g_1, g_2 .....g_n)$ and $H = (h_1....h_n)$. We then create a function $v = (v_1,....v_n)$, where $v_j(t, e^{i\theta}, \zeta) = g_j(t, e^{i\theta}, \zeta) - h_j(t, e^{i\theta})$. Note that this means that $v_j(t, e^{i\theta}, 0) = 0$.

v_j is continuous in the first and second coordinates and analytic in the third, so we can expand to

$$v_j(t, e^{i\theta}, \zeta) = \sum_{l=1}^{\infty} a_{j,l}(t, e^{i\theta})\zeta^l$$

Here $a_{j,l}$ is a continuous function on the cylinder $Cy = [0,1] \times \partial\Delta$. The sum starts at l=1 because we know that there is no constant term.

We now remove all terms in the sum above a certain limit, and thus get a finite sum

$$\bar{v_j}(t, e^{i\theta}, \zeta) = \sum_{l=1}^{N} a_{j,l}(t, e^{i\theta})\zeta^l$$

We adjust N so that $\parallel v_j - \bar{v_j} \parallel < \frac{\epsilon}{2n}$.

Next, by using the complex version of the Stone-Weierstrass theorem we get that any continuous function on Cy can be approximated uniformly by a sum of the type $\sum_{n,m,o=0}^{\infty} a_{n,m,o} z^n \bar{z}^m t^o$, where $a_{n,m,o}$ is a constant. This is because the two functions I(t,z)=z and T(t,z)=t separates points on Cy, so one can construct the necessary *-algebra from that. Finally, as $\bar{t} = t$ on Cy, one does not need to count in $\bar{t}$.

For each $a_{j,l}$ we can find a function $A_{j,l}$ where $\mid a_{j,l} - A_{j,l} \mid < \frac{\epsilon}{4nN}$ on $[0,1] \times \partial\Delta$ and where

$$A_{j,l}(t, e^{i\theta}) = \sum_{o,u,s=0}^{\infty} b_{j,l,u,s,o} t^o e^{iu\theta} e^{\bar{i}s\theta}$$

However, as we are on the unit circle in the second coordinate, $e^{\bar{i\theta}} = e^{-i\theta}$. Also, we can approximate $A_{j,l}$ by a finite sum

$$B_{j,l}(t, e^{i\theta}) = \sum_{o,u,s=0}^{M} b_{j,l,t,s} t^o e^{iu\theta} e^{-is\theta}$$

where we cut off o,u and s at M, where M is so large that $\mid A_{j,l} - B_{j,l} \mid < \frac{\epsilon}{4nN}$ for all l. As there are only N functions $A_{j,l}$ we only take the maximum of a finite set, so we can find such a maximum. By the triangle inequality we get that $\mid a_{j,l} - B_{j,l} \mid < \frac{\epsilon}{nN}$.

One can now replace each instance of $a_{j,l}$ in the formula for $\bar{v}_j$ with $B_{j,l}$. Naming this new series C_j, we get

$$C_j(t, e^{i\theta}, \zeta) = \sum_{l=1}^{N} B_{j,l}(t, e^{i\theta}) \zeta^l$$

Putting in the definition of $B_{j,l}$ gives:

$$C_j(t, e^{i\theta}, \zeta) = \sum_{l=1}^{N} (\sum_{o,u,s=0}^{M} b_{j,l,t,s} t^[o e^{iu\theta} e^{-is\theta}) \zeta^l$$

Using the triangle inequality repeatedly we get that $\parallel C_j - v_j \parallel < \frac{\epsilon}{n}$. Next, we set $\zeta = \eta^{kM}$, where k is an integer larger than or equal to 2. We put $e^{i\theta} = z$ for some $z \in \partial\Delta$, and finally we also extend C_j a little so that the function is defined on a half-open annulus R where one part of R's boundary is the unit circle, and this part of the boundary is in R, and the other part is a circle of radius r < 1. If you could not do this, then there would have to be singularities of C_j arbitrarily close to the circle, meaning that the absolute value of C_j would have to be arbitrarily big at some point on the circle. But C_j must be bounded on the circle, so that is impossible.

We now have an analytic function C_j: $[0,1] \times$ R $\times \Delta \to \mathbb{C}$. Next define a function q_j:

$$q_j : [0,1] \times R \to \mathbb{C} = C_j(t,z,z)$$

This gives us

$$q_j = \sum_{l=1}^{N} (\sum_{o,u,s=0}^{M} b_{j,l,t,s} t^o z^t z^{-s}) z^{kMl}$$

which we rewrite to:

$$q_j = \sum_{l=1}^{N} (\sum_{o,u,s=0}^{M} b_{j,l,t,s} t^o z^t z^{M-s}) z^{(k-1)Ml}$$

The last rewriting of q_j makes it apparent that q_j can be extended to the entire closed disc, as none of the powers of z are negative, and q_j is a finite sum. Now we can define $Q(t,z) = (q_1, q_2...q_n)$. This is the function we wanted. First, for any disc with center in 0 and radius r less than 1, we can get $| q_j |< \frac{\epsilon}{n}$ by increasing the constant k. As the absolute value of z is less than r, increasing k by 1 will mean that the absolute value of q_j will lower by at least $| r^M |$. This can be done until the absolute value is low enough.

It is obvious that $Q(t,0) = (0,0...0)$, so $Q(t,0) + H(t,0) = p(t)$. Finally it needs to be shown that

$min \parallel Q(t,z) + H(t,z) - G(t,z,\partial\Delta) \parallel < \epsilon$ for any $\epsilon > 0$ and $z \in \partial\Delta$. Remember that $| q_j(t,z) - v_j(t,z,z^{kM}) | < \frac{\epsilon}{n}$ when $z \in \partial\Delta$. Using the triangle inequality and recalling that $v_j(t,e^{i\theta},\zeta) = g_j(t,e^{i\theta},\zeta) - h_j(t,e^{i\theta})$ we get the correct result. □

2.4.1 Taking a Family of Discs from Ω_n to Ω_{n-1}

This section will be about doing the same thing we did in Lemma 2.3.2, but for a family of discs. If we could for any continuous family of discs H with boundary of all the discs and first disc in Ω_n find a family of discs J with boundary in Ω_{n-1}, for which the centres in H lie in the centres of the discs in J, and in addition $J(0,z) \in \Omega_{n-1}$, then by the definition of Ω_n we would have that $\Omega_n = \Omega_{n-1}$. This would mean that Ω_1 is the envelope of Ω, which is what we wanted to prove.

Here we will construct J in a case where we have several assumptions made. These assumptions will be justified in Chapter 3. First, while we do not assume that all discs around the boundary of H are continuous, we assume that on each disc $H(t,\cdot)$ they can be made piecewise continuous, and that at the discontinuities we, like in the section "Taking a disc from Ω_n to Ω_{n-1}", have trees that we can use to create a continuous set of discs.

We also assume that there is a curve $a(t) : [0,1] \to \partial\Delta$ which takes values close to 1, and where along the curve $H(t, a(t))$ and in a small neighbourhood of the boundary around it which we will call Γ we have it so that the discs are very small discs embedded in Ω, and that this is always the same disc, just linearly shifted. An underlying assumption is then that $H(t, a(t))$ is in Ω. Finally, we assume that the first disc in H is embedded in Ω, and for that disc we can make the continuous set of discs into small discs embedded in Ω. Again, all these assumptions will be justified in the next chapter. Also note that by applying continuous rotations on the closed unit disc, we can make it so that $a(t) = 1$ for a rotated version of H.

Theorem 2.4.2. *Under the assumptions made above, we can for a family of discs H find a new family of discs J with first disc and boundary of all discs in Ω, and whose centres after a time follow the curve $H(t,0)$.*

We split the proof up into two lemmas.

- The first will create a piecewise continuous family of analytic discs which will be approximations of the discs in H, but with trees. Around each there should be a continuous set of discs.
- The second lemma will make this piecewise continuous family into a continuous one by adding trees in strategic places and creating new discs to "tie together" the discontinuities.

Lemma 2.4.1. *From the family of discs H previously described we can find a piecewise continuous family of analytic discs H_1, where the discontinuities happens in the*

Figure 2.16: What we wish to accomplish with Lemma 2.4.1. Each colour marks a continuous family of discs. At the discontinuity the two families of discs have approximately the same "main" part, but different trees attached. Each individual disc looks like Figure 2.17.

t-coordinate, and all points $H(t,0)$ are contained in the new family(and those are the only centres of the discs in H_1). There are finitely many discontinuity points, and at each discontinuity point one can find one disc from each of the two continuous families on either side that would be a continuous extension of each family. These two discs are basically the same, i.e they are the same basic disc but with different fattened trees. Also, at $H_1(t,a(t))$ we should have small discs along the boundary.

Each continuous piece of analytic discs has a continuous set of discs defined along the boundary. However, along $H_1(t,a(t))$ the boundary discs are small and embedded in Ω, and the discs are just the same tiny disc linearly shifted. This also holds in some small neighbourhood of the curve, which we call Γ.

Proof: To start, take any disc $d = H(k,\cdot)$. The boundary of the disc has a lift to Ω^1, and using Theorem 2.3.2 one constructs a disc e which almost contains the original d, as the change is less than δ. In addition there are a finite number of trees along the boundary. For this disc we have a continuous set of analytic discs G along the boundary of e.

