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# APPROXIMATION NUMBERS OF COMPOSITION OPERATORS ON $H^p$ SPACES OF DIRICHLET SERIES

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Abstract. By a theorem of the first named author, $\varphi$ generates a bounded composition operator on the Hardy space $\mathscr{H}^p$of Dirichlet series ($1 \leqslant p < \infty$) only if $\varphi(s) = c_0 s + \psi(s)$, where $c_0$ is a nonnegative integer and $\psi$ a Dirichlet series with the following mapping properties: $\psi$ maps the right half-plane into the half-plane $\operatorname{Re} s > 1/2$ if $c_0 = 0$ and is either identically zero or maps the right half-plane into itself if $c_0$ is positive. It is shown that the $n$th approximation numbers of bounded composition operators on $\mathscr{H}^p$ are bounded below by a constant times $r^n$ for some $0 < r < 1$ when $c_0 = 0$ and bounded below by a constant times $n^{-A}$ for some $A > 0$ when $c_0$ is positive. Both results are best possible. Estimates rely on a combination of soft tools from Banach space theory ($s$-numbers, type and cotype of Banach spaces, Weyl inequalities, and Schauder bases) and a certain interpolation method for $\mathscr{H}^2$, developed in an earlier paper, using estimates of solutions of the $\overline{\partial}$ equation. A transference principle from $H^p$ of the unit disc is discussed, leading to explicit examples of compact composition operators on $\mathscr{H}^1$ with approximation numbers decaying at a variety of sub-exponential rates. Finally, a new Littlewood–Paley formula is established, yielding a sufficient condition for a composition operator on $\mathscr{H}^p$ to be compact.

*Nombres d'approximation des opérateurs de composition sur les espaces $H^p$ des séries de Dirichlet*

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Résumé. Un théorème du premier auteur affirme que $\varphi$ définit un opérateur de composition borné sur l'espace de Hardy $\mathscr{H}^p$ des séries de Dirichlet ($1 \leqslant p < \infty$) dès lors que $\varphi(s) = c_0 s + \psi(s)$, où $c_0$ est un entier positif ou nul et $\psi$ est une série de Dirichlet qui envoie le demi-plan droit sur le demi-plan $\operatorname{Re} s > 1/2$ lorsque $c_0 = 0$ et est ou bien identiquement nulle ou bien envoie le demi-plan droit dans lui-même si $c_0 > 0$. Nous prouvons que le $n$-ième nombre d'approximation de ces opérateurs de composition est minoré, à une constante multiplicative près, par $r^n$, $0 < r < 1$ si $c_0 = 0$ et par $n^{-A}$, $A > 0$, si $c_0 > 0$. Ces minorations sont optimales et reposent sur une combinaison d'outils venant à la fois de la théorie des espaces de Banach (type et cotype, inégalités de Weyl, bases de Schauder) et sur une méthode d'interpolation pour $\mathscr{H}^2$ utilisant des estimations des solutions d'une équation $\overline{\partial}$. Un principe de transfert avec les espaces $H^p$ du disque est discuté, conduisant à des exemples explicites d'opérateurs de composition ayant des nombres d'approximation avec divers types de décroissance sous-exponentielle. Enfin, une nouvelle formule de Littlewood-Paley est établie, conduisant à une condition suffisante de compacité pour un opérateur de composition sur $\mathscr{H}^p$.

## 1. Introduction and statement of main results

In the recent work [26], we studied the rate of decay of the approximation numbers of compact composition operators on the Hilbert space $\mathscr{H}^2$, which consists of all ordinary Dirichlet series $f(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ such that

$$\|f\|_{\mathscr{H}^2}^2 := \sum_{n=1}^{\infty} |b_n|^2 < \infty.$$

The general motivation for undertaking such a study is that the decay of the approximation numbers is a quantitative way of studying the compactness of a given operator, yielding more precise information than what for instance its membership in a Schatten class does. The purpose of the present paper is to take the natural next step of making a similar investigation in the case when $\mathscr{H}^2$ is replaced by the Banach spaces $\mathscr{H}^p$ for $1 \leqslant p < \infty$; here we follow [2] and define $\mathscr{H}^p$ as the completion of the set of Dirichlet polynomials $P(s) = \sum_{n=1}^{N} b_n n^{-s}$ with respect to the norm

$$\|P\|_{\mathscr{H}^p} = \left( \lim_{T\to\infty} \frac{1}{T} \int_0^T |P(it)|^p dt \right)^{1/p}.$$

We consider this a particularly interesting case because operator theory on these spaces so far is poorly understood and appears intractable by standard methods.

Our starting point is the first named author's work on $\mathscr{H}^p$ and boundedness and compactness of the composition operators acting on these spaces [2, 3]. A basic fact proved in [2] is that $\mathscr{H}^p$ consists of functions analytic in the half-plane $\sigma := \operatorname{Re} s > 1/2$. This means that $C_\varphi f := f \circ \varphi$ defines an analytic function whenever $f$ is in $\mathscr{H}^p$ and $\varphi$ maps this half-plane into itself. But more is clearly needed for $C_\varphi$ to map $\mathscr{H}^p$ into $\mathscr{H}^p$. In particular, we need to consider other half-planes as well and introduce therefore again the notation

$$\mathbb{C}_\theta := \{s = \sigma + it : \ \sigma > \theta\},$$

where $\theta$ can be any real number. Following the work of Gordon and Hedenmalm [10], we say that an analytic function $\varphi$ on $\mathbb{C}_{1/2}$ belongs to the Gordon–Hedenmalm class $\mathscr{G}$ if it can be represented as

$$\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n \, n^{-s} =: c_0 s + \psi(s),$$

where $c_0$ is a nonnegative integer and $\psi$ is a Dirichlet series that is uniformly convergent in each half-plane $\mathbb{C}_\varepsilon$ ($\varepsilon > 0$) and is either identically 0 or has the mapping properties $\psi(\mathbb{C}_0) \subset \mathbb{C}_0$ if $c_0 \geqslant 1$ and $\psi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$ if $c_0 = 0$. This terminology is justified by the result from [10] saying that $C_\varphi$ is bounded on $\mathscr{H}^2$ if and only if $\varphi$ belongs to $\mathscr{G}$. (See [26, Theorem 1.1] for this particular formulation of the result.) The $\mathscr{H}^p$ version of the Gordon–Hedenmalm theorem reads as follows [3].

Theorem 1.1. — *Assume that $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ is an analytic map and that $1 \leqslant p < \infty$.*

(a) *If $C_\varphi$ is bounded on $\mathscr{H}^p$, then $\varphi$ belongs to $\mathscr{G}$.*
(b) *$C_\varphi$ is a contraction on $\mathscr{H}^p$ if and only if $\varphi$ belongs to $\mathscr{G}$ and $c_0 \geqslant 1$.*
(c) *If $\varphi$ belongs to $\mathscr{G}$ and $c_0 = 0$, then $C_\varphi$ is bounded on $\mathscr{H}^p$ whenever $p$ is an even integer.*

The curious fact that part (c) of this theorem only covers the case when $p$ is an even integer can be directly attributed to an interesting feature of $\mathscr{H}^p$. To see this, we recall that $\mathscr{H}^p$ can be identified isometrically with $H^p(\mathbb{T}^\infty)$, which is the $H^p$ space of the infinite-dimensional polydisc $\mathbb{T}^\infty$, via the so-called Bohr lift. This space is a subspace of the Lebesgue space $L^p(\mathbb{T}^\infty)$ with respect to normalized Haar measure on $\mathbb{T}^\infty$. The main difference with classical $H^p$ spaces is that, even for $1 < p < \infty$, $p \neq 2$, $\mathscr{H}^p = H^p(\mathbb{T}^\infty)$ is not complemented in $L^p(\mathbb{T}^\infty)$ [9]. As a consequence, there seems to be little hope to obtain a useful description of the dual space of $\mathscr{H}^p$. This is a serious obstacle and makes it hard to employ familiar techniques such as interpolation in the Riesz–Thorin or Lions–Peetre sense. We refer to [33, 20] for further details about the anomaly of $\mathscr{H}^p$. There are additional obstacles as well, since familiar Hilbert space techniques such as orthogonal projections, frames and Riesz sequences, and equality of various $s$-numbers (like approximation, Bernstein, Gelfand numbers) are no longer available.

We have found ways to circumvent these difficulties to obtain results that, at least partially, parallel those from [26]. To state our first result, we recall that the $n$th approximation number $a_n(T)$ of a bounded operator on a Banach space $X$ is the distance in the operator norm from $T$ to operators of rank $< n$. One of our main theorems is a direct analogue and indeed an improvement of [26, Theorem 1.1], showing again the crucial dependence on the parameter $c_0$:

Theorem 1.2. — *Assume that $c_0$ is a nonnegative integer, $p \geqslant 1$, and that $\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s}$ is a nonconstant function that generates a compact composition operator $C_\varphi$ on $\mathscr{H}^p$.*

(a) *If $c_0 = 0$, then $a_n(C_\varphi) \gg \delta^n$ for some $0 < \delta < 1$.*
(b) *If $c_0 \geqslant 1$, then $a_n(C_\varphi) \geqslant \delta_p (n \log n)^{-\operatorname{Re} c_1}$, where $\delta_p > 0$ only depends on $p$. In particular, if $\operatorname{Re} c_1 > 0$ (which is the case as soon as $\psi$ is not constant), then*

$$\sum_{n=1}^{\infty} [a_n(C_\varphi)]^{1/\operatorname{Re} c_1} = \infty. \tag{1.1}$$

*These lower bounds are optimal.*

As in [26], the notation $f(n) \ll g(n)$ or equivalently $g(n) \gg f(n)$ means that there is a constant $C$ such that $f(n) \leqslant C g(n)$ for all $n$ in question.

The proof of Theorem 1.2 follows the same pattern as the proof of [26, Theorem 1.1], with an additional ingredient allowing a sharper estimate (new even for $p = 2$) in the case $c_0 \geqslant 2$. The general strategy is based on fairly soft functional analysis, which turns out to remain valid in the context of our spaces $\mathscr{H}^p$. It leads to certain general estimates for approximation numbers of $C_\varphi$, which are made effective thanks to a hard technical device involving $\overline{\partial}$ correction arguments. It is perhaps surprising to find such methodology in this context, but it proved to be quite efficient in the case $p = 2$ [26]. Unexpectedly, this auxiliary Hilbert space result can be used again in the present setting without any essential changes.

Contrary to what happens for composition operators on $H^p(\mathbb{D})$, continuity or compactness of composition operators on $\mathscr{H}^p$ can not be immediately inferred from what is known when $p = 2$. Indeed, no nontrivial sufficient condition for a composition operator to be compact on $\mathscr{H}^p$, $p \neq 2$, has been found in previous studies. We have therefore chosen to include in the present paper a sufficient condition when $c_0 \geqslant 1$, similar to the condition given in [3] for composition operators on $\mathscr{H}^2$. As in that paper, our condition will depend on a Littlewood–Paley formula, but the proof of [3] can not be easily adapted since it depends crucially on the Hilbert space structure of $\mathscr{H}^2$. To state our condition, we need to recall that the Nevanlinna counting function of a function $\varphi$ in $\mathcal{G}$ is defined by

$$\mathcal{N}_\varphi(s) = \begin{cases} \sum_{w\in\varphi^{-1}(s)} \operatorname{Re} w & \text{if } s \in \varphi(\mathbb{C}_0) \\ 0 & \text{otherwise.} \end{cases}$$

When $c_0 \geqslant 1$, this counting function satisfies a Littlewood type inequality [3]

$$\mathcal{N}_\varphi(s) \leqslant \frac{1}{c_0} \operatorname{Re} s,\ \ s \in \mathbb{C}_0.$$

An enhancement of this property gives our sufficient condition for compactness.

Theorem 1.3. — *Assume that $\varphi(s) = c_0 s + \psi(s)$ is in $\mathcal{G}$ with $c_0 \geqslant 1$ and that*

(a) $\operatorname{Im}\psi$ *is bounded on* $\mathbb{C}_0$*;*
(b) $\mathcal{N}_\varphi(s) = o\big(\operatorname{Re} s\big)$ *when* $\operatorname{Re} s \to 0^+$.

*Then $C_\varphi$ is compact on $\mathscr{H}^p$ for $p \geqslant 1$.*

Note that in the special case when $\varphi$ is univalent, we have either $\mathcal{N}_\varphi(s) = \operatorname{Re}\varphi^{-1}(s)$ or $\mathcal{N}_\varphi(s) = 0$. This means that assumption (b) can be rephrased as saying that $\operatorname{Re}\psi(s)/\operatorname{Re} s \to +\infty$ when $\operatorname{Re} s \to 0^+$.

In the final part of the paper, we will discuss a transference principle that was established in [26], showing that symbols of composition operators on $H^p(\mathbb{D})$, via left and right composition with two fixed analytic maps, give rise to composition operators on $\mathscr{H}^p$. The idea is to use this principle to transfer estimates for approximation numbers from $H^p(\mathbb{D})$. This leads to satisfactory results when $p = 1$, but, surprisingly, for no other values of $p \neq 2$. As an example, we mention that it allows us to prove the following result.

