Use of the program FOURIER for steady waves

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Abstract

This document has accompanied the package of programs for several years. In February 2012 both programs and this document received a major upgrade. The home page is http://johndfenton.com/Steady-waves/Fourier.html. This document is http://johndfenton.com/Steady-waves/Instructions.pdf; the package of programs and files is at http://johndfenton.com/Steady-waves/Fourier.zip.

Revision History of Instructions.pdf	
→ February 2012	Series of versions, becoming slowly incompatible with the program package
February 2012 →	Combined and simplified program, this document corrected to agree with the program
March 2012	For the theory the origin of η changed to mean water level (back to Fenton (1988)) and many incorrect references to $2N+5$ variables and equations changed to $2N+10$. All the section on theory in the Appendix has been re-written, including more on numbers of variables and equations.
24 May 2012	Espen Engebretsen noted that it was not clear how the results in the output file Flowfield.res (see Table 6-2) were made dimensionless. This is now shown in the table, and in the program output.

1. Introduction

Throughout coastal and ocean engineering the convenient model of a steadily-progressing periodic wave train is used to give fluid velocities, surface elevations and pressures caused by waves, even in situations where the wave is being slowly modified by effects of viscosity, current, topography and wind or where the wave propagates past a structure with little effect on the wave itself. In these situations the waves do seem to show a surprising coherence of form, and they can be modelled by assuming that they are propagating steadily without change, giving rise to the so-called steady wave problem, which can be uniquely specified and solved in terms of three physical length scales only: water depth, wave length and wave height. In many practical problems it is not the wavelength which is known, but rather the wave period, and in this case, to solve the problem uniquely or to give accurate results for fluid velocities, it is necessary to know the current on which the waves are riding. In practice, the knowledge of the detailed flow structure under the wave is so important that it is usually considered necessary to solve accurately this otherwise idealised model.

The main theories and methods for the steady wave problem which have been used are: Stokes theory, an explicit theory based on an assumption that the waves are not very steep and which is best suited to waves in deeper water; and cnoidal theory, an explicit theory for waves in shallower water. The accuracy of both depends on the waves not being too high. In addition, both have a similar problem, that in the inappropriate limits of shallower water for Stokes theory and deeper water for cnoidal theory, the series become slowly convergent and ultimately do not converge.

An approach which overcomes this is the Fourier approximation method, which does not use series expansions based on a small parameter, but obtains the solution numerically. It could be described as a nonlinear spectral approach, where a series is assumed, each term of which satisfies the field equation, and then the coefficients are found by solving a system of nonlinear equations. This is the basis of the computer program FOURIER. It has been widely used to provide solutions in a number of practical and theoretical applications, providing solutions for fluid velocities and pressures for engineering design. The method provides accurate solutions for waves up to very close to the highest.

A review and comparison of the methods is given in Sobey, Goodwin, Thieke & Westberg (1987) and Fenton (1990).

The aim of this article is to

- present an introduction to the theory so that input data supplied will be satisfactory
- describe the data format required by the program FOURIER
- describe the output files which are produced and how they might be used, including some graph-plotting files, and,
- to describe the basis of the Fourier method and the numerical techniques used.

2. History and critical appraisal

The usual method for periodic waves, suggested by the basic form of the Stokes solution, is to use a Fourier series which is capable of accurately approximating any periodic quantity, provided the coefficients in that series can be found. The analytical solution is obtained by using perturbation expansions for the coefficients in the series and solving linear equations at each order of approximation (Fenton 1985). For high waves, the series have trouble with convergence. A reasonable procedure, then, is to calculate the coefficients numerically by solving the full nonlinear equations. This approach would be expected to be more accurate than either of the perturbation expansion approaches, Stokes and cnoidal theory, because its only approximations would be numerical ones, and not the essential analytical ones of the perturbation methods. Also, increasing the order of approximation would be a relatively trivial numerical matter without the need to perform extra mathematical operations.