We can shift e and all discs around its boundary a little in all directions. Now, as H is a continuous family of discs, one can find other discs close to d in H_2 so that e shifted by a constant is an approximation which never differ by more than δ, except that it has trees. These shifted discs can be chosen so that they have the same centre as the discs in H they are approximating. Thus e and linear shifts of e are approximations of discs close to d. These approximations are done by linear shifts, but one could possibly cover more discs by letting e change continuously. However, this is not important for the proof.

This way one finds an interval around the index k of d of indexes of discs, where all the discs can have the same trees, only continually changing, and the same discs, except again changing continuously. Taking these intervals around each disc one gets an open cover of the interval $[0,1]$, or of all the discs. Take a finite cover, and each cover gives a continuous family of discs with continuous discs along the boundary. It is easy to see that this construction fulfils all the requirements in the first paragraph of the description of the Lemma.

What remains then is to prove the second paragraph of the lemma, that which concerns itself with $H_3(t,a(t))$. We remember from Theorem 2.3.2 that we place trees in the intersection between two of the open intervals in the cover of the boundary we create. That means that we have some choice of where the trees are put, so we can make sure that the trees are not on $H_1(\cdot,a(t))$. In addition, by a compactness-argument, we can make a small interval around $H_1(\cdot,a(t))$ where there are no trees.

By assumption we can choose the discs around the boundary so that they are small around $H_1(t,1)$. The lemma follows. □

Lemma 2.4.2. *The family of discs H_1 described in Lemma 2.4.1 can be made into a continuous family of closed analytic discs H_2 where the points $H_2(t,0)$ are contained in the centres of H_1 and vice versa. Also, the boundary of the discs is each the center of an analytic disc with boundary in Ω, and those discs varies continuously with the boundary.*

Proof: The problem here is to, at any discontinuity point t_0 in H_1, find a way to create a continuous family of discs that connects the two families of discs that meet at the discontinuity point(we can call the first disc, first being determined by index, d^-

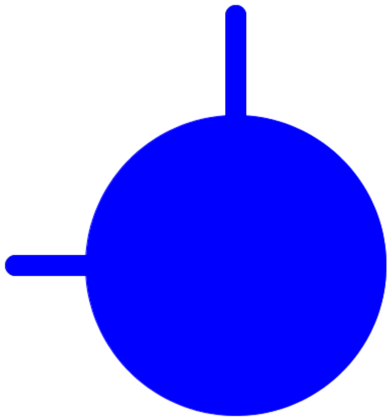


Figure 2.17: An image of D.

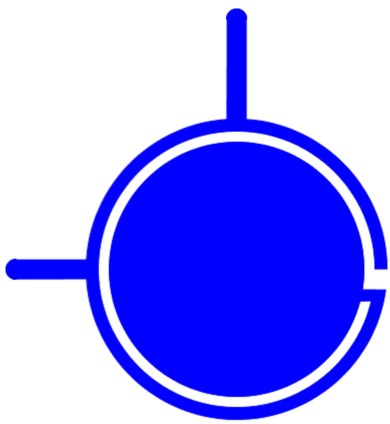


Figure 2.18: The result of "peeling" a disc: removing points around $b(t)$ until you get a slightly smaller disc with a tree going around the disc.

and the other d^+), so that a continuous set of analytic discs with boundary in Ω can be found along the boundary. The process that we will use here can be imagined as "peeling". If one imagines the first disc d^- as an unpeeled orange, and d^+ as a peeled one, then one starts with the unpeeled orange and then makes cut in the disc in a circle around the boundary, "revealing" d^+ underneath. Then one will get something close to d^+ except with a tree attached to it and all other discs in the same continuous piece(the tree being the orange peel still barely attached). This process is then repeated for all discontinuity points.

Note that we do not end up with d^+, but with an approximation of d^+ with a tree added. Approximations of the continuous family of discs d^+ belongs to will have this tree continuously added to it.

First, let us assume the simple case where d^+ has no trees, meaning that the construction in Lemma 2.4.1 was straightforward and a continuous set of analytic discs with boundary in Ω could be found around the boundary without any trouble.

Let $F : \bar{\Delta} \to \mathbb{C}^n$ and $G : \bar{\Delta} \to \mathbb{C}^n$ be the continuous functions(holomorphic on the interior) that defines d^- and d^+ respectively. Let $F_1, G_1 : \partial\Delta \times \bar{\Delta} \to \mathbb{C}^n$ be the function giving the continuous set of discs along the boundary of each analytic disc.

As said before, the discs d^- and d^+, as well as those in F_1, G_1 can be moved by some small constant, let us say by a value smaller than ϵ for both of them. We also want all trees to be able to be shifted by ϵ, and as there are finitely many one can indeed find such an ϵ.

We remember from the proof of Theorem 2.3.1 that we construct a disc in the plane which is the closed unit disc together with a finite number of trees attached at the boundary. The defining functions from the unit disc to the analytic disc (F and G in the cases above) were constructed by taking a composite of a function from the disc with trees and the function η gotten from applying the Riemann mapping theorem to the unit disc and the closed unit disc with trees. As the discs d^- was constructed this way, we can find the function F^1, which is the function from the unit disc to d^-. For the case of d^-, call the closed unit disc with trees D.

Let us assume that $a(t) = 1$ at the discontinuity, and that $\eta(1) = 1$, which we can say by the use of rotations. We also assume that 1 is on the boundary of D, that is to say that no fattened trees are at 1 or in some small neighbourhood around it. Then, take a curve $b(t)$ in D starting at 1 and then going slightly in towards 0 to a point $r < 1$, before counter-clockwise along $re^{i\theta}$ almost the entire way around back to r again. This curve should be so close to $\partial\Delta \in D$ that the image of $F^1(b(t))$ lies less than $\dfrac{\epsilon}{3}$ away from the image of $F^1(\partial\Delta)$ and the image of $G(\partial\Delta)$. Recall that d^+ and d^- are approximations of the same disc, so if that approximation is made good enough, this holds.

We now remove the points around $b(t)$ that lies so close that the the image of the points under F^1 differs from the image of $F^1(b(t))$ by less than $\frac{\epsilon}{100}$. This is done continuously, that is to say we create a continuous family of analytic discs, the first being D, and the others have a small part removed along $b(t)$. We remove part of D all the way around to the end of $b(t)$. The endpoint of $b(t)$ is determined to be $re^{i\theta}, \theta < 0$, where r is as close to 1 as previously mentioned and θ being so that $F^1(e^{i\theta})$ maps to the part of the boundary of d^- so close to 1 that the discs defined around the boundary are small discs, that is to say that $F^1(e^{i\theta})$ is in Γ. We must make sure that D with this set removed is still connected, but that is possible.

It follows from 2.2.1 that the Riemann mapping theorem gives a continuous family of discs when the domain the unit disc is mapped to changes continuously. That means that when one continuously removes $b(t)$ and the points close to it as described above, it gives us a continuous family of analytic discs $D(t,z)$ lying in $\mathbb{C}$. Taking $B(t,z) = F^1(D(t,z))$ we get a continuous family of analytic discs in $\mathbb{C}^n$, where $B(0,z)$ coincides with d^-.

We need to find a continuous set of analytic discs with boundary in Ω along the discs $B(t,z)$, so that for $D(0,z)$ the discs match those defined for d^-, in way that would help "tie" together the discs d^- and d^+. We split the boundary of $B(t,z)$ into three parts: we call the boundary of D for ∂D, and the boundary of the inner part of the removed set along $b(t)$ will be split up into the parts b_1 and b_2. b_1 marks the part of the curve closest to ∂D, and b_2 the other half. We make it so that $b_2(t)$ is simply a curve following $r_1 e^{i\theta}$ for some r_1 (this is useful in the case where d^+ has trees), and let b_1 make a small curve at the end to tie together it and b_2. Now we define the analytic discs along each part.

- For $F^1(\partial D)$ we define the discs in the same way that they were defined for d^-, that is to say taking F_1.
- For $F^1(b_1)$ take the discs in G_1. These are defined along the boundary of $\partial\Delta$ (remember that we first assumed that there were no trees on d^+), so we can just shift them slightly to get a similar family of discs defined along $F^1(b_1)$. So if $b_1(t) = re^{iv}$, then take the disc that is on the point $G_1(e^{iv})$.
- $F^1(b_2$ is defined in the same way as $F^1(b_1)$.

We need to check that this is continuous at the part where these different definitions meet. As the second and third parts of the boundary only meets the third at the place close to 1, where the discs defined is the same, small disc, just linearly shifted, they are continuous. Where $F^1(b_1)$ and $F^1(b_2)$ meet, the discs are defined by the same disc in G_1, so again we have continuity.

Remember that the constructions in previous lemmas were such that d^- and d^+ were approximations of the same analytic disc, just with different fattened trees added to them. As long as δ was small enough, they were both approximations that on the "main body", the part without the trees, the approximations were better than δ. That means that aside from at the trees d^- and d^+ approximate each other better than 2δ.

Here we assume that d^+ does not have any trees, in which case we can take d^+ as an analytic disc from $\bar{\Delta} \in D$. Now, if r is big enough and δ is small enough, then the values that $B(1,z)$ takes at the inner part, that is to say the disc formed by completing the arc b_2 to a complete circle, gives us an approximation of d^+. Thus, $F^1(B(1,z))$ is an approximation of d^+ with a tree, the tree being the image under F^1 if the part of D separated by the boundary of D and b_1. We can always continuously make it so that d^+ goes from the new, smaller disc instead of from the unit disc.