Theorem 1.4. — *There exists a function $\varphi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ in $\mathscr{G}$ such that $C_\varphi$ is bounded on $\mathscr{H}^1$ with approximation numbers verifying*

$$a\, e^{-b\sqrt{n}} \leqslant a_n(C_\varphi) \leqslant a'\, e^{-b'\sqrt{n}}.$$

This theorem follows, via our transference principle, from estimates from [17] for composition operators on $H^p(\mathbb{T})$ associated with lens maps on $\mathbb{D}$.

More general statements about the range of possible decay rates for approximation numbers of $C_\varphi$ on $\mathscr{H}^1$ can be worked out, using our transference principle and the methods of our recent work [27]. However, at present, we are not able to reach the same level of precision for approximation numbers of slow decay. This is related to a certain local embedding inequality, known to be valid only when $p$ is an even integer; this is also the reason for the constraint in part (c) of Theorem 1.1. As to the success of our method when $p = 1$, the point is that we are able to resort to estimates for interpolating sequences for $\mathscr{H}^2$, in contrast to what can be done for general $p > 1$. Indeed, essentially nothing is known about the interpolating sequences for $\mathscr{H}^p$ when $1 < p < \infty, p \neq 2$.

Our study requires a fair amount of background material and function and operator theoretic results pertaining to $\mathscr{H}^p$. More specifically, the following list shows what will be covered in the remaining part of the paper:

- Section 2 is devoted to some preliminaries and definitions on operators and Banach spaces.
- In Section 3, we recall some general properties of $\mathscr{H}^p$: Schauder basis, Bohr lift, etc...
- In Section 4, we have collected a number of basic functional analytic results as well as more specific results about Carleson measures and interpolating sequences, including estimates relying on our previous work [26].
- Section 5 is devoted to a new Littlewood–Paley type formula for $\mathscr{H}^p, p \neq 2$, which is subsequently used to prove Theorem 1.3.
- Section 6 establishes general lower bounds for $a_n(C_\varphi)$ using norms of Carleson measures and constants of interpolation.
- Section 7 gives several proofs of Theorem 1.2.
- Section 8 shows the optimality of the previous bounds.
- The final Section 9 is devoted to our transference principle and to the proof of Theorem 1.4. We also discuss two basic problems (the local embedding inequality and interpolating sequences for $\mathscr{H}^p$) that hinder further progress in our particular context as well as in our general understanding of $\mathscr{H}^p$.

Throughout the paper, we insist on exhibiting methods, and sometimes we give several proofs of the same result.

## 2. Preliminaries

### 2.1. $s$-numbers

Let $T : X \to X$ be a compact operator from a Banach space into itself, and let $(\lambda_n(T))_{n\geqslant 1}$ be the sequence of its eigenvalues, arranged in descending order. We attach to this operator three sequences of so-called $s$-numbers, which dominate in a vague sense the sequence $(\lambda_n(T))_{n\geqslant 1}$ and whose decay is designed to evaluate the degree of compactness of $T$ in a quantitative way:

(a) Approximation numbers

$$a_n(T) = \inf\{\|T - R\| : \ \text{rank } R < n\}$$

(b) Bernstein numbers

$$b_n(T) = \sup_{\dim E = n} \Big[\inf_{x\in S_E} \|Tx\|\Big]$$

($S_E$ is the unit sphere of $E$)

(c) Gelfand numbers

$$c_n(T) = \inf\{\|T|_E\| : \ \text{codim } E < n\}.$$

Note that the first and third sequences are defined as min–max and the second as a max–min. We have

$$a_n(T) = b_n(T) = c_n(T) = a_n(T^*) = \lambda_n(|T|)$$

if $X$ is a Hilbert space and $|T|$ denotes $\sqrt{T^*T}$ and

$$a_n(T) \geqslant \max(b_n(T), c_n(T)) \quad \text{and} \quad a_n(T) \geqslant a_n(T^*) \tag{2.1}$$

if $X$ is a Banach space, where the latter inequality is an equality when $T$ is compact. Moreover,

$$a_n(T) \leqslant 2\sqrt{n}\, c_n(T), \tag{2.2}$$

which follows from the fact that any $n$-dimensional subspace $E$ of a Banach space $X$ is complemented in $X$ through a projection of norm at most $\sqrt{n}$ [22, p. 114].

Finally, we will combine a basic property of composition operators with a matching property of approximation numbers:

(i) (Non abelian semi-group property)

$$C_{\varphi_1\circ\varphi_2} = C_{\varphi_2} \circ C_{\varphi_1}$$

(ii) (Ideal property)

$$a_n(ATB) \leqslant \|A\| a_n(T) \|B\|.$$

For detailed information on $s$-numbers, we refer to the articles [22, 23] or to the books [6, 24].

### 2.2. The Weyl inequalities of Johnson–König–Maurey–Retherford and Pietsch

We borrow the following theorem from the famous paper [14], which extended for the first time, under an additive form, the Weyl inequalities for Hilbert spaces to a Banach space setting. We recall that an operator $T$ is power-compact if there exists an integer $k$ such that $T^k$ is compact.

Theorem 2.1 (Johnson–König–Maurey–Retherford). — *Let $T : X \to X$ be a bounded and power-compact linear operator from a Banach space $X$ to itself and $0 < r < \infty$. Then*

$$\|(\lambda_j(T))\|_{\ell^r} \leqslant c_r \|(a_j(T))\|_{\ell^r}. \tag{2.3}$$

Soon after this result was established, Pietsch found the following multiplicative improvement (see [22, p. 156] or [23, Lemma 13]) which we will use as well.

Theorem 2.2 (Pietsch). — *Let $T : X \to X$ be a bounded and power-compact linear operator from a Banach space $X$ to itself. Then*

$$|\lambda_{2n}(T)| \leqslant e \Big( \prod_{j=1}^{n} a_j(T) \Big)^{1/n}. \tag{2.4}$$

## 3. General properties of $\mathscr{H}^p$

### 3.1. A Schauder basis for $\mathscr{H}^p$ and the partial sum operator

We will make use of the following result, first found by Helson [12] (see also [28, p. 220]) in the framework of ordered groups and then reproved in a more concrete way in the context of $\mathscr{H}^p$-spaces by Aleman, Olsen, and Saksman [1].

Theorem 3.1 (Helson–Aleman–Olsen–Saksman). — *For $1 < p < \infty$, the sequence $(e_n) = (n^{-s})$ is a (conditional) Schauder basis for $\mathscr{H}^p$.*

For every positive integer $N$, we define the partial sum operator $S_N : \mathscr{H}^p \to \mathscr{H}^p$ by the relation

$$S_N\left(\sum_{n=1}^{\infty} x_n e_n\right) := \sum_{n\leqslant N} x_n\, e_n.$$

In [1], Theorem 3.1 was obtained as a consequence of the uniform boundedness of $S_N$, i.e., the fact that there exists a constant $C = C_p$ such that

$$\|S_N f\|_{\mathscr{H}^p} \leqslant C\|f\|_{\mathscr{H}^p}$$

for every $f$ in $\mathscr{H}^p$. It is a general functional analytic fact that a complete sequence is a Schauder basis for a Banach space $X$ if and only if the associated partial sum operators $S_N$ are uniformly bounded [16]. This leads to the following result.

Lemma 3.2 (Contraction principle for Schauder bases). — *Let $X$ be a Banach space and assume that $(e_n)_{n\geqslant 1}$ is a Schauder basis for $X$. Let $N$ be a positive integer and $(\lambda_n)_{n\geqslant N}$ a nonincreasing sequence of nonnegative numbers. Then for every $f = \sum_{n=1}^{\infty} x_n\, e_n$ in $X$,*

$$\left\|\sum_{n\geqslant N} \lambda_n\, x_n\, e_n\right\| \leqslant 2C\lambda_N\|f\|, \tag{3.1}$$

*where $C = \sup_n \|S_n\|$.*

*Proof.* — We write $x_n e_n = S_n f - S_{n-1} f$ so that we get

$$\sum_{n\geqslant N} \lambda_n\, x_n\, e_n = -\lambda_N S_{N-1} f + \sum_{n\geqslant N} (\lambda_n - \lambda_{n+1}) S_n f.$$

□

The sequence $(n^{-s})$ fails to be a basis for $\mathscr{H}^1$, but not by much, as expressed by the following theorem.

Theorem 3.3. — *There exists a constant $C$ such that*

$$\|S_N f\|_{\mathscr{H}^1} \leqslant C\, \log N\, \|f\|_{\mathscr{H}^1} \tag{3.2}$$

*for every $f$ in $\mathscr{H}^1$.*

*Proof.* — We can appeal to Helson's work [12] which deals with a compact connected group $G$ and its ordered dual $\Gamma$. We take $G = \mathbb{T}^\infty$, $\Gamma = \mathbb{Z}^{(\infty)}$ along with the order

$$\alpha \leqslant \beta \quad \text{if} \quad \sum_{j\geqslant 1} \alpha_j \log p_j \leqslant \sum_{j\geqslant 1} \beta_j \log p_j.$$

Here $\alpha = (\alpha_j)$, $\beta = (\beta_j)$, and $(p_j)$ denotes the sequences of primes.

Given $0 < p < 1$, Helson's result implies fairly easily that

$$\|S_N f\|_{\mathscr{H}^p}^p \ll (\cos \pi p/2)^{-1} \|f\|_{\mathscr{H}^1}^p \ll (1-p)^{-1} \|f\|_{\mathscr{H}^1}^p$$

when $f$ is in $\mathscr{H}^p$. Let now $f(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ be in $\mathscr{H}^1$. Observe that $|b_n| \leqslant \|f\|_{\mathscr{H}^1}$ which implies the pointwise estimate $|S_N f| \leqslant N \|f\|_{\mathscr{H}^1}$. Using the Bohr lift and integrating over $\mathbb{T}^\infty$ with its Haar mesure $m_\infty$, we obtain

$$\begin{aligned}
\|S_N f\|_{\mathscr{H}^1} &= \int |S_N f| dm_\infty = \int |S_N f|^{1-p} |S_N f|^p dm_\infty \\
&\leqslant N^{1-p} (\|f\|_{\mathscr{H}^1})^{1-p} \int |S_N f|^p dm_\infty \\
&\ll N^{1-p} (\|f\|_{\mathscr{H}^1})^{1-p} (1-p)^{-1} (\|f\|_{\mathscr{H}^1})^p \\
&\ll N^{1-p} (1-p)^{-1} \|f\|_{\mathscr{H}^1}.
\end{aligned}$$

Now choosing $p = 1 - 1/\log N$, we arrive at (3.2) since then $N^{1-p} = e$. □

We will now give an alternate and self-contained proof of Theorem 3.3. This proof is an application of a simple but powerful identity shown by Eero Saksman to the second-named author [29], which most likely will have other applications in the study of the spaces $\mathscr{H}^p$. The initial application Saksman had in mind concerned the conjugate exponent $p = \infty$ or, in other words, the Banach algebra $\mathscr{H}^\infty$ which consists of Dirichlet series that define bounded analytic functions on $\mathbb{C}_0$. Equipped with the natural $H^\infty$ norm on $\mathbb{C}_0$, $\mathscr{H}^\infty$ is isometrically equal to the multiplier algebra of $\mathscr{H}^p$ [2, 11].

We will use the notation

$$\widehat{\psi}(\xi) = \int_{-\infty}^{\infty} e^{-it\xi} \psi(t) dt$$

for the Fourier transform of a function $\psi$ in $L^1(\mathbb{R})$. If $\psi$ is in $L^1(\mathbb{R})$ and $f(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ is a Dirichlet series that is absolutely convergent in some half-plane $\mathbb{C}_\theta$, then the function

$$P_\psi f(s) := \sum_{n=1}^{\infty} b_n n^{-s} \widehat{\psi}(\log n) = \int_{-\infty}^{\infty} f(s+it) \psi(t) dt, \quad s \in \mathbb{C}_\theta, \tag{3.3}$$

is again a Dirichlet series. It is clear that $P_\psi f$ is absolutely convergent wherever $f$ is absolutely convergent. If $\psi$ is in $L^1(\mathbb{R})$ with $\widehat{\psi}$ compactly supported and $f(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ is a Dirichlet series that is bounded in some half-plane, then we have the identity between $P_\psi f(s)$ and the integral on the right-hand side of (3.3) throughout that half-plane. If $f$ is in $\mathscr{H}^p$ for some $p$, then we may write

$$P_\psi f = \int_{-\infty}^{\infty} (T_t f)\, \psi(t) dt, \tag{3.4}$$

where the right-hand side denotes a vector-valued integral in $\mathscr{H}^p$ and $T_t : \mathscr{H}^p \to \mathscr{H}^p$ is the vertical translation-operator defined by $T_t f(s) = f(s + it)$. The utility of (3.3) rests on the fact that $T_t$ acts isometrically on $\mathscr{H}^p$ for every $1 \leqslant p \leqslant \infty$. Saksman's vertical convolution formula (3.3) can now be applied to yield the following result.