This approach originated with Chappelear (1961). He assumed a Fourier series in which each term identically satisfied the field equation throughout the fluid and the boundary condition on the bottom. The values of the Fourier coefficients and other variables for a particular wave were then found by numerical solution of the nonlinear equations obtained by substituting the Fourier series into the nonlinear boundary conditions. He used the velocity potential ϕ for the field variable and instead of using surface elevations directly he used a Fourier series for that too. Dean (1965) instead used the stream function ψ for the field variable and point values of the surface elevations, and obtained a rather simpler set of equations and called his method ”stream function theory”. Rienecker & Fenton (1981) presented a method based exclusively on Fourier approximation, whereas earlier work had used other lower-order numerical methods in part. The nonlinear equations were solved by Newton’s method. The presentation emphasised the importance of knowing the current on which the waves travel if the wave period is specified as a parameter.

Results from these numerical methods show that accurate solutions can be obtained with Fourier series of 10-20 terms, even for waves close to the highest, and they seem to be the best way of solving any steady water wave problem where accuracy is important. Sobey et al. (1987)made a comparison between different versions of the numerical methods. They concluded that there was little to choose between them.

A simpler method and computer program was given by Fenton (1988), where the necessary matrix of partial derivatives is obtained numerically. In application of the method to waves which are high, in common with other versions of the Fourier approximation method (Dalrymple & Solana 1986), it was found that it is sometimes necessary to solve a sequence of lower waves, extrapolating forward in height steps until the desired height is reached. For very long waves all these methods can occasionally converge to the wrong solution, that of a wave one third of the length, which is obvious from the Fourier coefficients which result, as only every third is non-zero. This problem can be avoided by using a sequence of height steps.

It is possible to obtain nonlinear solutions for waves on shear flows for special cases of the vorticity distribution. For waves on a constant shear flow, Dalrymple (1974*a*), and a bi-linear shear distribution (Dalrymple 1974*b*) used a Fourier method based on the approach of Dean (1965). The ambiguity caused by the specification of wave period without current seems to have been ignored, however.

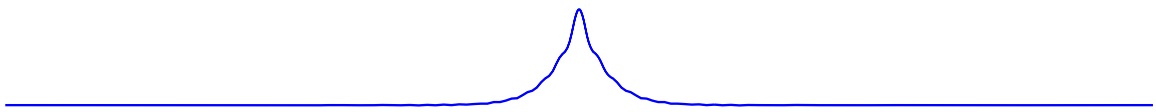

Figure 2-1. Free surface for a wave of length $L/d = 50$ and a height of $H/d = 0.786$, 98% of the maximum height possible for that length. There were $N = 70$ terms in the Fourier series, and the highest wave was computed from a sequence of 20 waves, using initial solutions extrapolated from two previous solutions.

The present Fourier approach breaks down in the limit of very long and high waves, when the spectrum of coefficients becomes broad-banded and many terms have to be taken, as the Fourier approximation has to approximate both the short rapidly-varying crest region and the long trough where very little changes, *and* approaching the highest wave, the sharp crest. For example figure 2-1 shows results for the surface profile using the Fourier program for a wave of length $L/d = 50$ and a height of $H/d = 0.786$, 98% of the maximum height possible for that length. It can be seen with the very long wave and the crest approaching sharp makes the program have to work very hard indeed, but it has obtained a solution. Fenton (1995) developed a numerical cnoidal theory so that long waves could be treated without difficulty, however for wavelengths as long as 50 times the depth, the Fourier method provides good solutions.

3. The physical problem

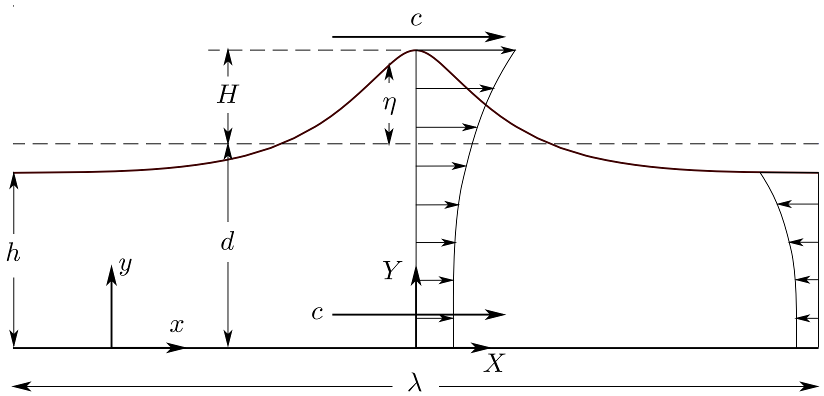

Figure 3-1. One wave of a steady train, showing principal dimensions, co-ordinates and velocities