Now we want to create a continuous family of discs going from $F^1(B(1,z))$ and along the continuous family of discs that d^+ belongs to, while keeping the centres intact and making sure that these continuous analytic discs goes all the way to the next discontinuity point(or to the end of H_2 if this was the last discontinuity). To do this, we use Mergelyan's theorem on $B(1,z)$. We want to find a function on $B(1,z)$ that we can approximate with Mergelyan's theorem, where the function when added to $F^1(B(1,z))$ should approximate the family of discs that d^+ belongs to, while having the tree pretty much still, so that we can make sure it does not leave the domain or something similar.

We might not be able to find a function that can do this in one go, but we should be able to find one that lets us get a little bit further. Look at d^+ and the family it belongs to, which we will call $J(t,z)$. As said before $F^1(B(1,z))$ approximates $J(0,z)$ as closely as we want, say better than τ. τ should already be so small that we can make a straight line in Ω between any $F^1(B(1,e^{i\theta}))$ and $J(0,e^{i\theta})$.

Now, take the closed unit disc in the complex plane together with a tree connected to the boundary at the same place that $B(1,z)$ has a tree, except not fattened. The tree should otherwise be the same as the tree in $B(1,z)$. Now, on the closed unit disc, we can now define the function $f = J(0,z) - F^1(B(1,z))$, the difference between d^+ and its approximation. This function is holomorphic. On the tree we let f have the value 0, except at the part of the tree closes to the unit disc, where it linearly goes from 0 to the value the closed unit disc takes at the intersection of it and the tree. This line segment must be so short that the entire tree lies in Ω.

By Mergelyan's theorem we can approximate f as closely as we wish, say by an error smaller than $\dfrac{\alpha}{2}$, by a polynomial P. Extend P slightly to the fattened version of the tree, but not more than so that P on the fattened tree has a value so small that the tree on $F^1(B(1,z))$ can be moved that much without going out of Ω. It might be that the tree on

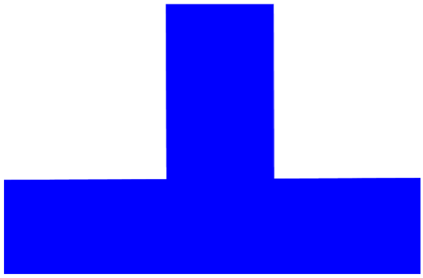


Figure 2.19: l grown out.

$B(1,z)$ is too fat, so that the extension of P gives too large values. In that case, create a continuous family of discs from $B(1,z)$ to a version where the tree has been "shrunk", so that the problem no longer exists.

Taking the family of discs $F^1(B(1,z)) + tP(z)$ we get a family of discs that when $t = 1$ approximates d^+ better than $\frac{\alpha}{2}$. Each disc constructed here will have the same centre as the discs in H_2, but if that were not the case, the fact that our approximation is better than $\frac{\alpha}{2}$ means that each disc need only be moved a maximum of $\frac{\alpha}{2}$ to get it to the centre of the disc we are approximating. The error then becomes α. Moving the discs like this can be done continuously, and will be useful for what is coming up. For now, we just see that we can get as good an approximation of d^+ as we want, except with a tree added.

Now we use the same process as in the last two paragraphs, except that f on the closed unit disc now is the difference between $F^1(B(1,z)) + P(z)$ and $J(t_1,z)$. When this is added to the approximation of a previous disc(for starters, the approximation of d^+), we move to an approximation of the $J(t_1,z)$. We then repeat the process, and by a compactness-argument we only need to do this a finite number of times to get to the next discontinuity point on H_2 (or the end of H_2). This solves the problem in the case that d^+ has no trees.

Now we take the case where d^+ has trees. Here we use a process we might call "growing twin trees". We recall that when we started cutting the curve $b(t)$ through D, we

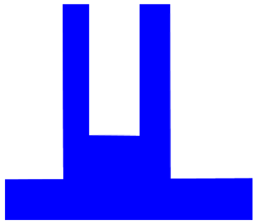


Figure 2.20: l being cut down the middle to create two twin trees.

needed to define the continuous set of discs along the boundary, and that for both b_1 and b_2 these were the discs defined on d^+. That means that when we do this process when d^+ has trees, trees must be added to make it so that the discs are continuous, and to get a good approximation of d^+.

As mentioned before, the construction of d^+ consists of taking the union of the closed unit disc and some trees, and then approximating the disc we wanted. Call the disc with trees in the case of d^+ E(recall that this was called D for d^-). In this construction, we have some choice about how the trees attached to the unit disc looks. Here, we want the trees to be so small that they fit well inside the part we dig out of D, note that this is not a problem, as we just redefine the function from the tree to $\mathbb{C}^n$ to make it fit. We can also make it so that at each point where several edges meet on the tree, only two meet. In the case that there would be several edges meeting, we can just move them slightly.

Finally, we make it so that the tree starts of with one edge, being the "main" part of the tree, and all edges connected to it are added to the left of that edge. This also holds for all sub-trees of the tree, where other edges meet edges connected to the "main" part. Call this tree T, and the function on it used to define the tree in $d^+$for g.

Now, when $b(t)$ has arrived at a place where d^+ has a root, we stop, and create a new continuous family of discs that at the end gives us the trees we need. As said before, we want to create to copies of T, one on b_1, one on b_2, and we easily see that the one on b_1 should be a mirrored version of T. The way we do this is like this: start with

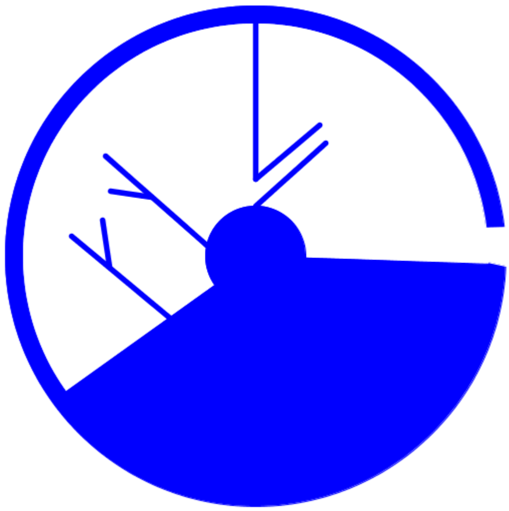


Figure 2.21: Inner trees being constructed. Two twin trees are being constructed at the place $b(t)$ has gotten to, and one pair of trees have already been constructed.

continuously growing a fattened edge l from where b_1 and b_2 meet, and let the function on this edge be an approximation of g on the main part. More on how exactly that works later.

The continuous family of discs defined along l are those defined along the right side of T, on both sides of l. This makes it easy to see that the discs along the boundary are continuous. Now, in the case where T only consisted of one single edge, we would at this point cut l in to. To be specific, once l had grown completely, we would continuously remove a small set of points along the central line of l, thus creating a Y-shaped tree, until it reached the bottom, when it would become two trees. The discs defined on this inner boundary would on both sides be the discs T has along the left boundary. In this way we end up with two copies of T, one mirrored, and can continue the process of removing points along $b(t)$. Figure 2.21 shows what we end up with.

In the case where there T is more complicated, the process starts as before, with making l and then cutting it. But then we come to the point where we can not cut any longer, because T has an edge s there. We then stop cutting l, and instead let a new edge grow from the center of the Y-shape. Recall that there is only one edge attached at any point of the main part. Now, we repeat the process we have done with l on this new edge, except that this one estimates s. We let the edge grow, have the function on the edge be the one g is on s, and let the discs along the boundary be the the ones on $g(s)$'s right side, and then start cutting. Repeat this process until all edges have been cut, and you are left with two mirrored versions of T, with the discs properly defined.

When we repeat this process all the way around, we get a disc that approximates d^+ together with a tree as before. The rest of the process is as it was previously. Thus, the only remaining issue is the extension of F^1 unto the copies of T. If we were to extend F^1 itself, we would only get the values F^1 had on D in the removed parts. What we instead do is to once again use Mergelyan's theorem.

When we reach the point where we want to start growing T, we construct the disc with a line segment from which l could be considered a fattening. On this construct, we create a polynomial C that approximates F^1 on the remaining part of D, and approximates g on the line segment. We then fatten, making sure that the boundary of l is so close that all necessary boundary discs can be centred where they should. We make sure that C takes the same value that F^1 does at 0, and so we create a family of continuous discs $(1-t)F^1 + tC$, that all have the same centre. When we extend C to l we get what we want. Note that when we cut l, we must make the part we remove so small that what remains is still a good approximation. Repeat the process for all edges on T when we reach those.

It might be that the trees we originally had on E were too big to fit inside the small part of D we are digging out, after all we made E in the previous lemma, before we knew how much discs could be shifted. In that case, simply create a new version of E, E_1, with trees with the same structure, but small enough to fit in the removed part of D. Then construct a function from E_1 to E, where the closed unit disc is mapped to the closed unit disc, and the trees are mapped unto the corresponding tree on E. Then use Mergelyan's theorem to approximate this function. Then, when we want to move along the family of d^+ as mentioned before, we first construct the function on E, then use the constructed function.