Lemma 3.4. — *If $\psi$ is in $L^1(\mathbb{R})$ and $f(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ is in $\mathscr{H}^p$ for some $1 \leqslant p \leqslant \infty$, then*

$$\|P_\psi f\|_{\mathscr{H}^p} \leqslant \|f\|_{\mathscr{H}^p}\, \|\psi\|_1. \tag{3.5}$$

*Proof.* — By (3.4) and the vertical translation invariance of the norm in $\mathscr{H}^p$, we get

$$\|P_\psi f\|_{\mathscr{H}^p} \leqslant \int_{-\infty}^{\infty} \|T_t f\|_{\mathscr{H}^p}\, |\psi(t)| dt = \|f\|_{\mathscr{H}^p}\, \|\psi\|_1.$$

□

We now turn to Saksman's alternate proof of Theorem 3.3. The idea is to choose $\psi$ so that $\widehat{\psi}$ is smooth and $P_\psi f$ is a good approximation to $S_N f$. A suitable trade-off between these requirements can be made as follows. Let $\Delta_h$ be the indicator function of the interval $[-h, h]$ for $h > 0$, and let $\Lambda = (N/2)(\Delta_1 * \Delta_{1/N})$ be the even trapezoidal function with nodes at $1 - 1/N$ and $1 + 1/N$. Then $\Lambda(x) = 1$ for $x$ in $[-(1 - 1/N),\ 1 - 1/N]$. We see that

$$\widehat{\Lambda}(\xi) = (N/2)\, \widehat{\Delta_1}(\xi)\widehat{\Delta_{1/N}}(\xi) = (N/2)\, \frac{\sin\xi \sin\xi/N}{\xi^2},$$

which implies that $\|\widehat{\Lambda}\|_1 \ll \log N$. We choose $\psi$ by requiring that $\widehat{\psi}(x) = \Lambda(x/\log N)$. Then, by the Fourier inversion formula, we have $\|\psi\|_1 \ll \log N$. In addition, we observe that $P_\psi f$ differs from $S_N f = \sum_{n=1}^{N} b_n n^{-s}$ by at most

$$O\big[N(e^{\log N/N} - e^{-\log N/N})\big] = O(\log N)$$

terms that are all of size $\ll \|f\|_{\mathscr{H}^1}$ since $|b_n| \leqslant \|f\|_{\mathscr{H}^1}$. In view of (3.5), this ends the second proof of Theorem 3.3.

### 3.2. The Bohr lift.

By a fundamental observation of Bohr, $\mathbb{C}_{1/2}$ can be be embedded in the infinite-dimensional polydisc $\mathbb{D}^\infty = \{(z_1, z_2, \dots);\ |z_i| < 1\}$ in the following way. Every positive integer $n$ can be factored as $n = \prod_j p_j^{\kappa_j}$, where $(p_j)$ denotes the sequence of prime numbers. This means that we may represent $n$ by the multi-index $\kappa(n) = (\kappa_1, \kappa_2, \dots)$ associated with its prime factorization. Now let $\varphi$ be a function in $\mathscr{G}$ with $c_0 = 0$, and assume that $\overline{\varphi(\mathbb{C}_0)}$ is a bounded subset of $\mathbb{C}_{1/2}$. The Bohr lift $\Phi$ of $\varphi$ is the function

$$\Phi(z) := \sum_{n=1}^{\infty} c_n z^{\kappa(n)} = \sum_{n=1}^{\infty} c_n z_1^{\kappa_1} z_2^{\kappa_2} \cdots .$$

Defining $\Delta : \mathbb{C}_{1/2} \to \mathbb{D}^\infty \cap \ell^2$ by $\Delta(s) := (p_j^{-s})$, we observe that $\Phi$ is an analytic map $\mathbb{D}^\infty \cap \ell^2 \to \mathbb{C}_{1/2}$ satisfying

$$\Phi \circ \Delta = \varphi. \tag{3.6}$$

We set $\Phi^*(z) := \lim_{r\to 1} \Phi(rz)$, where $z$ is a point in $\mathbb{T}^\infty$. We denote by $m_\infty$ the Haar measure of $\mathbb{T}^\infty$ and define the pullback measure $\mu_\varphi$ by

$$\mu_\varphi(E) := m_\infty(\{z \in \mathbb{T}^\infty : \Phi^*(z) \in E\}) = m_\infty\left((\Phi^*)^{-1}(E)\right). \tag{3.7}$$

We will need the following basic result about the Bohr lift [26].

Theorem 3.5. — *Suppose that $\varphi \in \mathscr{G}$ induces a bounded composition operator on $\mathscr{H}^p$ with $c_0 = 0$ and that $\overline{\varphi(\mathbb{C}_0)}$ is a bounded subset of $\mathbb{C}_{1/2}$. Then, for every $f$ in $\mathscr{H}^p$, $1 \leqslant p < \infty$, we have*

$$\|C_\varphi(f)\|_{\mathscr{H}^p}^p = \int_{\mathbb{T}^\infty} |f(\Phi^*(z))|^p dm_\infty(z) = \int_{\overline{\varphi(\mathbb{C}_0)}} |f(s)|^p d\mu_\varphi(s).$$

### 3.3. Type and cotype of $(\mathscr{H}^p)^*$

Let $(\varepsilon_j)$ denote a Rademacher sequence and $\mathbb{E}$ the expectation. We recall that a Banach space $X$ is of type $p^*$ with $1 \leqslant p^* \leqslant 2$ if, for some constant $C \geqslant 1$,

$$\mathbb{E}\left\|\sum \varepsilon_j x_j\right\| \leqslant C\left(\sum \|x_j\|^{p^*}\right)^{1/p^*}$$

for every finite sequence $(x_j)$ of vectors from $X$. The smallest constant $C$, denoted by $T_{p^*}(X)$, is called the type $p^*$-constant of $X$. Similarly, a Banach space $Y$ is said to be of cotype $q^*$ with $2 \leqslant q^* \leqslant \infty$ if

$$\mathbb{E}\left\|\sum \varepsilon_j y_j\right\| \geqslant C^{-1}\left(\sum \|y_j\|^{q^*}\right)^{1/q^*}$$

for every finite sequence $(y_j)$ of vectors of $Y$. The smallest constant $C$, denoted by $C_{q*}(Y)$, is called the cotype $q^*$-constant of $Y$.

As already mentioned, the space $X = \mathscr{H}^p$ is well understood as the subspace $H^p(\mathbb{T}^\infty)$ of $L^p(\mathbb{T}^\infty)$, but it is not complemented in $L^p(\mathbb{T}^\infty)$. This means that its dual $Y$ is something rather mysterious. But the following fact will be sufficient for our purposes.

Lemma 3.6. — *The Banach space $Y = (\mathscr{H}^p)^*$ is of cotype* $\max(q, 2)$ *where $q$ is the conjugate exponent of $p$.*

*Proof.* — $\mathscr{H}^p$ is isometric to the subspace $H^p(\mathbb{T}^\infty)$ of $L^p(\mathbb{T}^\infty)$. The latter space is of type $p^* = \min(p, 2)$ by Fubini's theorem and Khintchin's inequality. According to a result of Pisier (from [25], see also [18], [8, p. 220], or [16, p. 165]), its dual $Y$ is of cotype $q^* = \max(q, 2)$ $(1/p + 1/q = 1)$ and, moreover,

$$C_{q*}(Y) \leqslant 2T_{p^*}(\mathscr{H}^p). \tag{3.8}$$

Hence the conclusion of the lemma follows. □

### 3.4. Description of the spectrum of compact composition operators

A complete description of the eigenvalues of $C_\varphi$ on $\mathscr{H}^2$ was given in [3]. We will show that this result is valid for $\mathscr{H}^p$ as well:

Theorem 3.7. — *Let $\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n\, n^{-s}$ induce a compact operator on $\mathscr{H}^p$, $1 \leqslant p < \infty$. Then, the eigenvalues of $C_\varphi$ have multiplicity one and are*

(a) $\lambda_n = [\varphi'(\alpha)]^{n-1}$, $\quad n = 1, 2, \dots$, *when $c_0 = 0$, where $\alpha$ is the fixed point of $\varphi$ in $\mathbb{C}_{1/2}$.*
(b) $\lambda_n = n^{-c_1}$, $\quad n = 1, 2, \dots$, *when $c_0 = 1$.*

*Proof.* — To part (a), we adapt the proof of [3] to $\mathscr{H}^p$. To begin with, we observe that when $c_0 = 0$, $\varphi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$ and $\varphi(+\infty) \neq +\infty$. Hence, $\varphi$ admits a (necessarily unique) fixed point in $\mathbb{C}_{1/2}$. We then set $E := \big\{[\varphi'(\alpha)]^{n-1},\ n = 1, 2, \dots\big\}$ and let $\sigma(C_\varphi)$ be the spectrum of $C_\varphi$, which coincides with $\sigma(C_\varphi^*)$ since $T$ is invertible if and only if $T^*$ is. For a given integer $m \geqslant 1$, consider the space $\mathcal{K}_m$ defined by

$$\mathcal{K}_m = \text{ span } (\delta_\alpha, \delta'_\alpha, \cdots, \delta_\alpha^{(m)}) \subset (\mathscr{H}^p)^*,$$

where by definition $\delta_\alpha^{(k)}(f) = f^{(k)}(\alpha)$. This space is invariant under $C_\varphi^*$ since $\varphi(\alpha) = \alpha$. It is also finite-dimensional, and the matrix of $(C_\varphi^*)_{|\mathcal{K}_m}$

on the natural basis of $\mathcal{K}_m$ is upper triangular with diagonal elements $[\varphi'(\alpha)]^{n-1}$, $1 \leqslant n \leqslant m+1$. This shows that $E \subset \sigma(C_\varphi^*) = \sigma(C_\varphi)$. For the reverse inclusion, we use the compactness of $C_\varphi$ and Königs's theorem [34, pp. 90–91] which in particular claims that $f \circ \varphi = \lambda f$ with $f$ a nonzero analytic function in $\mathbb{C}_{1/2}$ implies that $\lambda = [\varphi'(\alpha)]^k$ for some nonnegative integer $k$ and that the corresponding eigenfunctions generate a one-dimensional space of $\mathscr{H}^p$.

As for (b), set $E = \{n^{-c_1},\ n \geqslant 1\}$. The proof of [3] still gives $\sigma(C_\varphi) \subset E \cup \{0\}$ for $\mathscr{H}^p$. Moreover, for a fixed integer $m$, the vector spaces $\mathcal{K}_m$ and $\mathcal{L}_m$ respectively generated by $1, 2^{-s}, \dots, m^{-s}$ and $j^{-s},\ j > m$ are complementary, $\mathcal{L}_m$ is stable by $C_\varphi$, $\mathcal{K}_m$ is finite-dimensional, and the matrix of $C_{\varphi|\mathcal{K}_m}$ is lower triangular with diagonal elements $j^{-c_1}, 1 \leqslant j \leqslant m$. Therefore, $E \subset \sigma(C_\varphi)$ as in [3]. □

The result of [3] stating that $\sigma(C_\varphi) = \{0, 1\}$ when $c_0 \geqslant 2$ also extends from $\mathscr{H}^2$ to $\mathscr{H}^p$, but this case will not be needed in this work and is omitted here.

### 3.5. Vertical translates

$\mathbb{T}^\infty$ may be identified with the dual group of $\mathbb{Q}_+$, where $\mathbb{Q}_+$ denotes the multiplicative discrete group of strictly positive rational numbers: given a point $z = (z_j)$ on $\mathbb{T}^\infty$, we define a character $\chi$ on $\mathbb{Q}_+$ by its values at the primes by setting

$$\chi(2) = z_1,\ \chi(3) = z_2,\ \dots,\ \chi(p_m) = z_m,\ \dots$$

and by extending the definition multiplicatively. In the sequel, we will associate the character $\chi$ with the point $(z_j)$ and refer to $\chi$ as well as a point on $\mathbb{T}^\infty$. In particular, we will be interested in properties that hold for almost all characters $\chi$ with respect to the Haar measure $m_\infty$ on $\mathbb{T}^\infty$.

Characters are connected with vertical limit functions of $\mathscr{H}^p$. Indeed, fix an arbitrary element $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ of $\mathscr{H}^p$. The vertical translates of $f$ are the functions $T_\tau f(s) := f(s + i\tau)$. To every sequence $(\tau_n)$ of translations there exists a subsequence, say $(\tau_{n(k)})$, such that $T_{\tau_{n(k)}} f$ converges uniformly on compact subsets of the domain $\mathbb{C}_{1/2}$ to a limit function, say $\tilde{f}(s)$. We will say that $\tilde{f}$ is a vertical limit function of $f$. In [2], it is proved that the vertical limit functions of $f$ coincide with the functions $f_\chi(s) := \sum_{n=1}^\infty a_n \chi(n) n^{-s}$, $\chi$ being a character.

We can also view vertical translates as a flow of rotations acting on $\mathbb{T}^\infty$. More precisely, given $z$ in $\mathbb{T}^\infty$ and $t$ in $\mathbb{R}$, we set $\mathcal{T}_t z = (2^{it} z_1, \dots, p_j^{it} z_j, \dots)$.

Then a simple application of Fubini's theorem and the invariance of $m_\infty$ under rotation show the following lemma.