The problem considered is that of two-dimensional periodic waves propagating without change of form over a layer of fluid on a horizontal bed, as shown in Figure 3-1. A co-ordinate system (x, y) has its origin on the bed, and waves pass through this frame with a velocity c in the positive x direction. It is this stationary frame which is the usual one of interest for engineering and geophysical applications. Consider also a frame of reference (X, Y) moving with the waves at velocity c, such that $x = X + ct$, where t is time, and $y = Y$. It is easier to solve the problem in this moving frame in which all motion is steady and then to compute the unsteady velocities. If the fluid velocity in the (x, y) frame is (u, v), and that in the (X, Y) frame is (U, V), the velocities are related by $u = U + c$ and $v = V$.

4. The program FOURIER.EXE

In my previous package, up until February 2012, there were actually two programs – one ran the first to solve the wave problem, then the second to obtain solutions for the wave profile, velocities, and accelerations. This can still be accessed *via* the package at http://johndfenton.com/Steady-waves/OldFourier.zip.

In February 2012 I have streamlined and unified the programs, and have corrected this document so that it more accurately describes what the program does. All the files necessary can be found at http://johndfenton.com/Steady-waves/Fourier.zip. The executable program is FOURIER.EXE.

5. Input data

It is well-known that a steadily-progressing periodic wave train is uniquely specified by three length scales, the water depth d, the wave height H, and the wavelength λ, or, in terms of only two dimensionless quantities involving these, such as dimensionless wave height H/d and dimensionless wavelength λ/d. The program allows for the specification of these, however in many practical situations it is not the wavelength which is known, but the wave period τ. If this is the case, it is not enough to uniquely specify the wave problem, as if there is a current, any current, then the period will be Doppler-shifted. Hence, it is necessary also to specify the current in such cases. The value of this current will also affect the horizontal velocity components, and users of the program should be aware of this and if it is unknown, some maximum and minimum values might be tried and their effects evaluated.

All input data are to be specified in terms non-dimensionalised with respect to gravitational acceleration g and mean depth d.

There are three files necessary, which should be in the same directory as FOURIER.EXE:

5.1 DATA.DAT

The wave data is of the form as given in the first column of Table 5-1. Any other information, such as that in column 2, can be placed after that on each line, such as we have done here, to label each line. A blank is also allowed.

Test wave (A title line to identify each wave)	
0.5	H/d
Wavelength	Measure of length: "Wavelength" or "Period"
10.	Value of that length: λ/d or $T\sqrt{g/d}$ respectively
1	Current criterion (1 or 2)
0.	Current magnitude, $\bar{u}_1/\sqrt{gd}$ or $\bar{u}_2/\sqrt{gd}$
20	Number of Fourier components
1	Number of height steps to reach H/d
Any number of other wave data can be placed here, each occupying 8 lines as above	
FINISH	Must be used to tell the program to stop - the file can continue after this

Table 5-1. Form of data to be supplied for each wave

Here we describe the nature of each element of the input data.

5.1.1 Description

A line containing any identifier or description of the wave, up to 100 characters

5.1.2 Wave height

The relative wave height H/d is specified. There is a formula for the maximum wave height H_m/d for a particular wavelength λ/d, given as equation (A-1) in Appendix A. In many problems, where it is the period that is specified it is not possible to calculate the highest possible wave height *a priori*. The program, after it solves a wave, prints out the theoretical maximum H_m/d for the calculated wave length. The user could then reconsider the value of H/d to specify.

5.1.3 Wavelength or Period

If "Wavelength" is chosen, then a value of λ/d is then specified in the next line; if "Period" then a value of dimensionless period $\tau\sqrt{g/d}$ is to be given.