This solves the case where there are trees on d^+. □

Proof of Theorem 2.4.2:

Using lemmas 2.4.1 and 2.4.2 we get a continuous family of discs A with the first disc embedded in Ω and with a continuous family of discs with boundary in Ω along the boundary of A, and whose centres follow the curve $a(t) = H(t,0)$ after a certain point. Now, simply use Theorem 2.4.1 with an ϵ so small that all of the boundary of the new family of discs are in Ω. This is possible by a compactness-argument. We can make it so that the discs along the boundary of the first disc has a very small radius, so that the disc is still embedded in Ω when we construct our new family. Theorem 2.4.1 keeps the centres in place, and thus we have the theorem. □

Chapter 3

Burglind Jöricke's paper

This chapter deals with Burglind Jöricke's paper (Jöricke (2009)). The main idea is this: we take the set of all analytic discs of a certain type with boundary in a domain Ω, and create an equivalence relation between them. We then give the equivalence classes the structure of a Riemann domain, and show that that Riemann domain is the envelope of holomorphy. Finally we show that the domain is in fact Ω_1.

3.1 A Riemann Domain Based on Discs in Ω

The purpose of this section is this: to define a set of a particular type of analytic disc which we will call Ω^0, and then construct an equivalence relation on Ω^0 to create a new set called Ω^1. Ω^1 will then be given the structure of a Riemann domain. Everything here comes from Jöricke (2009).

One thing to note is that unlike Jöricke's work, here we only work with domains embedded in $\mathbb{C}^n$.

First, for a domain Ω, define Ω^0 as the set of analytic discs d in $\mathbb{C}^n$ with the following properties:

Definition 3.1.1. *An analytic disc is in Ω^0 if there is a continuous family of discs $G(t,z)$:*

$[0,1] \times \bar{\Delta} \to \mathbb{C}^n$, *where* $G(0,\cdot)$ *is embedded in* Ω, $G(1,\cdot) = d$, *and* $G(t, e^{i\theta}) \in \Omega$ *for all* $t \in [0,1]$ *and* $\theta \in [0,2\pi]$.

One can think of these discs as the final disc in the continuous family of analytic discs used in for instance Theorem 2.2.3, or as the set of discs for which there exists a homotopy of discs with boundary in Ω that connectes the disc to a small disc embedded in Ω. A disc in Ω^0 will be called a Ω^0 disc. Next, we define an equivalence relation on Ω^0.

Definition 3.1.2. *The equivalence relation is defined to be the smallest, that is to say the one with the fewest possible equivalent discs, so that these two requirements hold:*

- *Two* Ω^0 *discs that are embedded into* Ω *and have common center are equivalent.*
- *The equivalence relation is preserved under homotopies of equally centered pairs of* Ω^0 *discs.*

We will simply refer to this relation as the "equivalence relation", as there are no other equivalences on discs used here, so there is no confusion.

Another way of looking at the second requirement is that if for two equally centered Ω^0 discs there is a path from the center to another point, and you for each disc create a continuous family of discs, $F_1(t,z)$ and $F_2(t,z)$, where the center goes along the path and the families are so constructed that they have the same center for any $t \in [0,1]$, that is to say $F_1(t,0) = F_2(t,0)$, and the discs at the end point of the continuous families are equivalent, that is to say $F_1(1,z)$ is equivalent to $F_2(1,z)$, then the two first discs are equivalent.

Here is a short lemma about this equivalence relation:

Lemma 3.1.1. *Any pair of equivalent discs can be constructed by performing the following operations a finite number of times:*

- *take a pair of equally centered discs embedded in* Ω.

- *take a pair of discs that is homotopic through a pair of equally centred discs to a pair of equivalent discs, as in part 2 of definition 3.1.2.*
- *let d_1, d_2, d_3 be Ω^0 discs where d_1 and d_2 are equivalent, and d_2 and d_3 are equivalent. Then take the pair d_1, d_3.*

Proof: First, notice that step 1 and 2 gives us equivalent discs as defined in definition 3.1.2. Also notice that this way of matching pairs of discs is symmetric and reflexive. From step 3 one gets transitivity. As such we get an equivalence relation. Because any equivalence relation that fulfils step 1 and 2 must have transitivity which step 3 gives, this construction is indeed the minimal equivalence relation we previously defined. □

Next, here is a lemma that shows that for any pair of equivalent disc, there exists a certain homotopy of Ω^0 discs between them. First we define a tree the same way we did in the previous chapter:

Definition 3.1.3. *A rooted tree in the complex plane is a graph without simple closed paths and one vertex chosen as root, that is to say, a finite set of line segments connected only at their endpoints and with no loops (the tree is simply connected). From here on a tree can also be a continuous function from such a rooted tree in $\mathbb{C}$ to $\mathbb{C}^n$, and in that case the root of the tree is the point that the root of the rooted tree is mapped to. A leaf of a tree is an endpoint of a line segment, other than the root, that is not connected to any other line segment.*

Lemma 3.1.2. *For any pair of equivalent discs, one can find a tree with root at their common center and all leaves in Ω, so that for any point on the tree, there is a pair of equivalent discs defined with centres in that point. We will say that one of these discs is on the left side, and one on the right, and so that at the leaves of the tree there is a disc embedded into Ω with center at that point. This can be done so that if you start at the root of the tree, and go around the tree in either in clockwise or counterclockwise direction, picking the "right" discs for one side and the "left" side for the other, you get a homotopy of discs connecting the two discs at the root.*

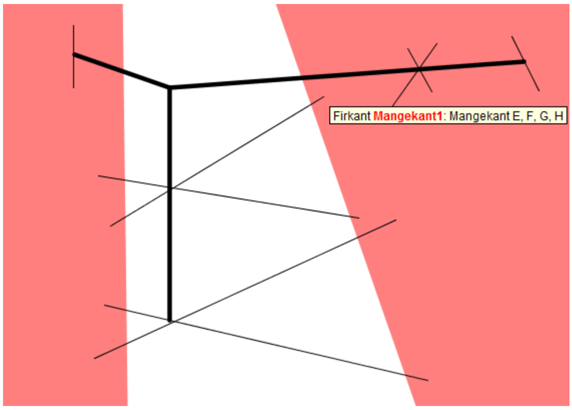


Figure 3.1: A simplified imagining of a tree. The thick structure is the tree, while the thinner black lines are the analytic discs. Along every point on the tree there are two discs, and when ordering one of each pair to one "side", each side gets a continuous family of discs. At the leaves of the trees there is an embedded disc. The red part is Ω.

Basically, one can find a tree and Ω^0 *discs with centres on the tree so that the discs create a continuous family of discs going from one of the discs in the pair of equivalent discs to the other.*

Proof: Lemma 3.1.1 gives us a way of constructing all equivalent discs. So take two discs and the finite sequence of steps necessary to get us from one to the other. For each step, one constructs a segment of a tree in the following way: for step 1, the embedded discs, one constructs a leaf of the tree or simply a point if you will. It is easy to find a continuous family of discs between these discs, all with common center. Just shrink both discs until they are small enough that you can rotate them and manipulate them as you wish.

For step 2, the equivalent discs goes along a line from a pair of equivalent discs, so simply let the tree be the line segment of the common centres. As for step 3, take the union of the trees, gluing them together at their common root. How to construct the discs around the tree is obvious from this construction. □

Lemma 3.1.3. *Let* d *be a* Ω^0 *disc, and let* U *be the connected component of* $\bar{\Delta}$ *such that* *for $z \in U$, $d(z) \in \Omega$ and that $\partial\Delta$ is in U. Then d together with an automorphism A on the closed unit disc that moves the center of d to a point in U i equivalent to a small embedded disc.*

Proof: Call d with moved centre d_1. Let e be a small embedded disc in Ω with the same centre as d_1. Let $D(t,z)$ be the continuous family of discs which d is a part of. Now create a continuously changing automorphism $A(t,z)$ on the closed unit disc where $A(1) = A$ and which moves the centres of D so that they lie in Ω. As the discs in D must have a continuous component in Ω near the boundary, and because of how U is defined, we can find such an $A(t)$. Then simply create a continuous family of small embedded discs from e along the centres of $D(t, A(t,z))$. As all discs embedded in Ω with the same centre are equivalent, $D(0, A(0,z))$ is equivalent to the small disc we have there. Since equivalence is preserved under homotopies of analytic discs with common centre, we have our result. □

Next, an example of non-obvious equivalent discs in a domain:

Example 3.1.1. *Here we again write $z = x + iy$ and $w = a + bi$, $(z,w) \in \mathbb{C}^2$. Let G be a domain in $\mathbb{C}^2$ constructed by the union of the following sets:*

$$G_1 = \{(z,w) \in \mathbb{C}^2 : |z| < 2, -2 < a < -1.5, b \in (-0.1, 0.1)\}$$

$$G_2 = \{(z,w) \in \mathbb{C}^2 : 1.5 < |z| < 2, -1.5 \le a < 0.5, b \in (-0.1, 0.1)\}$$

$$G_3 = \{(z,w) \in \mathbb{C}^2 : |z| < 4, 1.5 < a < 2, b \in (-0.1, 0.1)\}$$

$$G_4 = \{(z,w) \in \mathbb{C}^2 : 3.5 < |z| < 4, -0.5 < a \le 1.5, b \in (-0.1, 0.1)\}$$

$$G_5 = \{(z,w) \in \mathbb{C}^2 : |(k,0) - (z,w)| < 0.1, k \in [-2, -3.5]\}$$

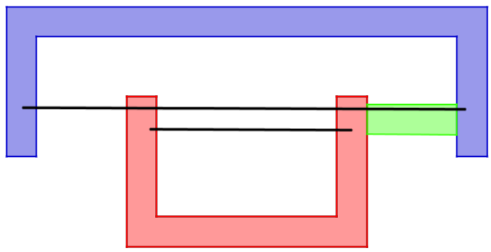


Figure 3.2: A two-dimensional cut of G, cut along the x-a plane. The red part is G_1 and G_2, the blue part G_3 and G_4 and the green part G_5. The two black lines are analytic disc, one with boundary in G_2 and one in G_4. These are mean to have equivalent centres, but are slightly apart for visual reasons.