Lemma 3.8. — *Let $\mu$ be a finite Borel measure on $\mathbb{R}$ and $F \in L^1(\mathbb{T}^\infty)$. Then*

$$\int_{\mathbb{T}^\infty} \int_{\mathbb{R}} F(\mathcal{T}_t z) d\mu(t) dm_\infty(z) = \mu(\mathbb{R}) \|F\|_{L^1(\mathbb{T}^\infty)}.$$

In [11] and [2], it was explained that it is useful to consider $f_\chi$ to get a more profound understanding of the function theoretic properties of $f$. For example, for almost all characters $\chi$ , the function $f_\chi$ can be extended to $\mathbb{C}_0$. Moreover, setting $F(z) = |f_\chi(0)|^p$ (we identify again $\mathbb{T}^\infty$ and $\mathbb{Q}_+$), we obtain immediately from Lemma 3.8 the following way to compute the $\mathscr{H}^p$ norm of $f$ (see also [11, theorem 4.1] or [2, lemma 5]):

Lemma 3.9. — *Let $\mu$ be a finite Borel measure on $\mathbb{R}$. Then :*

$$\|f\|^p_{\mathscr{H}^p} \mu(\mathbb{R}) = \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_\chi(it)|^p d\mu(t) dm_\infty(\chi).$$

We shall need to extend our notation of vertical translates to the class of functions of the form $\varphi(s) = c_0 s + \psi(s) \in \mathcal{G}$. For such functions, $\varphi_\chi$ will mean

$$\varphi_\chi(s) = c_0 s + \psi_\chi(s).$$

The connection between the composition operators $C_\varphi$ and $C_{\varphi_\chi}$ is clarified in [10], where it is proved that for every $f$ in $\mathscr{H}^p$ and every $\chi$ in $\mathbb{T}^\infty$,

$$(f \circ \varphi)_\chi(s) = f_{\chi^{c_0}} \circ \varphi_\chi(s),\ s \in \mathbb{C}_{1/2},$$

where $\chi^{c_0}$ is the character taking the value $[\chi(n)]^{c_0}$ at $n$. Moreover, for almost all $\chi$ in $\mathbb{T}^\infty$, this relation remains true in $\mathbb{C}_0$.

## 4. Carleson measures and sequences, and interpolating sequences

### 4.1. The case $p < \infty$

We will denote by $\delta_s$ the functional of point evaluation on $\mathscr{H}^p$ at the point $s = \sigma + it$ in $\mathbb{C}_{1/2}$, so that $\delta_s(f) = f(s)$. The norm of $\delta_s$ was computed by Cole and Gamelin [7] (see also [2]):

$$\|\delta_s\| = [\zeta(2\sigma)]^{1/p}, \tag{4.1}$$

where $\zeta(s)$ is the Riemann zeta function. It should be noted that $\zeta(2\sigma) \approx (2\sigma - 1)^{-1}$ when $s$ is restricted to a set on which $\sigma$ is bounded.

Let now $\mu$ be a nonnegative Borel measure on the half-plane $\mathbb{C}_{1/2}$. We say that $\mu$ is a Carleson measure for $\mathscr{H}^p$ if there exists a positive constant $C$ such that

$$\int_{\mathbb{C}_{1/2}} |f(s)|^p d\mu(s) \leqslant C\|f\|_{\mathscr{H}^p}^p$$

holds for every $f$ in $\mathscr{H}^p$. The smallest possible $C$ in this inequality is called the $\mathscr{H}^p$ Carleson norm of $\mu$. We denote it by $\|\mu\|_{\mathcal{C},\mathscr{H}^p}$ and declare that $\|\mu\|_{\mathcal{C},\mathscr{H}^p} = \infty$ if $\mu$ fails to be a Carleson measure for $\mathscr{H}^p$. Let next $S = (s_j)$ be a sequence of distinct points of $\mathbb{C}_{1/2}$. We say that this sequence is a Carleson sequence if the discrete measure

$$\mu_S = \sum_j \frac{\delta_{s_j}}{\|\delta_{s_j}\|^p}$$

is a Carleson measure for $\mathscr{H}^p$ in the above sense. The Carleson $\mathscr{H}^p$ norm $\|\mu_S\|_{\mathcal{C},\mathscr{H}^p}$ is called the Carleson $\mathscr{H}^p$ constant of $S$.

We will need the following estimate.

Lemma 4.1. — *Let $\mu$ be a nonnegative Borel measure in the half-plane $\mathbb{C}_{1/2}$ whose support is contained in $\overline{\mathbb{C}_\theta}$ for some $\theta > 1/2$. Then*

$$\|\mu\|_{\mathcal{C},\mathscr{H}^p} \leqslant \begin{cases} [\zeta(2\theta)]^{(p-2)/p}\|\mu\|_{\mathcal{C},\mathscr{H}^2}, & p \geqslant 2 \\ [\zeta(2\theta)]\|\mu\|, & 1 \leqslant p < 2. \end{cases}$$

*Proof.* — For an arbitrary point $s = \sigma + it$ in the support of $\mu$, we have

$$|f(s)| \leqslant [\zeta(2\sigma)]^{1/p}\|f\|_{\mathscr{H}^p} \leqslant [\zeta(2\theta)]^{1/p}\|f\|_{\mathscr{H}^p}.$$

We infer from this that

$$\begin{aligned} \int_{\mathbb{C}_{1/2}} |f(s)|^p d\mu(s) &= \int_{\mathbb{C}_{1/2}} |f(s)|^{p-2}|f(s)|^2 d\mu(s) \\ &\leqslant [\zeta(2\theta)]^{(p-2)/p}\|f\|_{\mathscr{H}^p}^{p-2} \int_{\mathbb{C}_{1/2}} |f(s)|^2 d\mu(s). \end{aligned}$$

Now the result follows since $\|f\|_{\mathscr{H}^2} \leqslant \|f\|_{\mathscr{H}^p}$ when $p \geqslant 2$. The (poor) estimate in the case $p < 2$ is obvious. □

The $\mathscr{H}^p$ constant of interpolation $M_{\mathscr{H}^p}(S)$ is defined as the infimum of the constants $K$ with the following property: for every sequence $(a_j)$ of complex numbers such that

$$\sum_j |a_j|^p \|\delta_{s_j}\|^{-p} < \infty,$$

there exists a function $f$ in $\mathscr{H}^p$ such that

$$f(s_j) = a_j \quad \text{for all } j \text{ and } \quad \|f\|_p \leqslant K \left( \sum_j |a_j|^p \|\delta_{s_j}\|^{-p} \right)^{1/p}.$$

For a sequence $S$ of distinct points in $\mathbb{C}_0$, we will denote by $M_{\mathscr{H}^\infty}(S)$ the best constant $K$ such that, for every bounded sequence $(a_j)$ of complex numbers, there exists a function $f$ in $\mathscr{H}^\infty$ such that

$$f(s_j) = a_j \quad \text{for all } j \text{ and } \quad \|f\|_\infty \leqslant K \sup_j |a_j|.$$

In the next lemma, we will use the following notation. Given a sequence $S = (\sigma_j + it_j)$ and a real number $\theta$, we write $S + \theta := (\sigma_j + \theta + it_j)$.

Lemma 4.2. — *Suppose that $\theta > 1/2$, $\delta > 0$ and that $S = (s_j = \sigma_j + it_j)_{j=1}^n$ is a finite sequence in the half-plane $\mathbb{C}_{1/2+\delta}$. Then*

$$\begin{aligned} M_{\mathscr{H}^p}(S) \quad &\leqslant \quad [\zeta(2\theta)]^{1/\min(2,p)} \left( \frac{\zeta(1+2\delta)}{\zeta(1+2(\delta+\theta))} \right)^{1/\min(2,p)} n^{1/\min(2,p)-1/p} \\ &\qquad \times \big(M_{\mathscr{H}^2}(S+\theta)\big)^{2/\min(p,2)} \end{aligned}$$

*for $1 \leqslant p \leqslant \infty$.*

*Proof.* — Given a sequence $(a_j)_{j=1}^n$, we find the minimal norm solution $F$ to the interpolation problem $F(s_j + \theta) = a_j$ in $\mathscr{H}^2$. From the definition of the constant of interpolation and (4.1), we get the basic estimate

$$\|F\|_{\mathscr{H}^2} \leqslant M_{\mathscr{H}^2}(S+\theta) \left( \sum_{j=1}^n |a_j|^2 [\zeta(2(\sigma_j+\theta))]^{-1} \right)^{1/2}. \tag{4.2}$$

By our restriction on $S$ and the fact that $\zeta'/\zeta$ increases on $(1,\infty)$, we have

$$\frac{\zeta(2\sigma_j)}{\zeta(2(\sigma_j+\theta))} \leqslant \frac{\zeta(1+2\delta)}{\zeta(1+2(\delta+\theta))},$$

which gives

$$\|F\|_{\mathscr{H}^2} \leqslant \left( \frac{\zeta(1+2\delta)}{\zeta(1+2(\delta+\theta))} \right)^{1/2} M_{\mathscr{H}^2}(S+\theta) \left( \sum_{j=1}^n |a_j|^2 [\zeta(2\sigma_j)]^{-1} \right)^{1/2}$$

when inserted into (4.2). We first assume $1 \leqslant p < 2$. Using (4.1) which gives $|a_j| \leqslant \sqrt{\zeta(2\theta)}\|F\|_2$, we then get

$$\begin{aligned}\|F\|_{\mathscr{H}^2} &\leqslant \left(\frac{\zeta(1+2\delta)}{\zeta(1+2(\delta+\theta))}\right)^{1/2} M_{\mathscr{H}^2}(S+\theta)[\zeta(2\theta)]^{1/2-p/4}\|F\|_{\mathscr{H}^2}^{1-p/2} \\ &\qquad \times \left(\sum_{j=1}^n |a_j|^p[\zeta(2\sigma_j)]^{-1}\right)^{1/2}.\end{aligned}$$

This yields

$$\begin{aligned}\|F\|_{\mathscr{H}^2} &\leqslant \left(\frac{\zeta(1+2\delta)}{\zeta(1+2(\delta+\theta))}\right)^{1/p} \big(M_{\mathscr{H}^2}(S+\theta)\big)^{2/p}[\zeta(2\theta)]^{1/p-1/2} \\ &\qquad \times \left(\sum_{j=1}^n |a_j|^p\|\delta_{s_j}\|^{-p}\right)^{1/p}.\end{aligned}$$

When $2 \leqslant p < +\infty$, we simply use Hölder's inequality to obtain

$$\begin{aligned}\|F\|_{\mathscr{H}^2} &\leqslant \left(\frac{\zeta(1+2\delta)}{\zeta(1+2(\delta+\theta))}\right)^{1/2} M_{\mathscr{H}^2}(S+\theta)n^{1/2-1/p} \\ &\qquad \times \left(\sum_{j=1}^n |a_j|^p\|\delta_{s_j}\|^{-p}\right)^{1/p}.\end{aligned}$$

For $p = \infty$, we get

$$\|F\|_{\mathscr{H}^2} \leqslant M_{\mathscr{H}^2}(S+\theta)n^{1/2}\sup_j |a_j|.$$

We now observe that the shifted function $G(s) = F(s+\theta)$ is in $\mathscr{H}^\infty$ since $\theta > 1/2$, and that $G(s_j) = a_j$ as well. Hence

$$\|G\|_{\mathscr{H}^p} \leqslant \|G\|_{\mathscr{H}^\infty} \leqslant \sqrt{\zeta(2\theta)}\|F\|_{\mathscr{H}^2}.$$

□

The preceding lemma is useful because we have good estimates for constants of interpolation in the $\mathscr{H}^2$ setting, thanks to the following key result from [26]. Here we use the notation $S_R$ for the subsequence of points $s_j$ from $S$ that satisfy $|\operatorname{Im} s_j| \leqslant R$. Moreover, $H^2(\mathbb{C}_{1/2})$ is the classical $H^2$ space of the half-plane $\mathbb{C}_{1/2}$, and the constant of interpolation $M_{H^2(\mathbb{C}_{1/2})}(S)$ is defined in the obvious way via the reproducing kernel of $H^2(\mathbb{C}_{1/2})$.

Lemma 4.3. — *Suppose $S = (s_j = \sigma_j + it_j)$ is an interpolating sequence for $H^2(\mathbb{C}_{1/2})$ and that there exists a number $\theta > 1/2$ such that $1/2 < \sigma_j \leqslant$*

*$\theta$ for every $j$. Then there exists a constant $C$, depending on $\theta$, such that*

$$M_{\mathscr{H}^2}(S_R) \leqslant C[M_{H^2(\mathbb{C}_{1/2})}(S)]^{2\theta+6} R^{2\theta+7/2} \tag{4.3}$$

*whenever $R \geqslant \theta + 1$.*

This result is based on [32] and relies on quite involved estimates for solutions of the $\overline{\partial}$ equation.

We mention finally the remarkable fact that, in general

$$M_{\mathscr{H}^1}(S) \leqslant [M_{\mathscr{H}^2}(S)]^2 \tag{4.4}$$

whenever $S$ is an interpolating sequence for $\mathscr{H}^2$. This bound follows from the observation (see [21]) that we may solve $f(s_j) = a_j$ in $\mathscr{H}^1$ by first solving $g(s_j) = \sqrt{a_j}$ in $\mathscr{H}^2$ and then setting $f = g^2$. Inequality (4.4) is the reason $\mathscr{H}^1$ stands out as a distinguished case in our context; for other values of $p$, we do not know how to obtain a nontrivial bound for $M_{\mathscr{H}^p}(S)$ when $\operatorname{Re} s_j \to 1/2$, and essentially nothing is known about the $\mathscr{H}^p$ interpolating sequences.