An idea of how the union G looks like can be gotten by Figure 3.3. The two black lines on the figure(note that they are supposed to have equal center even though that is not clear on the figure) represents these two discs:

$d_1 : z \in \bar{\Delta} \rightarrow (1.8z, 0)$

$d_2 : z \in \bar{\Delta} \rightarrow (3.8z, 0)$

It is clear that these two have equal center $(0,0)$*, but it is not obvious that they are equivalent, as the boundaries are in different parts of the domain. To show that they are in fact equivalent, first consider the case where we through an automorphism on each disc, labelled* B^1, B^2 *have moved their centre to, for instance* $(1.7,0)$ *in* Ω*, the same for both. By Lemma 3.1.3* $d_1(B^1(z))$ *is equivalent to a small disc with centre* $(1.7,0)$ *embedded in* Ω*, and the same for* $d_2(B^2(z))$*. As all embedded disc with equal centre are equivalent, and by transitivity, the discs are equivalent.*

While this is valid, for this example the homotopy and tree will be constructed. Start with the disc $d_1(B^1) = (1.8B^1(z), 0)$*. Create a homotopy given by* $D_1(t,z) = (1.8B^1, -1.6t)$*, so that* $D_1(0,z) = d_1$ *and* $D_1(1,z)$ *is embedded in G with centre in* $(1.7,-1.6)$*. Now create a new homotopy which shrinks the disc,* $D_2(t,z) = ((1-\frac{999t}{1000})1.8B^1(z) + 1.7\frac{999t}{1000}, -1.6+0.1t)$

giving us at $D_2(1,z)$ *a disc with radius* $\frac{1.8}{1000}$ *and center* $(1.7,-1.7)$. *Call this disc* $S(z) = (s(z), -1.7)$

The next homotopy $D_3(t,z) = (s(z), -1.7 + 1.7t$. *This creates homotopies of discs, all with centres in* $(1.7, -1.7t)$, *meaning that that* $d_1(B^1)$ *and* $(s(z), 0)$ *are equivalent. In a similar way one constructs two homotopies along the line going from* $(1.7,0)$ *to* $(3.7,0)$ *, one being made using automorphisms on* d_2 *moving the centre along the straight line between the two points, and having small discs going along the same curve. Then we create another pair of homotopies going to* $(3.7,3.7)$ *by shifting* d_2 *and the small discs upwards, where the two discs are equivalent. This shows that* $d_2(B^2)$ *is equivalent to a small disc embedded in* Ω *with centre* $(1.7,0)$. *Finding a tree that creates a homotopy between the two small discs is easy, and again one uses transitivity to get the result. The tree is made of the line segment from* $(1.7,0)$ *to* $(1.7,-1.7)$, *the one going from* $(1.7,0)$ *to* $(3.7,0)$ *and then to* $(3.7,3.7)$.

Now to prove the original discs are equivalent. Simply take the straight line going from $(0,0)$ *to* $(1.7,0)$. *It is easy to see that you can use Blaschke factors on the disc to create a continuous family of* Ω^0 *discs going from* d_ν *to* $d_\nu(B^\nu), \nu = 1,2$. *Simply take* $B(t,z) = \dfrac{z - a(t)}{1 - \bar{a(t)}z}$, *for a fitting curve* $a(t)$. *Thus part 2 of the definition of the equivalence class states that* d_1 *and* d_2 *are equivalent. The tree is the line segment from* $(0,0)$ *to* $(1.7,0)$ *plus the tree mentioned in the previous paragraph.*

Note that this would not work if the corridor connecting the two glasses goes in a curve on the outside of the glasses, which we know would give non-shclicht results. In that case we could not say that the largest of the two discs was equivalent to a small disc lying in the small glass, because moving centres from close to the boundary of the big disc into the small glass would force you to leave the domain.

Definition 3.1.4. *The quotient space on* Ω^0 *given by the equivalence relation mentioned above will be referred to as* Ω^1.

Ω^1 can be given the structure of a Riemann domain in the following way:

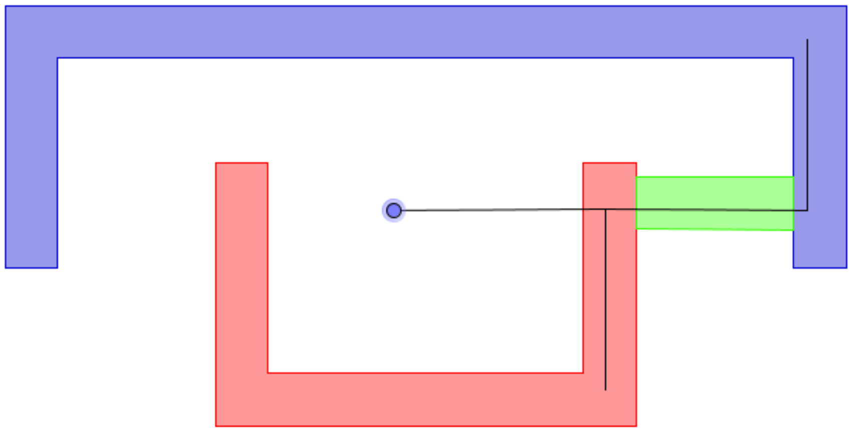


Figure 3.3: G with the tree referred to in the example. The point is the center of the discs d_1 and d_2.

First, a basis for the topology: A basis element around some element k in Ω^1 is created by taking a representative disc f for the element k, and then for each point in a small ball around the center of d, taking a disc that is homotopic to f through a family of continuous Ω^0 discs. For simplicity, one can take discs that are simply linear shifts of f.

It is obvious that any disc $f : \bar{\Delta} \to \mathbb{C}^n$ in Ω^0 can be moved slightly by a constant $\epsilon > 0$ smaller than some constant σ to get a Ω^0 disc $f + \epsilon$. This is because the boundary of the disc is compact, and so its minimal distance from the boundary of Ω is larger than some positive constant. One can thus create a "ball" of size σ of discs moved slightly in any direction from the original, where the centres form an open ball in the topology of Ω. Take each of these discs to their equivalence class, and say that this is an element in the basis.

Several things needs to be checked here: first, is this definition well-defined? Consider two equivalent discs with the same center, and that you shift both of them small distance in the same direction. The two discs you end up with must be equivalent, as we can take a straight line between the first centre and the centre of the new discs and create two homotopies of shifted discs along them where discs pairwise has the same centre. The definition of the equivalence relation gives us that the shifted discs must be equivalent.

This procedure can be done for any disc, so one finds that the basis indeed covers every element of the equivalence class. Also, taking two balls that overlap, one can take an element in the intersection and shift a representative slightly in all directions, and one gets a ball lying inside the intersection. As such, one has a basis of the topology, and this basis makes it so that a continuous family of discs lifts to a continuous curve in Ω^1 when taking each disc to their equivalence class.

It is clear that the space is connected: simply remember that we are working with Ω^0 discs, which means that there is a continuous homotopy to a disc embedded in Ω. This lifts to a continuous curve with our topology. Any two embedded discs can be continuously changed into another, and so one easily finds that any two elements in Ω^1 have a line between them.

To prove that Ω^1 is Hausdorff, we prove that for any two elements in Ω^1 we can find one open set around each so that the open sets do not intersect. Take two different elements of Ω^1. Either the discs in those classes have different centres, or they have same center but are not equivalent. If the first is the case, one can create balls in Ω of so small radius around each so that they do not intersect. If these balls are small enough one can for each linearly shift a representative around in it and get discs which lift to an open ball in the topology of Ω^1, and these can not intersect as none of the discs share centres.

In the case that the discs do share centres, take a small ball around the centre and shift representatives of the two elements around in that ball. Lift both of these sets of balls to get open balls in Ω^1. Imagine if there was an intersection between them. Then take a line segment from the centre of the original discs to the center of the intersecting element. Along this line segment construct the continuous discs that are a constant shift of the two originals. Now there is a homotopy from the original elements to a pair of equivalent discs, and from the definition of the equivalence class one has that the original discs are equivalent. Therefore, this can not happen.

The projection $\pi : \Omega^1 \to \mathbb{C}^n$ is simply taking any element to the centre of any disc in the equivalence class, which all have common center. It is clear that this is a local homeomorphism, as a basis element in Ω^1 maps to a basis element in Ω(or the Stein manifold it is in), and vice versa. This means that Ω^1 is a Riemann domain.

We can summarize the previous paragraphs as:

Theorem 3.1.1. *Ω^1 is a Riemann domain.*

In the next example we construct G^1 for a domain G.