### 4.2. The case $p = \infty$

If $H^\infty$ denotes the Banach space of bounded analytic functions on $\mathbb{C}_0$ and if $S = (s_j)$ is a sequence of points in $\mathbb{C}_0$, we define the interpolation constant $M_{H^\infty}(S)$ as the infimum of constants $C$ such that for any bounded sequence $(a_j)$, the interpolation problem $a_j = f(s_j)$, $j = 1, 2, \ldots$, has a solution $f \in H^\infty$ such that $\|f\|_\infty \leqslant C \sup_j |a_j|$. We will make use of a result of the third-named author [31], which can be rephrased as follows:

Theorem 4.4. — *Let $S$ be a subset of $\mathbb{C}_0$ such that $|s| \leqslant K$ for every $s$ in $S$. Then there exists a positive constant $\gamma$, depending only on $K$, such that*

$$M_{\mathscr{H}^\infty}(S) \leqslant C\big[M_{H^\infty}(S)\big]^\gamma, \tag{4.5}$$

*where $C$ is an absolute constant.*

This result is implicit in [31]; it is obtained by combining the interpolation theorem of Berndtsson and al. [4] (see [31, Lemma 3]) with [31, Lemma 4].

## 5. A Littlewood–Paley formula for $\mathscr{H}^p$ and proof of Theorem 1.3

Our new Littlewood–Paley formula reads as follows (abbreviating $\|\ \ \|_{\mathscr{H}^p}$ to $\|\ \ \|_p$.)

THEOREM 5.1. — *Let $\mu$ be a probability measure on $\mathbb{R}$ and $p \geqslant 1$. Then*

$$\|f\|_p^p \asymp |b_1|^p + \int_{\mathbb{T}^\infty}\int_0^{+\infty}\int_{\mathbb{R}} \sigma|f_\chi(\sigma+it)|^{p-2}|f_\chi'(\sigma+it)|^2 d\mu(t)d\sigma dm_\infty(\chi) \tag{5.1}$$

*holds for every Dirichlet series $f(s)=\sum_{n\geqslant 1} b_n n^{-s}$ in $\mathscr{H}^p$ .*

The notation $u(f)\asymp v(f)$ means as usual that there exists a constant $C\geqslant 1$ such that for every $f$ in question, $C^{-1}u(f)\leqslant v(f)\leqslant Cu(f)$.

*Proof.* — We start from the Littlewood–Paley formula for $H^p(\mathbb{D})$, which appears for instance in [35]: We have

$$\|g\|_{H^p(\mathbb{D})}^p \asymp |g(0)|^p + \int\int_{\mathbb{D}} (1-|z|^2)|g(z)|^{p-2}|g'(z)|^2 d\lambda(z)$$

when $g$ is in $H^p(\mathbb{D})$, where now $d\lambda$ denotes Lebesgue area measure on $\mathbb{D}$. Next we let $f$ be a Dirichlet polynomial, $\xi>0$, and consider the Cayley transform

$$\omega_\xi(z)=\xi\frac{1+z}{1-z},\quad \omega_\xi^{-1}(s)=\frac{s-\xi}{s+\xi}.$$

By Lemma 3.9,

$$\|f\|_p^p = \int_{\mathbb{T}^\infty}\left(\int_{\mathbb{R}}|f_\chi(it)|^p\frac{\xi}{\pi(\xi^2+t^2)}dt\right)dm_\infty(\chi). \tag{5.2}$$

For fixed $\chi$ on $\mathbb{T}^\infty$, we find that

$$\begin{aligned}&\int_{\mathbb{R}}|f_\chi(it)|^p\frac{\xi}{\pi(\xi^2+t^2)}dt\\ &\quad=\|f_\chi\circ\omega_\xi\|_{H^p(\mathbb{D})}^p\\ &\quad\asymp|f_\chi(\xi)|^p+\int\int_{\mathbb{D}}(1-|z|^2)|f_\chi\circ\omega_\xi(z)|^{p-2}|f_\chi'\circ\omega_\xi(z)|^2|\omega_\xi'(z)|^2d\lambda(z).\end{aligned}$$

By using the change of variables $s=\sigma+it=\omega_\xi(z)$, we get

$$\begin{aligned}&\int_{\mathbb{R}}|f_\chi(it)|^p\frac{\xi}{\pi(\xi^2+t^2)}dt\\ &\quad\asymp|f_\chi(\xi)|^p+\int_0^{+\infty}\int_{\mathbb{R}}\left(1-\frac{|s-\xi|^2}{|s+\xi|^2}\right)|f_\chi(s)|^{p-2}|f_\chi'(s)|^2dtd\sigma\\ &\quad\asymp|f_\chi(\xi)|^p+\int_0^{+\infty}\int_{\mathbb{R}}\frac{\sigma\xi}{(\sigma+\xi)^2+t^2}|f_\chi(s)|^{p-2}|f_\chi'(s)|^2dtd\sigma.\end{aligned}$$

We integrate this over $\mathbb{T}^\infty$. In view of (5.2), this gives

$$\|f\|_p^p \asymp \int_{\mathbb{T}^\infty} |f_\chi(\xi)|^p dm_\infty(\chi) + \int_0^{+\infty} \frac{\sigma\xi}{\sigma+\xi} \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_\chi(s)|^{p-2} |f_\chi'(s)|^2 \frac{\sigma+\xi}{\pi\big((\sigma+\xi)^2+t^2\big)} dt dm_\infty(\chi) d\sigma.$$

Using Lemma 3.8 for the two probability measures $d\mu(t)$ and $\frac{\sigma+\xi}{\pi\big((\sigma+\xi)^2+t^2\big)}dt$, we obtain that

$$\int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_\chi(s)|^{p-2} |f_\chi'(s)|^2 \frac{\sigma+\xi}{\pi\big((\sigma+\xi)^2+t^2\big)} dt dm_\infty(\chi) = \int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f_\chi(s)|^{p-2} |f_\chi'(s)|^2 d\mu(t) dm_\infty(\chi).$$

Finally, we conclude by letting $\xi$ tend to infinity. □

In the sequel, it will be convenient to set $f(+\infty) := b_1$ in (5.1).

*Proof of Theorem 1.3.* — Let $(f_n)$ be a sequence in $\mathscr{H}^p$ that converges weakly to zero. We will let $f_{n,\chi}$ denote the vertical limit function of $f_n$ with respect to the character $\chi$. By assumption (a) of Theorem 1.3, there exists a positive number $A$ such that $|\operatorname{Im}\psi| \leqslant A$. This implies that $|\operatorname{Im}\psi_\chi| \leqslant A$ for any $\chi$ on $\mathbb{T}^\infty$. By the Littlewood–Paley formula applied with $d\mu(t) = \mathbf{1}_{[0,1]}dt$, setting $w = u + iv$, we get

$$\|C_\varphi(f_n)\|_p^p \asymp |C_\varphi f_n(+\infty)|^p + \int_{\mathbb{T}^\infty} \int_0^{+\infty} \int_0^1 u |f_{n,\chi^{c_0}}(\varphi_\chi(w))|^{p-2} |f_{n,\chi^{c_0}}'(\varphi_\chi(w))|^2 |\varphi_\chi'(w)|^2 dv\, du\, dm_\infty(\chi).$$

Our assumption on $f_n$ implies that $|f_n(+\infty)|$ and hence $C_\varphi f_n(+\infty)$ tend to zero. In the innermost integral, we use the non-univalent change of variables $s = \sigma + it = \varphi_\chi(u+iv)$. Observe that, for every $v$ in $[0,1]$ and every $u > 0$, $-A \leqslant \operatorname{Im} s \leqslant A + c_0$, whence

$$\|C_\varphi(f_n)\|_p^p \ll o(1) + \int_{\mathbb{T}^\infty} \int_0^{+\infty} \int_{-A}^{A+c_0} |f_{n,\chi^{c_0}}(s)|^{p-2} |f_{n,\chi^{c_0}}'(s)|^2 \mathcal{N}_{\varphi_\chi}(s) dt\, d\sigma\, dm_\infty(\chi).$$

We now use assumption (b) of Theorem 1.3 in the following way. For any given $\varepsilon > 0$, we let $\theta > 0$ be such that $\mathcal{N}_\varphi(s) \leqslant \varepsilon \operatorname{Re} s$ whenever $\operatorname{Re} s < \theta$. We split the integral over $\mathbb{R}_+$ into $\int_0^\theta + \int_\theta^{+\infty}$. For the first integral, say $I_0 := \int_0^\theta$, we use that $\mathcal{N}_{\varphi_\chi}(s) \leqslant \varepsilon \operatorname{Re} s$ for any $\chi$ on $\mathbb{T}^\infty$ and any $s$ with $\operatorname{Re} s < \theta$ (see [3, Proposition 4]). Using again the Littlewood–Paley formula, we get that there exists some constant $C>0$ such that

$$\begin{aligned} I_0 &= \int_{\mathbb{T}^\infty}\int_0^\theta\int_{-A}^{A+c_0} |f_{n,\chi^{c_0}}(s)|^{p-2}|f'_{n,\chi^{c_0}}(s)|^2\mathcal{N}_{\varphi_\chi}(s)dtd\sigma dm_\infty(\chi)\\ &\leqslant C\varepsilon\|f_n\|_p^p. \end{aligned}$$

For the second integral, say $I_\infty := \int_\theta^\infty$, we observe that $\mathcal{N}_{\varphi_\chi}(s)\leqslant \frac{1}{c_0}\operatorname{Re} s$ (see [2, Proposition 3]), so that

$$\begin{aligned} I_\infty &= \int_{\mathbb{T}^\infty}\int_\theta^{+\infty}\int_{-A}^{A+c_0} |f_{n,\chi^{c_0}}(s)|^{p-2}|f'_{n,\chi^{c_0}}(s)|^2\mathcal{N}_{\varphi_\chi}(s)dtd\sigma dm_\infty(\chi)\\ &\leqslant \frac{1}{c_0}\int_{\mathbb{T}^\infty}\int_\theta^{+\infty}\int_{-A}^{A+c_0} \sigma|f_{n,\chi^{c_0}}(s)|^{p-2}|f'_{n,\chi^{c_0}}(s)|^2 dtd\sigma dm_\infty(\chi)\\ &= \frac{1}{c_0}\int_{\mathbb{T}^\infty}\int_\theta^{+\infty}\int_{-A}^{A+c_0} \sigma|f_{n,\chi}(s)|^{p-2}|f'_{n,\chi}(s)|^2 dtd\sigma dm_\infty(\chi)\\ &\leqslant \frac{1}{c_0}\int_{\mathbb{T}^\infty}\int_{\theta/2}^{+\infty}\int_{-A}^{A+c_0} (\sigma+\theta/2)|f_{n,\chi}(s+\theta/2)|^{p-2}\\ &\qquad\times|f'_{n,\chi}(s+\theta/2)|^2 dtd\sigma dm_\infty(\chi)\\ &\leqslant \frac{2}{c_0}\int_{\mathbb{T}^\infty}\int_{\theta/2}^{+\infty}\int_{-A}^{A+c_0} \sigma|f_{n,\chi}(s+\theta/2)|^{p-2}|f'_{n,\chi}(s+\theta/2)|^2 dtd\sigma dm_\infty(\chi)\\ &\ll \|f_n(\cdot+\theta/2)\|_p^p, \end{aligned}$$

and this last quantity goes to zero since the horizontal translation operator $f(s)\mapsto f(s+\theta/2)$ acts compactly on $\mathscr{H}^p$. □

## 6. Two general lower bounds

We will let $q$ denote the conjugate exponent of $p$. The evaluation $\delta_s$ at $s$ is in $(\mathscr{H}^p)^*$ and, by (4.1), $\|\delta_s\| = [\zeta(2\operatorname{Re} s)]^{1/p}$ when $p$ is any real number $\geqslant 1$. Observe that $\delta_s/\|\delta_s\|$ converges weakly to 0 as $\operatorname{Re} s \xrightarrow{>} 1/2$ and that $C_\varphi^*(\delta_s)=\delta_{\varphi(s)}$, so that a necessary condition for compactness of $C_\varphi:\mathscr{H}^p\to\mathscr{H}^p$ is that

$$\lim_{\operatorname{Re} s\xrightarrow{>}1/2}\frac{\|\delta_{\varphi(s)}\|}{\|\delta_s\|} = \lim_{\operatorname{Re} s\xrightarrow{>}1/2}\left(\frac{\zeta(2\operatorname{Re}\varphi(s))}{\zeta(2\operatorname{Re} s)}\right)^{1/p} = 0.$$

It is therefore not surprising to see the latter quotient appearing in our general estimates for $a_n(C_\varphi)$ in the two theorems given below. These results represent two different ways of obtaining lower bounds for the quantities $a_n(C_\varphi)$ via respectively $\mathscr{H}^\infty$ interpolation and $\mathscr{H}^p$ interpolation.

THEOREM 6.1. — *Suppose that $\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s}$ determines a compact composition operator $C_\varphi$ on $\mathscr{H}^p$, $1 \leqslant p < \infty$. Let $S = (s_j)$ and $S' = (s'_j)$ be finite sets in $\mathbb{C}_{1/2}$, both of of cardinality $n$, such that $\varphi(s'_j) = s_j$ for every $j$. Then we have*

$$a_n(C_\varphi) \geqslant \rho_p n^{-(1/\min(2,p)-1/p)} [M_{\mathscr{H}^\infty}(S)]^{-1} \, \|\mu_{S'}\|_{\mathcal{C},\mathscr{H}^p}^{-1/p} \times \inf_{1\leqslant j\leqslant n} \left( \frac{\zeta(2\operatorname{Re} s_j)}{\zeta(2\operatorname{Re} s'_j)} \right)^{1/p}, \tag{6.1}$$

*where $\rho_p$ is a constant depending only on $p$.*

*Proof.* — As already noted, the transpose $C_\varphi^* : (\mathscr{H}^p)^* \to (\mathscr{H}^p)^*$ still verifies in an obvious way the mapping equation

$$C_\varphi^*(\delta_a) = \delta_{\varphi(a)},$$

which was used extensively in our previous work [26]. We will use the inequality (2.1) in the form

$$a_n(C_\varphi) \geqslant a_n(C_\varphi^*) \geqslant b_n(C_\varphi^*)$$

and minorate the latter quantity. For clarity, we separate the proof into two parts.