Example 3.1.2. *As usual, let $z = x + iy$ and $w = a + bi$. Let G be the domain in $\mathbb{C}^2$ which is the union of these three sets:*

$$G_1 = \{(z, w) \in \mathbb{C}^2 | a, b \in (-1, 1), |z| < 4\}$$

$$G_2 = \{(z, w) \in \mathbb{C}^2 | a \in [1, 5), b \in (-1, 1), 3 < |z| < 4\}$$

$$G_3 = \{(z, w) \in \mathbb{C}^2 | a \in [1, 5), b \in (-1, 1), |z| < 1\}$$

Figure 3.4 and Figure 3.5 gives some idea about what this domain looks like.One thing to note is that for any point (z, w) in the domain, the straight line between (z, w) and $(z, 0)$ lies in the domain. This means that any analytic disc $f = (f_1, f_2)$ with boundary in G is a G^0 disc, since $F(t, z) = (f_1, tf_2)$ creates a homotopy between f and a disc embedded in G, where the boundary of all discs lie in G. Note that f_1 maps any point to a point with absolute value less than 4, as the boundary is mapped to points with absolute value less

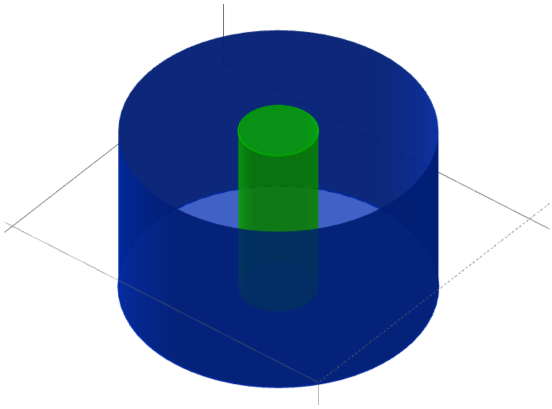


Figure 3.4: A 3D cut of G. G_3 is marked in green, and the other parts in blue.

than 4.

The homotopies mentioned in the previous paragraph makes it so that two discs with equal center have equal centres along the entire family F. This means that they are equivalent, as the first discs are embedded in G. So all discs with equal centre and boundary in G are equivalent, for instance $g = (0.5z, 3)$ and $h = (3.5z, 3)$ despite one having boundary in G_2 and in G_3.

Let $G_4 = \{(z, w) \in \mathbb{C}^2 | a \in (-1, 5), b \in (-1, 1), |z| < 4\}$. Any point $(z_0, w_0) \in G_4$ is in the centre of a G^0 disc. For a point in G this is obvious, and otherwise $j(z) = (3.9z, w_0)$ together with an automorphism makes this clear. No point outside of $G \cup G_4$ can be reached by a G^0 disc, though, as G_4 is convex, and thus pseudoconvex. In addition, by the disc property all analytic functions on G can be extended, so G_4 is the envelope of holomorphy.

As G_4 is pseudoconvex, no G^0 disc can have image outside of G_4. So G^1 is a domain with one sheet per point, and the image of its projection equals G_4. Thus it is isomorphic to G_4, which is also the envelope of holomorphy. What we want to show in this chapter is that G^1 is in fact always the envelope.

Figure 3.5: A 2D cut of G. Colours are as in Figure 3.4

3.2 Ω^1 as an Extension of Ω

This section will prove that Ω^1 is an analytic extension of Ω, the way it was defined in Chapter 2. For this it needs to be shown that for any analytic function on Ω can be extended to Ω^1, and an inclusion of Ω into Ω^1 is needed.

We start with the inclusion map ϕ: for any point $p \in \Omega$ one can find a small analytic disc d embedded in Ω with center p. Let $\phi(p) = [d]$. As all embedded discs around a common point are equivalent this is well defined, and it is easy to see that it is a local analytic isomorphism.

For the extension of an analytic function $f \in H(\Omega)$, the following lemma is needed.

Lemma 3.2.1. *Take two equivalent discs, d_1 and d_2. As they are Ω^0 discs we can extend any holomorphic function on Ω to the centre of those discs. Their value at their centre is the same for any analytic function on Ω. Another way to say this is that if you lifted both discs to $\widetilde{\Omega}$, then their centre in the envelope would still be the same.*

Proof: Take the tree which creates a homotopy between d_1 and d_2. The continuity principle gives us that any analytic function on Ω can be extended to the entire tree. Here is how one shows that the analytic functions extend to the same value for each step: for step 1 the centre is in Ω, so its value is given and equal for both discs. For step 2: assuming that the starting discs have equal value, both homotopies of discs have the same centres all the way, and the analytic function only has one extension along that line. So the function must have equal value at the centre of both discs. In step 3, assuming that d_1 and d_2 have equal values in the centre, and so does d_2 and d_3, then obviously the centres of d_1 and d_3 also share their value for any analytic function. d_2 can only have one unique value in the centre, as its value is determined by the value along the boundary. □

Using this lemma it is possible to define an extension of an analytic function f on Ω^1. For an element $p \in \Omega^1$, take a representative disc d. As one knows the values on the boundary(which is in Ω), one can calculate the value at the center, and because of the way a Ω^0 disc is defined, we know that f has a value there. Also, by the previous lemma the choice of d does not matter. Let f_1 be the function on Ω^1 defined by choosing the value f has in each centre.

We must show that f_1 is analytic. Taking π as Ω^1's projection, look at $f_1 \circ \pi^{-1}$ on an open ball $U \subset \mathbb{C}^n$ so small that π^{-1} is injective on U and that. f^1 locally takes the exact values f takes on some extension to U. So f^1 is analytic and matches f when we use the inclusion from Ω into Ω^1.

Theorem 3.2.1. *Ω^1 is an extension of Ω, when the inclusion ϕ of Ω into Ω^1 is the function which takes a point p in Ω to the equivalence class of embedded discs with p as center.*

3.3 Pulling a Family of Discs to Ω

This section will prove that for a continuous family of analytic discs, where the first disc and the boundary of all the discs have a lift to Ω^1, we can find Ω^0 discs going through the centres of all the discs in the original family. This is perhaps the most important part of proving that Ω^1 is envelope of holomorphy of Ω, as the theorem will be used to find an Ω^0 disc which is a representative of a certain point in Ω^1.

Theorem 3.3.1. *Let Ω be a domain in $\mathbb{C}^n$ and let $H(t,z) : [0,1] \times \bar{\Delta} \to \mathbb{C}^n$ be a continuous family of discs, where $H(0,z) \cup H(t,e^{i\theta})$ has a lift to Ω^1 when $t \in [0,1], z \in \bar{\Delta}, \theta \in [0,2\pi]$.*

Then we can find a continuous family $G(t,z)$ of Ω^0 discs so that every centre in H is the centre of at least one disc in G. Also, if we let $a(t)$ be the curve of the centres of H, then starting at some disc $G(t_0,z)$, the discs in G will follow the same curve.

The proof of Theorem 3.3.1 consists of two lemmas that together will give us another set of discs that satisfy conditions of Theorem 2.4.2. Each lemma is constructive, and constructs a new family of analytic discs based on the construction in the previous lemma, keeping the properties of the previous lemma, but adding additional properties. Simply put:

- The first lemma will give us the property that the first disc is embedded in Ω, and the lift of that disc will be identical to ϕ.
- The second lemma will make it so that each disc has part of its boundary in Ω, with lift being identical to ϕ. This part of the boundary varies continuously with t.

These lemmas and Theorem 2.4.2 together will prove the theorem.

Lemma 3.3.1. *Let Ω be a connected domain in $\mathbb{C}^n$. Let H be a continuous family of closed analytic discs such that the first disc and the boundary of each disc has a continuous lift to Ω^1. Then one can construct a new family of discs, H_1, where for the discs $H_1(t,z)$ with t close to one, the discs would be the same as in H, and with the properties that the first disc and the boundary of all discs have a lift to Ω^1. Also, the first disc, $H_1(0,z)$, should be embedded into Ω, and the lift of that disc to Ω^1 should be in the image of $\phi(\Omega)$, where ϕ is the inclusion of Ω into Ω^1 defined previously.*

Proof: Recall that Ω^1 is a connected Riemann domain. Take the lift of $H(0,z)$ and call it K. Say that the lift of $H(0,0)$ is $k \in \Omega^1$. Then there is a line segment l in Ω^1 connecting k with some element in $\phi(\Omega)$. Shrink K to a small disc around k. The projection of Ω^1 down to $\mathbb{C}^n$ gives us a continuous family of shrinking discs.

Further, taking the small disc around k and moving it along l we eventually gets a disc embedded in $\phi(\Omega)$. This is possible when the disc is shrunk enough. Again project-

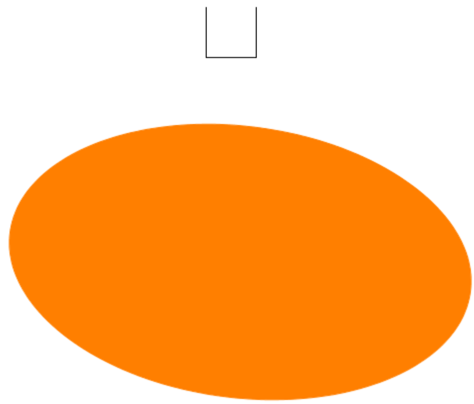


Figure 3.6: A picture of the initial situation in Ω^1. The three black lines are meant to be the lift of the first disc(the horizontal line) and the boundary of the other discs (the vertical lines) in H. The orange part is meant to be $\phi(\Omega)$.

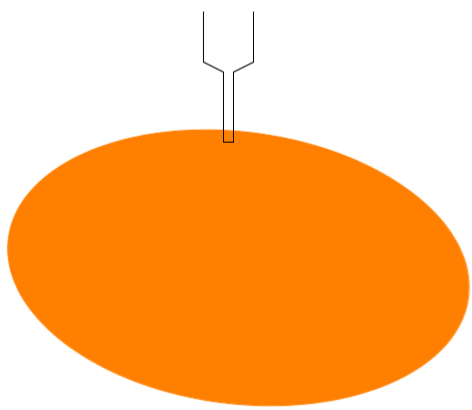


Figure 3.7: A picture of what lemma 3.3.1 accomplishes. We shrink the first disc in H, then move it into $\phi(\Omega)$ so that the first disc is embedded there. The orange part is $\phi(\Omega)$.