### Case 1: $p \geqslant 2$

Let $E$ be the space generated by $\delta_{s'_1}, \ldots, \delta_{s'_n}$. This is an $n$-dimensional space. Let $L = \sum_{j=1}^n \lambda_j \delta_{s'_j}$ be an element in the unit sphere $S_E$ of $E$. If $f$ is in $\mathscr{H}^p$, then the Cauchy-Schwarz inequality in $\mathbb{C}^n$ implies that

$$|L(f)| = \Big| \sum_{j=1}^n \lambda_j f(s'_j) \Big| = \Big| \sum_{j=1}^n \Lambda_j F_j \Big| \leqslant \|\Lambda\|_2 \, \|F\|_2,$$

where we have set

$$\Lambda_j := \lambda_j \|\delta_{s'_j}\| \quad \text{and} \quad F_j := f(s'_j) \|\delta_{s'_j}\|^{-1}$$

as well as

$$\Lambda := (\Lambda_1, \ldots, \Lambda_n) \quad \text{and} \quad F := (F_1, \ldots, F_n).$$

Since $L$ is assumed to be in the unit sphere of $E$, Hölder's inequality now implies that

$$1 \leqslant n^{1/2-1/p} \, \|\Lambda\|_2 (\|\mu_{S'}\|_{\mathcal{C},\mathscr{H}^p})^{1/p}. \tag{6.2}$$

Next we observe that the sequence $\delta_{s_j}$ is unconditional with constant $\leqslant M_{\mathscr{H}^\infty}(S) =: M_S$, i.e.,

$$M_S^{-1}\left\|\sum \omega_j\lambda_j\delta_{s_j}\right\| \leqslant \left\|\sum \lambda_j\delta_{s_j}\right\| \leqslant M_S\left\|\sum \omega_j\lambda_j\delta_{s_j}\right\| \tag{6.3}$$

for any choice of scalars $\lambda_j$ and unimodular scalars $\omega_j$. To see this, we first set

$$\Phi = \sum_{j=1}^n \lambda_j\delta_{s_j},\ \ \Phi_\omega = \sum_{j=1}^n \omega_j\lambda_j\delta_{s_j}.$$

If $h$ in $\mathscr{H}^\infty$ verifies $h(s_j) = \omega_j,\ 1 \leqslant j \leqslant n$, and $\|h\|_\infty \leqslant M_S$, then we see that, for every $f \in \mathscr{H}^p$, $\Phi_\omega(f) = \Phi(hf)$. Since $\mathscr{H}^\infty$ is isometrically equal to the multiplier algebra of $\mathscr{H}^p$, we therefore get that

$$|\Phi_\omega(f)| \leqslant \|\Phi\|\|hf\|_{\mathscr{H}^p} \leqslant \|\Phi\|\|h\|_\infty\|f\|_{\mathscr{H}^p} \leqslant M_S\,\|\Phi\|\|f\|_{\mathscr{H}^p}.$$

This gives the left-hand inequality of (6.3). The right-hand inequality readily follows, replacing $\lambda_j$ by $\lambda_j\omega_j$ and $\omega_j$ by $\overline{\omega_j}$. Averaging with respect to independent choices of $\omega_j$ (Rademacher variables) and using Lemma 3.6 and (3.8), we get from the left-hand side of (6.3), setting $\rho_p = [2T_2(\mathscr{H}^p)]^{-1}$, that

$$\begin{aligned}\|\Phi\| &\geqslant M_S^{-1}\mathbb{E}\left\|\sum_{j=1}^n \omega_j\,\lambda_j\,\delta_{s_j}\right\| \geqslant M_S^{-1}\rho_p\left(\sum_{j=1}^n |\lambda_j|^2\|\delta_{s_j}\|^2\right)^{1/2}\\ &\geqslant M_S^{-1}\rho_p\inf_{1\leqslant j\leqslant n}\frac{\|\delta_{s_j}\|}{\|\delta_{s'_j}\|}\left(\sum_{j=1}^n |\lambda_j|^2\|\delta_{s'_j}\|^2\right)^{1/2}.\end{aligned}$$

By the mapping equation, $C_\varphi^*(L) = \Phi$, and hence we get

$$\|C_\varphi^*(L)\| \geqslant \rho_p\,M_S^{-1}\inf_{1\leqslant j\leqslant n}\left(\frac{\zeta(2\operatorname{Re}s_j)}{\zeta(2\operatorname{Re}s'_j)}\right)^{1/p}\|\Lambda\|_2.$$

Using (6.2), we finally obtain

$$\|C_\varphi^*(L)\| \geqslant \rho_p n^{-(1/2-1/p)}\,M_S^{-1}\|\mu_{S'}\|_{\mathcal{C},\mathscr{H}^p}^{-1/p}\inf_{1\leqslant j\leqslant n}\left(\frac{\zeta(2\operatorname{Re}s_j)}{\zeta(2\operatorname{Re}s'_j)}\right)^{1/p}.$$

This implies the desired result since $b_n(C_\varphi^*) \geqslant \inf_{L\in S_E}\|C_\varphi^*(L)\|$.

**Case 2:** $1 \leqslant p < 2$

We follow word for word the same route, with Hölder instead of Cauchy–Schwarz and $(\mathscr{H}^p)^*$ of cotype $q$ (see Lemma 3.6). In the special case $p = 1$, we have $q = \infty$, but then (6.3) implies

$$\sup_{1\leqslant j\leqslant n} |\lambda_j| \|\delta_{s_j}\| \leqslant M_S \| \sum_j \lambda_j \delta_{s_j} \|$$

so that

$$\inf_{1\leqslant j\leqslant n} \frac{\|\delta_{s_j}\|}{\|\delta_{s'_j}\|} \| \Lambda\|_\infty \leqslant \sup_{1\leqslant j\leqslant n} |\lambda_j| \|\delta_{s_j}\| \leqslant M_S \|C_\varphi^*(L)\|.$$

We thus obtain for all $1 \leqslant p < 2$ the two inequalities

$$1 \quad \leqslant \quad \|\Lambda\|_q (\|\mu_{S'}\|_{\mathcal{C},\mathscr{H}^p})^{1/p},$$

$$\begin{aligned} \|C_\varphi^*(L)\| \quad &\geqslant \quad \rho_p M_S^{-1} \|\Lambda\|_q \inf_{1\leqslant j\leqslant n} \frac{\|\delta_{s_j}\|}{\|\delta_{s'_j}\|} \\ &\geqslant \quad \rho_p M_S^{-1} \, \|\mu_{S'}\|_{\mathcal{C},\mathscr{H}^p}^{-1/p} \inf_{1\leqslant j\leqslant n} \left( \frac{\zeta(2 \operatorname{Re} s_j)}{\zeta(2 \operatorname{Re} s'_j)} \right)^{1/p}, \end{aligned}$$

where $\rho_p = (2T_p(\mathscr{H}^p))^{-1}$ for $1 < p < 2$ and $\rho_1 = 1$. This takes care of the second case and ends the proof of Theorem 6.1. □

We turn to the bound for $a_n(C_\varphi)$ using $\mathscr{H}^p$ interpolation.

THEOREM 6.2. — *Suppose that $\varphi(s) = c_0 s + \sum_{n=1}^\infty c_n n^{-s}$ determines a compact composition operator $C_\varphi$ on $\mathscr{H}^p$. Let $S = (s_j)$ and $S' = (s'_j)$ be finite sets in $\mathbb{C}_{1/2}$, both of of cardinality $n$, such that $\varphi(s'_j) = s_j$ for every $j$. Then we have*

$$a_n(C_\varphi) \quad \geqslant \quad n^{-(1/\min(2,p)-1/p)} [M_{\mathscr{H}^p}(S)]^{-1} \, \|\mu_{S'}\|_{\mathcal{C},\mathscr{H}^p}^{-1/p} \times \inf_{1\leqslant j\leqslant n} \left( \frac{\zeta(2 \operatorname{Re} s_j)}{\zeta(2 \operatorname{Re} s'_j)} \right)^{1/p}. \tag{6.4}$$

*Proof.* — The proof begins like that of Theorem 6.1, using the Bernstein numbers of the transpose of $C_\varphi : \mathscr{H}^p \to \mathscr{H}^p$. We have once again

$$1 \leqslant n^{1/\min(2,p)-1/p} \, \|\Lambda\|_2 (\|\mu_{S'}\|_{\mathcal{C},\mathscr{H}^p})^{1/p}. \tag{6.5}$$

From now on, we no longer appeal to cotype and $\mathscr{H}^\infty$ interpolation, but to $\mathscr{H}^p$ interpolation and a Boas-type lower bound, namely

$$\Big\| \sum_j \lambda_j \delta_{s_j} \Big\| \geqslant \big[M_{\mathscr{H}^p}(S)\big]^{-1} \big( \sum_j |\lambda_j|^q \, \|\delta_{s_j}\|^q \big)^{1/q} \geqslant \big[M_{\mathscr{H}^p}(S)\big]^{-1} \|\Lambda\|_2.$$

Here the latter inequality holds since $q \leqslant 2$ and therefore $\|\Lambda\|_{\ell^q} \geqslant \|\Lambda\|_{\ell^2}$. The first inequality is proved by duality as follows. Write

$$\Big(\sum_j |\lambda_j|^q \|\delta_{s_j}\|^q\Big)^{1/q} = \sum_j c_j \lambda_j \|\delta_{s_j}\|, \quad \text{where} \quad \sum_j |c_j|^p = 1.$$

Observe that $\sum_j \big(|c_j| \|\delta_{s_j}\|\big)^p \|\delta_{s_j}\|^{-p} = 1$ so that $c_j \|\delta_{s_j}\| = f(s_j)$ for some $f \in \mathscr{H}^p$ with norm $\leqslant M_{\mathscr{H}^p}(S)$. We finally get

$$\begin{aligned} \Big(\sum_j |\lambda_j|^q \|\delta_{s_j}\|^q\Big)^{1/q} &= \sum_j \lambda_j\, f(s_j) = \Phi(f) \\ &\leqslant \|f\|_{\mathscr{H}^p} \|\Phi\| \leqslant M_{\mathscr{H}^p}(S) \Big\| \sum_j \lambda_j \delta_{s_j} \Big\|. \end{aligned}$$

Using (6.5) and the bound $\|\Phi\| \geqslant [M_{\mathscr{H}^p}(S)]^{-1} \|\Lambda\|_2$, we conclude the proof in the same way as we did in the proof of Theorem 6.1. □

The difficulty in applying Theorems 6.1 or 6.2 is that it is in general difficult to get good estimates for $M_{\mathscr{H}^\infty}(S)$, $M_{\mathscr{H}^p}(S)$ or $\|\mu_S\|_{\mathcal{C},\mathscr{H}^p}$. We will later see some special cases in which this is in fact possible.

## 7. Proof of the bounds in Theorem 1.2

### Part (a)

We present two proofs: a sketchy one, based on the spectral properties of $C_\varphi$, and a detailed one, based on Theorem 6.1, illustrating the utility of $\mathscr{H}^\infty$ interpolation.

The first approach uses the spectrum $\sigma(C_\varphi)$ of $C_\varphi$ on $\mathscr{H}^p$ described in Theorem 3.7:

$$\sigma(C_\varphi) = \{0\} \cup \Big\{[\varphi'(\alpha)]^k,\ k = 0, 1, \dots\Big\}. \tag{7.1}$$

We can moreover assume that $r_0 = |\varphi'(\alpha)| > 0$, as in Lemma 6.1 of [26]. Finally, (7.1), Theorem 2.2, and a tauberian argument show that

$$a_n(C_\varphi) \gg r_0^{8n}.$$

The second approach goes as follows. Let $\Delta$ be an open disc whose closure is contained in $\mathbb{C}_{1/2}$. Clearly $\varphi(\Delta)$ contains a closed disc $\overline{D}(a, r) := \{s : |s - a| \leqslant r\}$ with $0 < r < 1$. Set

$$S = \{s_j := a + r\omega^j : 1 \leqslant j \leqslant n\}, \text{ where } \omega = e^{2i\pi/n},$$

and let $S' = \{s'_j\}$ be a set of $n$ distinct points from $\Delta$ such that

$$\varphi(s'_j) = s_j, \quad 1 \leqslant j \leqslant n.$$

We introduce the associated Blaschke product

$$B(s) := \prod_{1\leqslant j\leqslant n} \frac{s-s_j}{s+\overline{s_j}-1} = \frac{(s-a)^n - r^n}{(s+\overline{a}-1)^n - r^n}$$

and find, by an elementary computation, that the uniform separation constant

$$\delta(S) = \inf_{1\leqslant j\leqslant n} (2\sigma_j - 1)|B'(s_j)|$$

of $S$ verifies

$$\delta(S) \gg r^n. \tag{7.2}$$

It is known that $M_{H^\infty}(S) \leqslant (2e + 4e|\log\delta(S)|)/\delta(S)$ [15, p. 268], where $M_{H^\infty}(S)$ denotes the constant of interpolation for the space $H^\infty(\mathbb{C}_0)$ of bounded, analytic functions on $\mathbb{C}_0$.

After having made this choice of $S$ and $S'$, we now estimate each of the three terms appearing on the right-hand side of (6.1) of Theorem 6.1 with help of Theorem 4.4. We first claim that

$$M_{\mathscr{H}^\infty}(S) \ll r^{-(\gamma+\varepsilon)n} \tag{7.3}$$

for every $\varepsilon > 0$. Indeed, it follows from (7.2) and the relations between interpolation and uniform separation constants that $M_{H^\infty}(S) \ll [1/\delta(S)]^{1+\varepsilon} \ll r^{-(1+\varepsilon)n}$, and then (4.5) gives the result. We find next that

$$\|\mu_{S'}\|_{\mathcal{C},\mathscr{H}^p} \ll n. \tag{7.4}$$

This is a consequence of Lemma 4.1 since $S' = (s'_j)$ is uniformly bounded and lies far from the boundary $\operatorname{Re} s = 1/2$. Finally, we observe that

$$\inf_{1\leqslant j\leqslant n} \frac{\|\delta_{s_j}\|}{\|\delta_{s'_j}\|} \gg 1. \tag{7.5}$$

This is immediate since $\|\delta_{s_j}\| \geqslant 1$ and $\|\delta_{s'_j}\| = O(1)$ as $S'$ remains far from the boundary when $n$ increases.

Now part (a) of Theorem 1.2 follows from Theorem 6.1 if we put together (7.3), (7.4), and (7.5); taking into account the factors $n$ and $n^{-(1/\min(2,p)-1/p)}$, we observe that we may choose $\delta = r^{\gamma+\varepsilon}$ for an arbitrary $\varepsilon > 0$.

### Part (b): $c_0 = 1$

We first prove (1.1) by applying Theorem 2.1 with $r = 1/(\operatorname{Re} c_1)$. By Theorem 3.7, the left-hand side of (2.3) is infinite. It follows that the right-hand side is infinite as well, whence the result follows. Our proof of (1.1) given above does not lead to a pointwise estimate of $a_n(C_\varphi) =: a_n$. To achieve this, we use (2.4) with $N$ in place of $n$, where $N > 2n$ is an integer to be chosen later. We set $\gamma_1 = \operatorname{Re} c_1$ and use that $\lambda_j(C_\varphi) = j^{-c_1}$ to obtain

$$\begin{aligned} 2^{-\gamma_1} N^{-\gamma_1} = (2N)^{-\gamma_1} &\leqslant e\left(a_1 \cdots a_N\right)^{1/N} \\ &\leqslant e\left(a_1^n\, a_n^{(N-n)}\right)^{1/N} = e\, a_1^{n/N} a_n^{(N-n)/N}. \end{aligned}$$

This implies that

$$a_n \geqslant 2^{-\gamma_1 N/(N-n)}\, e^{-N/(N-n)} a_1^{-n/(N-n)} N^{-\gamma_1 N/(N-n)} \gg N^{-\gamma_1 N/(N-n)},$$

which we now write as

$$a_n \gg N^{-\gamma_1}\, N^{-\gamma_1 n/(N-n)} = N^{-\gamma_1}\, e^{-\gamma_1 n \log N/(N-n)}.$$

Choosing $N$ as the integer part of $n \log n + 2$ and noting that $\frac{n \log N}{N-n} \to 1$, we finally get

$$a_n \gg (n \log n)^{-\gamma_1}$$

as claimed.

It may be observed that the latter argument gives an alternate proof of (1.1).

## Part (c): $c_0 \geqslant 2$

In this case, the spectrum is reduced to $\{0, 1\}$ (see Subsection 3.4), so that the previous proof does not work. We will proceed differently and use Bernstein numbers and a properly chosen $n$-dimensional space $E$. This new argument will in fact work also when $c_0 = 1$ and give an alternate proof for that case.

Our proof is based on the following lemma which exploits the fact that the collection of linear functions constitute an infinite-dimensional subspace of $H^p(\mathbb{T}^\infty)$. In what follows, we let $\Omega(N)$ denote the number of prime factors in $N$ counted with their multiplicity, and we let $P_{c_0}$ denote orthogonal projection from $\mathscr{H}^2$ onto its subspace generated by the basis vectors $N^{-s}$ with $\Omega(N) = c_0$.

Lemma 7.1. — *Fix an integer $n \geqslant 1$. Suppose that $\varphi(s) = c_0 s + \sum_{j=1}^\infty c_j j^{-s}$ is in $\mathscr{G}$ with $c_0 \geqslant 1$. Let $E$ be the $n$-dimensional subspace of $\mathscr{H}^p$ spanned by the unit vectors $p_1^{-s}, p_2^{-s}, \ldots, p_n^{-s}$. Then for every $f(s) = \sum_{k=1}^n b_k p_k^{-s}$ in $E$, we have*

$$P_{c_0} C_\varphi f(s) = \sum_{k=1}^n b_k p_k^{-c_1} (p_k^{c_0})^{-s} \quad \text{and} \quad \|P_{c_0} C_\varphi f\|_{\mathscr{H}^p} \leqslant \|C_\varphi f\|_{\mathscr{H}^p}. \tag{7.6}$$

*Proof.* — We know from [10] that the following formal computation is allowed to determine the Dirichlet coefficients of $C_\varphi(p_k^{-s}), 1 \leqslant k \leqslant n$:

$$\begin{aligned} C_\varphi(p_k^{-s}) &= p_k^{-c_0 s} p_k^{-c_1} \prod_{j=2}^{\infty} (1 + \sum_{l=1}^{\infty} \frac{(-c_j \log p_k)^l}{l!} j^{-ls}) \\ &=: p_k^{-c_0 s} p_k^{-c_1} (1 + \sum_{m \geqslant 2} \alpha_{k,m} m^{-s}), \end{aligned}$$

so that $P_{c_0} C_\varphi f$ can be expressed as stated in (7.6). For the norm estimate, using the Bohr lift, we note that for $h(s) = \sum_{N=1}^{\infty} \beta_N N^{-s}$, the formula

$$Q(h)(s) = (1/2\pi) \int_0^{2\pi} \left( \sum_{N=1}^{\infty} \beta_N \, e^{i\Omega(N)\theta} N^{-s} \right) e^{-ic_0\theta} d\theta$$

defines a norm-one projection from $\mathscr{H}^p$ to its subspace generated by the vectors $N^{-s}$ with $\Omega(N) = c_0$. But this means that

$$\|P_{c_0} C_\varphi f\|_{\mathscr{H}^p} = \|Q C_\varphi f\|_{\mathscr{H}^p} \leqslant \|C_\varphi f\|_{\mathscr{H}^p},$$

which gives the second part of (7.6). □

We are now ready to prove part (c) of Theorem 1.2. Choose $E$ as in Lemma 7.1 and let $f$ be a vector in the unit sphere of $E$. By Lemma 7.1 and the Bohr lift , we get

$$\begin{aligned} \|C_\varphi(f)\|_{\mathscr{H}^p} &\geqslant \left\|P_{c_0} C_\varphi f\right\|_{\mathscr{H}^p} \\ &\geqslant \Big\|\sum_{k=1}^{n} c_k p_k^{-c_1} z_k^{c_0}\Big\|_{H^p(\mathbb{T}^\infty)} = \Big\|\sum_{k=1}^{n} c_k p_k^{-c_1} z_k\Big\|_{H^p(\mathbb{T}^\infty)}, \end{aligned}$$

where we for the last relation used the invariance of the Haar measure $m_\infty$ of $\mathbb{T}^\infty$ under the transformation $(z_j) \mapsto (z_j^{c_0})$. Applying the Khintchin inequality for the Steinhaus variables $z_j$ twice, we get

$$\begin{aligned} \|C_\varphi(f)\|_{\mathscr{H}^p} &\gg \Big\|\sum_{k=1}^{n} c_k p_k^{-c_1} z_k\Big\|_{H^2(\mathbb{T}^\infty)} \geqslant p_n^{-\operatorname{Re} c_1} \Big\|\sum_{k=1}^{n} c_k z_k\Big\|_{H^2(\mathbb{T}^\infty)} \\ &\gg p_n^{-\operatorname{Re} c_1} \Big\|\sum_{k=1}^{n} c_k z_k\Big\|_{H^p(\mathbb{T}^\infty)} = p_n^{-\operatorname{Re} c_1} \|f\|_{\mathscr{H}^p}. \end{aligned}$$

It follows that

$$a_n(C_\varphi) \geqslant b_n(C_\varphi) \gg p_n^{-\operatorname{Re} c_1}.$$

By the Tchebycheff form of the prime number theorem, $p_n \ll n \log n$, and so the desired estimate follows.

## 8. Optimality of the bounds in Theorem 1.2

The bounds (a), (b), (c) of Theorem 1.2 are optimal in view of the following theorem.

THEOREM 8.1. — *Suppose that* $c_0$ *is a nonnegative integer and that* $\varphi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s}$ *generates a bounded composition operator* $C_\varphi$ *on* $\mathscr{H}^p$ *for some* $1 \leqslant p < \infty$.

(a) *If* $c_0 = 0$ *and* $\Omega := \overline{\varphi(\mathbb{C}_0)} \subset \mathbb{C}_{1/2}$ *is compact, then* $a_n(C_\varphi) \ll \delta^n$ *for some* $0 < \delta < 1$.

(b) *If* $c_0 \geqslant 1$, *and if* $\varphi(\mathbb{C}_0) \subset \mathbb{C}_A$ *for some* $A > 0$, *then* $a_n(C_\varphi) \ll n^{-A}$ *if* $p > 1$ *and* $a_n(C_\varphi) \ll (\log n) n^{-A}$ *if* $p = 1$.

*Proof.* — We split the proof into three parts.

### Part (a)

We recall that the $n$th Gelfand number $c_n(C_\varphi)$ of an operator $T$ on $\mathscr{H}^p$ is

$$c_n(T) = \inf_E \|T|_E\|,$$

where $E$ runs over all subspaces of $\mathscr{H}^p$ of codimension $< n$. Let $E_0$ be the subspace of $\mathscr{H}^p$ defined by

$$E_0 = \{f \in \mathscr{H}^p : f(s_0) = f'(s_0) = \cdots = f^{(n-1)}(s_0) = 0\},$$

where $\operatorname{Re} s_0 \geqslant \theta$ and $\theta = \inf_{s\in\Omega} \operatorname{Re} s > 1/2$. This is a subspace of codimension $< n+1$. We will first prove that

$$\|C_\varphi|_{E_0}\|^p \leqslant \sup_{s\in\Omega} |B(s)|^p \zeta(1/2+\theta), \tag{8.1}$$

where $B$ is the “adapted" Blaschke product

$$B(s) = \left(\frac{s - s_0}{s - (1/2+\theta) + \overline{s_0}}\right)^n$$

which is of modulus 1 on the vertical line $\operatorname{Re} s = 1/4 + \theta/2$. We set

$$r := \sup_{s\in\mathbb{C}_\theta} \left|\frac{s - s_0}{s + \overline{s_0} - 1}\right| < 1$$

and $M := \sup_{s\in\Omega} |B(s)| = r^n$.

We now choose an arbitrary $f$ in $E_0$. This $f$ can be written $f = Bh$ with $h$ having the same supremum as $f$ on the vertical line $\operatorname{Re} s = 1/4 + \theta/2$. Using the maximum principle, we have that

$$\sup_{s\in\Omega} |h(s)| \leqslant \sup_{\operatorname{Re} s\geqslant 1/4+\theta/2} |h(s)| = \sup_{\operatorname{Re} s = 1/4+\theta/2} |h(s)| = \sup_{\operatorname{Re} s = 1/4+\theta/2} |f(s)|.$$

We then deduce

$$\begin{aligned}\sup_{s\in\Omega}|f(s)|^p \leqslant \sup_{s\in\Omega}|B(s)|^p \sup_{s\in\Omega}|h(s)|^p &\leqslant M^p \sup_{\operatorname{Re} s=1/4+\theta/2}|f(s)|^p\\ &\leqslant M^p\,\zeta(1/2+\theta)\|f\|^p.\end{aligned}$$

We use the pullback measure $\mu_\varphi$ defined by (3.7) and the set $\Omega := \overline{\varphi(\mathbb{C}_0)}$. Using the lifting identity (3.6) of Theorem 3.5, we then get

$$\|C_\varphi(f)\|^p_{\mathscr{H}^p} = \int_{\mathbb{T}^\infty}|f(\Phi^*(z))|^p dm_\infty(z) = \int_\Omega |f|^p d\mu_\varphi.$$

We infer from this that

$$\begin{aligned}\|C_\varphi(f)\|^p_{\mathscr{H}^p} &= \int_{\mathbb{T}^\infty}|f(\Phi^*(z))|^p dm_\infty(z) = \int_{\Phi^*(z)\in\Omega}|f(\Phi^*(z))|^p dm_\infty(z)\\ &\leqslant \Big[M^p\,\zeta(1/2+\theta)\Big]\|f\|^p_{\mathscr{H}^p},\end{aligned}$$

which gives (8.1). It follows that

$$[c_n(C_\varphi)]^p \leqslant \|C_\varphi|E_0\|^p \leqslant M^p\,\zeta(1/2+\theta).$$

Inequality (2.2), which states that $a_n(C_\varphi)\leqslant 2\sqrt{n}\,c_n(C_\varphi)$, finally gives

$$a_n(C_\varphi)\leqslant 2\sqrt{n}\,r^{n-1}[\zeta(1/2+\theta)]^{1/p}.$$

## Part (b), p>1

We consider first the special case in which $\varphi(s)=s+A$ and $A>0$. This function is seen to belong to $\mathscr{G}$. For a given integer $n\geqslant 2$, let $R$ the $(n-1)$-rank operator defined by

$$Rf := \sum_{j=1}^{n-1} j^{-A}x_j e_j,$$

where $f=\sum_{j=1}^\infty x_j e_j$ and $e_j(s)=j^{-s}$. It follows that $C_\varphi f=\sum_{j=1}^\infty j^{-A}x_j e_j$ and that

$$(C_\varphi - R)f = \sum_{j\geqslant n} j^{-A}x_j\,e_j.$$

Now the contraction principle (3.1) with $\lambda_j = j^{-A}$ gives

$$a_n(C_\varphi)\leqslant \|C_\varphi - R\|\leqslant 2Cn^{-A},$$

which settles our special case.