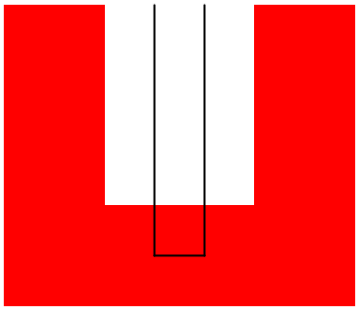


Figure 3.8: The initial situation for Lemma 3.3.2. The first disc and boundary of the family of discs H_1 is shown. The red part is Ω.

ing down we get a continuous family of analytic discs, the last one being embedded in Ω. Taking this constructed family together with the family H and re-parametrizing to get a family H_1, we get our result. $\square$

Lemma 3.3.2. *Let $H(t,z):[0,1]\times\bar{\Delta}\to\mathbb{C}^n$ be a continuous family of discs where $H(t,e^{i\theta})$ lifts to Ω^1, and say that we have created a new family of discs H_1 as in Lemma 3.3.1. Then one can create another family of analytic discs $H_2(t,z)$, where all point in $H_2(t,0)$ are centres of some disc in H_1, and all the centres in H_1 is $H_2(t,0)$ for some t. Also there is a curve $a(t)$ on the boundary of the unit disc so that $H_2(t,a(t))\in\Omega$.*

Adding to this, the results from Lemma 3.3.1 still applies, that is to say, the first disc should be embedded in Ω and the first disc and boundary of all discs should have a lift to Ω^1.

Proof: We now assume that we have the family H_1 from Lemma 3.3.1.

Let $c(t)=(t,1)$. Since H_1 has a lift to Ω^1, so does $H_1(c(t))$. Then use that to find a continuous family of discs along $H_1(c(t))$, and make sure that the first disc is a very small one, which is possible since the lift of the first disc is the inclusion of Ω, and small embedded discs are representatives of those elements. We add trees at a finite number of the discs indexed by H_1 to make sure that the discs are continuous. Each tree T has

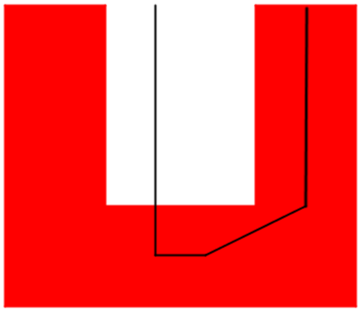


Figure 3.9: At the end of Lemma 3.3.2 we have a continuous family of discs where most of the boundary lies close to the boundary of H_1, the first disc is almost the same and the centres of the discs are the same, but where part of the boundary of each disc now lies in Ω. $H_2(t,1)$ lies in Ω for all t.

its root on the boundary of one disc d, indexed by t_0. We can now make a new disc by combining d and a fattened version of T in the same way as in Theorem 2.3.2.

To quickly go over the procedure used again, one takes the closed unit disc in the plane and the function f which defines the analytic disc d. Then one adds trees on the unit disc, and extends f continuously along the trees so that f takes the trees on the unit disc to the tree in $\mathbb{C}^n$ mentioned above. Use Mergelyan's theorem to approximate P, and then slightly fatten the tree on the unit disc, extending P to this new set K.

A small change to $c(t)$ needs to be made here. To make it easier to grow trees, we change $c(t)$ a little at each discontinuity point. At each t_0, we make $c(t)$ go along the boundary of that disc for a small distance, that is to say move along $(t_0, \partial\Delta)$, and the tree is constructed along that interval. If this line segment is small enough, we do not risk getting more discontinuities, as any disc can be moved a little. Call this new line $a(t)$.

We know that any Ω^0 disc can be shifted slightly in all directions. The fattening of the trees should be so small that any point in the boundary of the fattened tree is contained in the centre of an Ω^0 disc which is a linear shift of one of the discs that

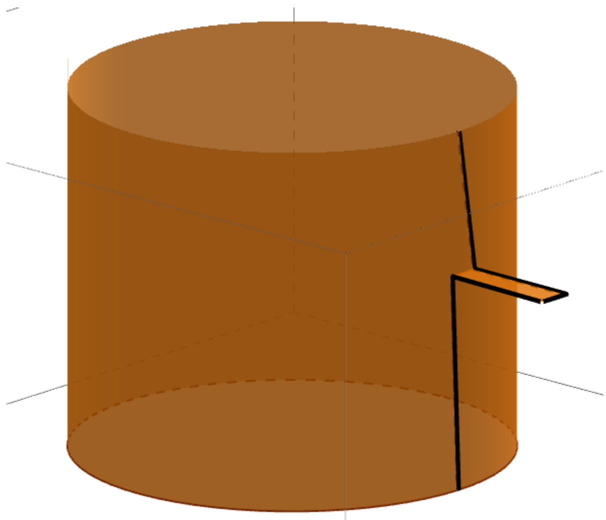


Figure 3.10: The continuous family of discs with a tree added at the discontinuity of discs along $H_1(t, a(t))$, to make Ω^0 discs along $a(t)$ (the black line) continuous.

creates a homotopy of discs on the tree. Now, make sure that K is simply connected, and then use the Riemann mapping theorem to show that $P : K \to \mathbb{C}^n$ together with a biholomorphism between the unit disc and K gives an analytic disc.

We want to put in new discs in the family H_1 at discontinuity points. These discs will allow us to grow trees, and will basically be the disc at the discontinuity point with a tree continuously growing and then the tree being shrunk back. We make sure that all these discs have the same centre, which is the centre of the original disc at the discontinuity. Figure 3.10 shows what we want to accomplish. The black line is $H_1(t, a(t))$. There is a tree constructed, and we have discs overlapping with the disc with the tree so that we get a continuous family of discs, by having discs slowly growing out the tree.

We can create a continuous family of closed analytic discs $E(t, z)$ where $E(0, z) = z$ and $Image(E(1, z)) = K$. This can be done because K consists of the closed unit disc and trees, and the trees are contractible, so we can use Lemma 2.2.1. $P(E(t, z))$ gives a new family of analytic discs where at $t = 0$ we approximate d as closely as we wish, and at $t = 1$ the disc is an approximation of d with T attached.

It is easy to create a homotopy of discs between d and $P(E(0,z))$ when you make the approximation extremely accurate, simple create discs $tP(E(0,z))+(1-t)d$ between them. After having gotten to $P(E(1,z))$ we add even more discs, the same discs as $P(E(t,z))$, but in reverse order, going back to d. In this way we get that the family of discs is still continuous with respect to the other discs in H. $a(t)$ will during these discs lie "still", that is to say that it will stay at approximately the point where it first reached the disc indexed by t_0. It will then cross over T when the tree has been fully grown.

In this way we can construct a continuous family of closed analytic discs H_1 containing all discs in the original family H and where along the curve $H_1(t,a(t))$ we have a continuously changing set of Ω^0 discs. We can also make $a(t)$ the image of $(t,1)$ by using continuous rotations on the unit disc.

Now, we want to use Theorem 2.4.1. That theorem would give us a family of discs with centres the same as in H_1, but where the along the curve $H_1(t,a(t))$ the discs along the curve would move the edges of the family of discs into Ω, giving us what we want. For this, though, we need to define discs along the entire boundary of H_1 so that they are continuous with respect to each other and the Ω^0 discs defined along $H_1(t,a(t))$.

This is easily done, but please note that the discs constructed along the boundary here are not necessarily Ω^0 discs outside of $H_1(t,a(t))$. Along a tiny neighbourhood of $H_1(t,a(t))$ we shrink the discs on $H_1(t,a(t))$. The shrinking of an analytic disc $f(z)$ is given by $f(rz)$, where r is continuously made so small that $f(rz)$ is smaller than some ϵ (the size of f is given by the maximum distance from the centre to the boundary), where ϵ is so small that for any point on the boundary of H_1, shifting it any direction would still give you a lift to Ω^1 in the same sheet as the current lift. We have to make sure that the shrinking happens in the same way around $H_1(t,a(t))$, so that the discs are continuous, and the shrinking should happen so fast that if λ is the distance we can move all of the Ω^0 discs so that they still have boundary in Ω, the shrinking should be finished before we have gotten $\frac{\lambda}{2}$ away from $a(t)$. Having done this, we get the family of analytic discs H_2 from Theorem 2.4.1. Note that H_2 has the same centres as H_1.

We next define the lift of the boundary and first discs of H_2 to Ω^1. For the first disc, we as before define the lift to be the inclusion ϕ. Note that this means that the first disc lies close to the lift of $H_1(0,z)$. Along the parts of the boundary where we attached discs of very small size, the boundary point $H_2(t,z)$ lies close to $H_1(t,z)$, so simply take a representative of the lift of $H_1(t,z)$ that can be shifted to $H_2(t,z)$ and use the equivalence class it belongs to. Locally, this is continuous as H_1's lift is continuous and the lifts are from centres close to each other.

That leaves us with points close to $H_2(a(t))$. For the points on $H_2(a(t))$, we use a similar definition to the one before: we let take the equivalence class of the disc we defined on $H_1(a(t))$, only shifted and with centre changed so it covers $H_2(a(t))$. Note that by Lemma 3.1.3 this is the same as taking the inclusion ϕ.