In the general case, we write $\varphi = T_A\circ\varphi_A$, where

$$\varphi_A(s)=\varphi(s)-A \quad\text{and}\quad T_A(s)=s+A.$$

Since $\varphi_A(\mathbb{C}_0) \subset \mathbb{C}_0$, we see from Theorem 1.1 that $C_{\varphi_A}$ maps $\mathscr{H}^p$ into itself. Now the semi-group property and the ideal property of approximation numbers (see Subsection 2.1), as well as the previous special case, give

$$a_n(C_\varphi) = a_n(C_{\varphi_A} \circ C_{T_A}) \leqslant \|C_{\varphi_A}\| a_n(C_{T_A}) \ll n^{-A}.$$

### Part (b), p=1

It is easy to conclude from Lemma 3.3. Indeed, repeating the proof of the contraction principle for Schauder bases (Lemma 3.2) and using that

$$\sum_{j \geqslant n} \frac{\log j}{j^{A+1}} \ll \frac{\log n}{n^A},$$

we obtain

$$\Big\| \sum_{j \geqslant n} j^{-A} x_j e_j \Big\|_1 \ll \frac{\log n}{n^A} \Big\| \sum_{j \geqslant 1} x_j e_j \Big\|_1.$$

We get $a_n(C_{T_A}) \ll (\log n)/n^A$ for $T_A(s) = s + A$ and conclude as before in the general case $\operatorname{Re}\varphi(s) > A$. □

## 9. A transference principle

In [26], we found a recipe for transferring a general composition operator on $H^2(\mathbb{D})$ to a composition operator on $\mathscr{H}^2$. The point was that, under this transference, decay rates for approximation numbers are preserved or at least not perturbed severely. The same transference makes sense in the $\mathscr{H}^p$ setting, but we succeed only partially in getting similarly precise results as in [26]. We will now present this state of affairs and briefly describe the two basic problems that prevent us from proceeding further.

We begin by describing the recipe from [26]. Given $1 \leqslant p < \infty$, we let $T$ be some conformal map from $\mathbb{D}$ into $\mathbb{C}_{1/2}$, which we will assume has the property that the operator $C_T$ is bounded from $\mathscr{H}^p$ to $H^p(\mathbb{D})$. We introduce the function

$$I(s) := 2^{-s}$$

which we view as an analytic map from $\mathbb{C}_0$ onto $\mathbb{D}\backslash\{0\}$. If $\omega$ is an analytic self-map of $\mathbb{D}$, then we define an analytic map $\varphi : \mathbb{C}_0 \to \mathbb{C}_{1/2}$ by the formula $\varphi := T \circ \omega \circ I$, which implies $C_\varphi = C_I \circ C_\omega \circ C_T$. The Dirichlet series $\varphi$ is then the symbol of a bounded composition operator $C_\varphi$ on $\mathscr{H}^p$ with $c_0 = 0$.

A natural choice is to set $T = T_0$, where

$$T_0(z) := \frac{1}{2} + \frac{1-z}{1+z},$$

so that $T$ maps $\mathbb{D}$ onto $\mathbb{C}_{1/2}$. Unfortunately, this forces us to require $p$ to be an even integer. This constraint comes, as in Theorem 1.1, from the local embedding

$$\sup_{a\in\mathbb{R}} \int_a^{a+1} |f(1/2+it)|^p \, dt \leqslant C\|f\|^p_{\mathscr{H}^p}, \tag{9.1}$$

which is only known to hold when $p$ is an even integer. This result relies on a well-known inequality in analytic number theory [19]. See [30] and also [21] for a thorough discussion of this inequality and its connections with Carleson measures. Assuming that $f$ is in $\mathscr{H}^p$ and using (9.1), we get that

$$\begin{aligned}
\|f\circ T_0\|^p_{H^p(\mathbb{D})} &= \int_{-\pi}^{\pi} |f\big(1/2 + i\tan(t/2)\big)|^p \, \frac{dt}{2\pi} \\
&= \int_{-\infty}^{\infty} |f(1/2+ix)|^p \frac{dx}{\pi(1+x^2)} \\
&= \sum_{k\in\mathbb{Z}} \int_k^{k+1} |f(1/2+ix)|^p \frac{dx}{\pi(1+x^2)} \\
&\ll \sum_{k\in\mathbb{Z}} \frac{1}{k^2+1} \|f\|^p_{\mathscr{H}^p} \ll \|f\|^p_{\mathscr{H}^p}.
\end{aligned}$$

It follows that the composition operator defined by the formula $C_T$ is a bounded operator from $\mathscr{H}^p$ to $H^p(\mathbb{D})$.

For other values of $p$, we may instead choose, for example,

$$T_\varepsilon(z) := \frac{1}{2} + \left(\frac{1-z}{1+z}\right)^{1-\varepsilon}$$

for some $0 < \varepsilon < 1$. Using the pointwise estimates $|f(\sigma+it)| \leqslant [\zeta(2\sigma)]^{1/p}\|f\|_{\mathscr{H}^p}$ along with

$$2\,\mathrm{Re}\, T_\varepsilon(z) - 1 \geqslant (2\sin \pi\,\varepsilon/2) \left|\frac{1-z}{1+z}\right|^{1-\varepsilon},$$

we may compute in a similar way as above to get that

$$\|f\circ T_\varepsilon\|^p_{H^p(\mathbb{D})} \ll \int_{-\infty}^{\infty} \|f\|^p_{\mathscr{H}^p} \max(1, |x|^{\varepsilon-1}) \frac{dx}{(1+x^2)} \ll \|f\|^p_{\mathscr{H}^p}.$$

We are prepared to state our first basic estimate for $C_\varphi$.

Theorem 9.1. — *Let $\omega$ be an analytic self-map of $\mathbb{D}$. Assume that $1 \leqslant p < \infty$ and that $C_T$ is bounded from $\mathscr{H}^p$ into $H^p(\mathbb{D})$, and set $\varphi := T \circ \omega \circ I$. Then*

$$a_n(C_\varphi) \leqslant \|C_T\| a_n(C_\omega).$$

*In particular, $C_\varphi$ is compact whenever $C_\omega$ is compact.*

*Proof.* — By Theorem 3.5, the operator $C_I$, defined by setting $C_I g(s) := g(I(s))$, is an isometry from $H^p(\mathbb{D})$ into $\mathscr{H}^p$. We use the ideal property of approximation numbers and their preservation under left multiplication by isometries to conclude that

$$a_n(C_\varphi) = a_n(C_\omega \circ C_T) \leqslant \|C_T\| a_n(C_\omega).$$

Here $\|C_T\|$ is finite by assumption. □

We would like to have a tight bound on $C_\varphi$ from below as well, but this is harder to achieve. We may adapt Theorem 6.2 to get the following general bound. Here we use the notation $\mu_Z := \sum_{j=1}^n (1-|z_j|^2)\,\delta_{z_j}$ for a sequence $Z = (z_j)$ in the unit disc.

Theorem 9.2. — *Let $\omega$ be an analytic self-map of $\mathbb{D}$ such that $\omega(\mathbb{D})$ has positive distance to $-1$. Assume that $1 \leqslant p < \infty$ and that $C_T$ is bounded from from $\mathscr{H}^p$ into $H^p(\mathbb{D})$, and set $\varphi := T \circ \omega \circ I$ and $\Phi = T \circ \omega$, the Bohr lift of $\varphi$. There exists a positive constant $c$ such that if $Z = (z_j)$ is any finite sequence with both $Z$ and $\omega(Z)$ consisting of $n$ distinct points in $\mathbb{D}$, then*

$$\begin{aligned} a_n(C_\varphi) \quad &\geqslant \quad c\, n^{-(1/\min(2,p)-1/p)}\, [M_{\mathscr{H}^p}(\Phi(Z))]^{-1} \|\mu_Z\|_{\mathcal{C},H^p(\mathbb{D})}^{-1/p} \\ &\qquad \times \inf_{1\leqslant j\leqslant n} \left(\frac{1-|z_j|^2}{1-|\omega(z_j)|^2}\right)^{1/p}. \end{aligned}$$

*Proof.* — Since $\Phi$ is bounded and $|1+\omega(z)| \gg 1$, we have (e.g. in the case $T = T_0$ and setting $s_j = \Phi(z_j)$ as well as $S = \Phi(Z)$)

$$\begin{aligned} \zeta(2\operatorname{Re} s_j) = \zeta(2\operatorname{Re}\Phi(z_j)) \quad &\geqslant \quad \frac{c}{2\operatorname{Re}\Phi(z_j) - 1} = \frac{c}{2}\frac{|1+\omega(z_j)|^2}{(1-|\omega(z_j)|^2)} \\ &\gg \quad (1-|\omega(z_j)|^2)^{-1}. \end{aligned}$$

Using this fact, and following the same reasoning as in the proof of [26, Theorem 9.1], we obtain the result from Theorem 6.2. □

This result is completely analogous to the bound from below in [26, Theorem 9.1], but at present only of interest when $p = 1$ because of (4.4) which says that $M_{\mathscr{H}^1}(S) \leqslant [M_{\mathscr{H}^2}(S)]^2$ for any $\mathscr{H}^2$ interpolating sequence $S$.

We now have what we need to present our leading example and thus prove Theorem 1.4.

*Proof of Theorem 1.4.* — When $\omega$ is a lens map, it is known from [17, Proposition 6.3] that the approximation numbers decay as $e^{-\sqrt{n}}$ to some positive power. By Theorem 9.1, the same upper bound holds for the decay of $a_n(C_\varphi)$ for the transferred operator[(1)] $C_\varphi = C_{T_\varepsilon} \circ C_\omega \circ C_I$ on $\mathscr{H}^p$ for all $p \geqslant 1$. When $p = 1$, we can use Theorem 9.2 to arrive at the bound from below. Indeed, if we take $z_j = 1 - \rho^j$ with $0 < \rho < 1$ and if $\theta$ denotes the parameter of the lens map $\omega$, then simple estimates, using in particular (4.4) and Lemma 4.3, show that

$$\begin{aligned} \|\mu_Z\|_{\mathcal{C},H^1(\mathbb{D})} &\ll e^{b/(1-\rho)}, \\ M_{\mathscr{H}^1}(S) &\ll [M_{\mathscr{H}^2}(S)]^2 \ll [M_{H^2(\mathbb{C}_{1/2})}(S)]^\alpha \ll e^{b/(1-\rho)}, \\ \inf_{1\leqslant j\leqslant n}\left(\frac{1-|z_j|^2}{1-|\omega(z_j)|^2}\right)^{1/p} &\gg \rho^{n(1-\theta)/p}. \end{aligned}$$

We now optimize the choice of $\rho$ by taking $\rho = 1 - 1/\sqrt{n}$, and we get the lower bound in Theorem 1.4 with the help of Theorem 9.2. □

We note that for general $p \neq 1, 2$, we are not able to get any better result from Theorem 9.2 than the general lower bound in part (a) of Theorem 1.2.

We observe the following limitation of our method when $p$ is not an even integer, and thus in particular also in the case $p = 1$. When the approximation numbers of $C_\omega$ decay more slowly than they do when $\omega$ is a lens map, the approximation numbers of $C_{T_\varepsilon} \circ C_\omega \circ C_I$ will still decay as a power of $e^{-\sqrt{n}}$ because of the map $T_\varepsilon$. Substituting $T_\varepsilon$ by a map $T$ which enjoys some smoothness at $z = 1$, we may remedy this situation to some extent. But it is clear that we are unable to obtain precise results when, for example, $(1 - |\omega|)^{-1}$ is non-integrable on $\mathbb{T}$.

We conclude that two rather fundamental open problems remain obstacles for extending the utility of our transference principle:

- Is the embedding inequality (9.1) valid for a continuous range of $p$, for instance all $1 \leqslant p < \infty$?
- What are the bounded interpolating sequences for $\mathscr{H}^p$ when $1 < p < \infty$, $p \neq 2$, and how can the constant of interpolation $M_{\mathscr{H}^p}(S)$ be estimated when the sequence $S$ approaches the vertical line $\operatorname{Re} s = 1/2$?

These questions await further investigation.

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(1) We mention without proof that, for lens maps, the choice $T = T_0$ would work as well.

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