For the points close to $H_2(a(t))$, where the discs around H_1 had not yet gotten to size ϵ, we again use the disc we defined on $H_2(a(t))$, let us take such a point $H_2(x,w)$. By the way we shrunk the discs $H_2(x,w)$ must be covered by a shifted version of a disc C on $H_1(a(t))$. Use a shifted C to define the class. This means that we get a continuous lift with respect to the defined lift of $H_2(a(t))$. Also, it is continuous with respect to the parts of H_2 which lie close to H_1, as C can be extended further to the parts which had small discs. This is because we can make a representative of the lift at the part with small discs be the same disc we use to define $H_2(x,w)$, only shifted, since we assumed that C should be able to be shifted that far. This must go to the right sheet because of the lift on H_1, and locally the topology is defined by continuously changing discs, like shifting C. Lemma 3.4.1 below tells us that Ω^1 is not unbranched, so we have to get to the right element. This means that we have continuity. □

Proof of Theorem 3.3.1: Using lemmas 3.3.1 and 3.3.2 we get a family of discs fulfilling the conditions of Theorem 2.4.2 while having the centres of the discs of the new family containing the centres of the original family of discs H. Note that when we choose a Ω^0 disc along the boundary of our family H_2 by using the lift the boundary has to Ω^1, we can make continuous changes in a small interval around that part of the boundary. Also note that close to the curve embedded in Ω, we can make the discs small because the lift there is the inclusion of Ω into Ω^1. Also, at any discontinuity we can because of the lift make a tree bridging the gap.

Thus, using Theorem 2.4.2, we get a continuous family of discs G with first disc and boundary of all the discs in Ω, where the centres of the discs at some point start following the curve made by the centres of H. □

3.4 The Envelope of Ω

This section will finally prove Jöricke's Jöricke (2009) theorem that Ω^1 is the envelope of Ω when Ω is a connected domain in $\mathbb{C}^n$. First, a lemma is needed.

Lemma 3.4.1. *Ω^1 is an unbranched Riemann domain.*

Proof: Assume that there was a point p in Ω^1 which was branched, that is to say that there is no neighbourhood around p which is homomorphic to a ball of n complex dimensions. Take a small ball U around p, so small that all elements in it can be taken as discs continually changed from p(this was how the topology was defined). We must have that the projection down to $\mathbb{C}^n$ is not injective, so there must be two elements $a, b \in \Omega^1$ being projected to the same element q. Take the straight line between q and $\pi(p)$. There is a homotopy of discs between a representative of a and a representative of p going along that line, and the same for b and p. By the definition of the equivalence class, this implies that $a = b$. □

Theorem 3.4.1. *Let Ω be a connected domain in $\mathbb{C}^n$. Then Ω^1 is Stein.*

Proof: This proof depends on a result by Doquier and Grauert (1960). They define p_7* convexity, and show that for an unbranched Riemann domain G over a Stein manifold, if the disc property is fulfilled it implies that G is Stein. In this case the disc property means that for any continuous family of analytic discs, if the first disc and the boundary of all the discs are in G, the discs never touch the boundary of G. The construction of a boundary, denoted $\widetilde{\partial}\Omega^1$ of a Riemann domain is found in Fritzsche and Grauert (2002), page 100-103. This is the same construction used in Doquier and Grauert (1960) and is also used in Jöricke (2009). It is referred to as the set of accessible boundary points.

So, let us take Ω^1 and let $F(t,z)$ be a continuous family of analytic discs in $\Omega^1 \cup \widetilde{\partial}\Omega^1$. Say that for all $t < 1$, $F(t,z)$ lies entirely in Ω^1, and the boundary of $F(1,\partial\Delta)$ is also in Ω^1. By Lemma 3.3.1 we can assume that $F(0,\cdot)$ lies in $\phi(\Omega)$.

Define $p = F(1,0)$. We want to show that $p \in \Omega^1$. We say that $F(0,\cdot)$ is embedded in $\phi(\Omega)$. By the same process as in Lemma 3.3.1 this can always be done. We want to show that we can find a disc which lifts to p, in the sense that when taking the limit of $F(t,0)$, $t \to 1$, we find an Ω^0 disc that can be lifted to that limit.

Take the projection on Ω^1, π, and apply it to F to get a family of analytic discs G in $\mathbb{C}^n$, $G = \pi \circ F$. Note that the construction of the boundary of Ω^1 in Fritzsche and Grauert (2002) gives us a projection of the boundary that is continuous with respect to the projection on Ω^1, and an analytic disc projected down using this projection would still be analytic.

G obviously has a lift to Ω^1 along the boundary of the discs, namely the boundary of the discs in F. Now we can use Theorem 3.3.1 and get a family of Ω^0 discs $H(t,z)$ where the set of centres of H are the same as the ones in G, and occur in the same order. The only difference is that for H we sometimes stay at one centre for a while(this happens at the discontinuities in Lemma 2.4.1).

Next, look at the curve $a(t) = F(t,0)$. We can project it down to a curve $a_1(t)$ in $\mathbb{C}^n$, which is equal to $G(t,0)$. For all t except 1 we can for any $a(t)$ get a Ω^0 disc which represents that particular element in Ω^1 and with centre in $a_1(t)$. What we wish to prove is that the equivalence classes of the discs in H is $a(t)$, meaning that the continuous family of discs H are representatives of $a(t)$.

First, let us look at $a(0)$, and recall that it lies in $\phi(\Omega)$. As $H(0,z)$ is embedded in Ω

and $H(0,0) = a_1(0)$, the first disc of H lifts to $a(0)$. Now, let $A(t)$ be representatives of $a(t)$ for all t except 1, where there might not be any representatives. Let $A(t)$ a piecewise continuous family of discs, done so that they share centres with H and always have same center, $H(t,0) = A(t,0)$. By the definition of the equivalence class, when we move along $a_1(t)$ with the discs $H(t,\cdot)$ and $A(t,\cdot)$, they must be equivalent, as the first discs were equivalent, they move along the same centres, and Ω^1 is unbranched. We easily see that this holds for all t. Thus $H(t,\cdot)$ are representatives of $a(t)$ for all $t < 1$.

Obviously $H(1,\cdot)$ is the limit of the discs in H as t goes to 1. Let k be the representative of the equivalence class in Ω^1 that H belongs to. Remembering that the topology on Ω^1 is locally determined by continuously changing functions, $a(t)$ converges to k. That means that $k = p$, as $a(1) = p$. But we can create an open ball in Ω^1 around k by taking $H(1,\cdot)$ and shifting it slightly in all directions, then taking those discs to their equivalence classes. Thus p is not on the boundary, which was what we wanted to show. This implies that Ω^1 is p_7* convex, and thus Stein by Doquier and Grauert (1960). □

Theorem 3.4.2. *Ω^1 is the envelope of holomorphy of Ω.*

Proof: This follows from Theorem 3.2.1 and Theorem 3.4.1. □

Theorem 3.4.3. *Ω_1 is isomorphic to Ω^1 through an analytic isomorphism that satisfies all requirements used by Narasimhan's definition of the envelope of holomorphy, meaning that Ω_1 is the envelope of holomorphy of Ω.*

Proof: We know that the points in Ω^1 are separated by analytic functions, as it is Stein. That means that we can create a bijection u between Ω^1 and Ω_1 by for any point $p \in \Omega^1$ taking the projection down to $\mathbb{C}^n$, and then projecting into the envelope of Ω(defined by using sheaf-theory) by using the analytic functions defined locally around p. This function is 1-1, as the analytic functions separate points on Ω^1, and it goes into $\Omega_1 \subset \tilde{\Omega}$, since the projection of p must lie in a family of continuous discs with boundary in Ω. Also note that $u(\phi(\Omega))$ equals the inclusion of Ω into Ω_1

We also have that u is surjective, as any point p in Ω_1 is in the centre of a family of analytic discs $G(t,z)$, and when projected down, this family gives rise to an element q in Ω^1. They project to the same point using the respective manifold's projections, and their value of the functions at q must match p's, as they are both given by the value that G takes around the boundary, by Cauchy's theorem. Thus q maps to p.

Locally u is equal to the projection of Ω^1 composite the inverse of the projection of Ω_1. As they are both local homeomorphisms, so is u, so u and its inverse are continuous. This also means that u and its inverse are locally holomorphic. We have that Ω_1 is biholomorphic to the envelope of Ω.

We can show the result using the definitions of envelope of holomorphy introduced in Chapter 2. For any extension Y of Ω we have a holomorphic map from Y into Ω^1 with properties described in the definition of envelope of holomorphy. Take that map together with u into Ω_1, and it is clear that we have a map that satisfies all requirements for Ω_1 being the envelope of holomorphy. $\square$

Chapter 4

Summary and Recommendations for Further Work

4.1 Summary and Conclusions

Any point in the envelope of a connected domain $\Omega \subset \mathbb{C}^n$ lies in the image of a continuous family of analytic discs with first disc and boundary in the inclusion of Ω into the envelope.

4.2 Discussion

The result proven here is slightly weaker than in Jöricke's paper. Here it is assumed that Ω is a domain in $\mathbb{C}^n$, not a general Riemann domain. However, for a domain in Ω, it provides a much shorter way of constructing the envelope than the standard method. On the other hand, this result holds for all $n > 1$, not just $n = 2$.

4.3 Recommendations for Further Work

The theorems in this thesis provides new ways to work with the envelope of holomorphy, which could lead to new results. It is for instance an open question whether the envelope of a domain i $\mathbb{C}^n$ with smooth boundary must have a finite number of sheets over any point, and this result might help with proving or disproving that.

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