5.1.4 Current

This is described in more detail in Appendix B.2 below. There are actually two definitions of currents, the first, identified by 1 here is the ”Eulerian mean current”, the time-mean horizontal fluid velocity at any point denoted by $\bar{u}_1$, the mean current which a stationary meter would measure. In irrotational flow this is constant everywhere. A second type of mean current is the depth-integrated mean current, the ”mass-transport velocity”, which we denote by $\bar{u}_2$. If there is no mass transport, such as in a closed wave tank, $\bar{u}_2 = 0$. Usually the overall physical problem will impose a certain value of current on the wave field, thus determining the wave speed. To apply the methods of this theory if wave period rather than length is known, to obtain a unique solution it is necessary to specify both the nature (1 or 2) and magnitude of that current. If the current is unknown, any horizontal velocity components calculated are approximate only.

5.1.5 Number of Fourier components

This is the primary computational parameter in the program, which we denote by N. The program now has no limit (previously it was $N = 32$), but for many problems, $N = 10$ is enough – results show that accurate solutions can be obtained with Fourier series of 10-20 terms, even for waves close to the highest, although for longer and higher waves it may be necessary to increase N. The adequacy of the particular value of N used can be monitored by examining the output file SOLUTION.RES, where the spectra of Fourier coefficients obtained as part of the solution is presented, the B_j which are at the core of the method, as presented in equation (B-5) for $j = 1, \ldots, N$, and the Fourier coefficients of the computed free surface, the E_j as presented in equation (C-9). The B_j decay rather more rapidly than do the E_j. The value of E_N should be sufficiently small (less than 10^{-4} say) that there would be no identifiable high-frequency wave apparent on the surface plotted from the solution (*cf.* figure 2-1 above).

5.1.6 Number of height steps

In application of the method to waves which are high and long, in common with other versions of the Fourier approximation method, the Fourier method may converge to a wave of 1/3 of the wavelength (Dalrymple & Solana 1986, with comments by Fenton and Sobey noted in the References), but this can be remedied by solving for lower waves of the same length and stepping upwards in height (Fenton 1988). This occurrence of this phenomenon is made obvious from the Fourier coefficients which result, as only every third is non-zero. The present program overcomes this by solving a sequence of lower waves, extrapolating forward in height steps until the desired height is reached. For waves up to about half the highest $H \approx H_m/2$ it is not necessary to do this, and a value of 1 in the eighth line of the data file is all right, but thereafter it is better to take 2 or more height steps. For waves very close to H_m for a given length it might be necessary to take as many as 20. The evidence as to whether enough have been taken is provided by the spectrum, as noted above.

5.2 CONVERGENCE.DAT

This is a three-line file which controls convergence of the iteration procedure, for example:

```
Control file to control convergence and output of results
20      Maximum number of iterations for each height step; 10 OK for ordinary waves, 40 for highest
1.e-4   Criterion for convergence, typically 1.e-4, or 1.e-5 for highest waves
```

5.3 POINTS.DAT

This controls how much information is to be printed out afterwards to show the velocity and acceleration fields. For example:

Control output (for graph plotting *etc.*)
50 M, Number of points on free surface (the program clusters them near crest)
8 Number of velocity/acceration profiles over half a wavelength to print out, including $x = 0$ and $x = \lambda/2$.
20 Number of vertical points in each profile, including points at bottom and surface.

6. Output files

The program produces output to the screen showing how the process of convergence is working. Three files are produced:

6.1 SOLUTION.RES

After a heading block, including the theoretical highest wave for this length of wave, the program prints out the global parameters of the wave train, where all quantities are shown first non-dimensionalised with respect to ρ, g and k, where $k = 2\pi/\lambda$ is the wavenumber of the wave, and then the value non-dimensionalised with respect to g and depth d. The results are:

Quantity	Dimensionless w.r.t.		Reference	
	k	d	This document	Fenton (1988)
Water depth	kd	$d/d = 1$		
Wave length	$k\lambda = 2\pi$	λ/d		
Wave height	kH	H/d		
Wave period	$\tau\sqrt{gk}$	$\tau\sqrt{g/d}$		
Wave speed	$c\sqrt{k/g}$	$c/\sqrt{gd}$		
Eulerian current	$\bar{u}_1\sqrt{k/g}$	$\bar{u}_1/\sqrt{gd}$	(B-13)	Symbol c_E, p358
Stokes current	$\bar{u}_2\sqrt{k/g}$	$\bar{u}_2/\sqrt{gd}$	(B-14)	Symbol c_S, p359
Mean fluid speed	$\bar{U}\sqrt{k/g}$	$\bar{U}/\sqrt{gd}$	(B-5)	
Wave volume flux, $q = \bar{U}d - Q$	$q\sqrt{k^3/g}$	$q/\sqrt{gd^3}$		p359
Bernoulli constant, $r = R - gd$	rk/g	r/gd		p360
Volume flux	$Q\sqrt{k^3/g}$	$Q/\sqrt{gd^3}$	(B-3)	p360
Bernoulli constant	Rk/g	R/gd	(B-4)	p360
Momentum flux	$Sk^2/\rho g$	$S/\rho g d^2$		p362
Impulse	$I\sqrt{k^3}/\rho\sqrt{g}$	$I/\rho\sqrt{gd^3}$		p362
Kinetic energy	$Tk^2/\rho g$	$T/\rho g d^2$		p362
Potential energy	$Vk^2/\rho g$	$V/\rho g d^2$		p362
Mean square of bed velocity	$u_b^2 k/g$	u_b^2/gd		p362
Radiation stress	$S_{xx}k^2/\rho g$	$S_{xx}/\rho g d^2$		p362
Wave power	$Fk^{5/2}/\rho g^{3/2}$	$F/\rho g^{3/2} d^{5/2}$		p362
Fourier coefficients (dimensionless)	B_j	E_j	(B-5,C-5)	p360, p362
	B_1	E_1		
	$\ldots$	$\ldots$		
	B_N	E_N		

Table 6-1. Quantities printed out at the head of file SOLUTION.RES

Following the global parameters shown in table 6-1, the spectra of the velocity potential coefficients B_j and the surface elevation coefficients E_j are given, for $j = 1, \ldots, N$, the two corresponding coefficients on each row. These spectra should be checked, as suggested above, to ensure that the coefficients have become small enough that the solution has converged satisfactorily, and that it has not converged to one which is $1/3$ of the wavelength.

6.2 SURFACE.RES

This file contains co-ordinates of points on the surface, all given non-dimensionally with respect to the water depth d. It contains three columns, the first $X_i/d = (i/(M/2))^2 \lambda/d/2$ for $i = -M/2.. + M/2$, where M is the number of surface points defined in the description of file POINTS.DAT above. The points extend over a range trough-crest-trough, and are clustered quadratically near the crest for plotting purposes. The second column is the free surface elevation (total water depth) η_i/d. The third column contains a check on calculations, the computed value of the pressure on the surface $p_i/\rho gd$ from equation (C-10), which should be zero, and they are indeed very close to zero, being typically of the order of the last surface Fourier component E_N.

6.3 FLOWFIELD.RES

This contains a number of profiles of velocity components and time derivatives, the number of profiles and the number of points in each profile determined by file POINTS.DAT. For a sequence of (here equi-spaced) X/d values between 0 (crest) and $\lambda/d/2$ (trough), and then for each for a number of y/d from 0 (the bed) to the local free surface elevation η/d, quantities output on each line are shown in table 6-2. Note that all are dimensionless with respect to g and d, the mean depth.

$\frac{y}{d}$	$\frac{u}{\sqrt{gd}}$	$\frac{v}{\sqrt{gd}}$	$\frac{\partial\phi/\partial t}{gd}$	$\frac{\partial u/\partial t}{g}$	$\frac{\partial v/\partial t}{g}$	$\frac{\partial u}{\partial x}\sqrt{\frac{d}{g}} = -\frac{\partial v}{\partial y}\sqrt{\frac{d}{g}}$	$\frac{\partial u}{\partial y}\sqrt{\frac{d}{g}} = \frac{\partial v}{\partial x}\sqrt{\frac{d}{g}}$	Bernoulli check, equation C-12

Table 6-2. Line of output in file FLOWFIELD.RES

6.4 Graphical output

The package includes a file that enables the plotting of data from the results files. When run with the Gnuplot program (http://www.gnuplot.info/), file FIGURES.PLT (which uses file SETOUTPUT.PLT) produces three figures as shown here in figure 6-1.

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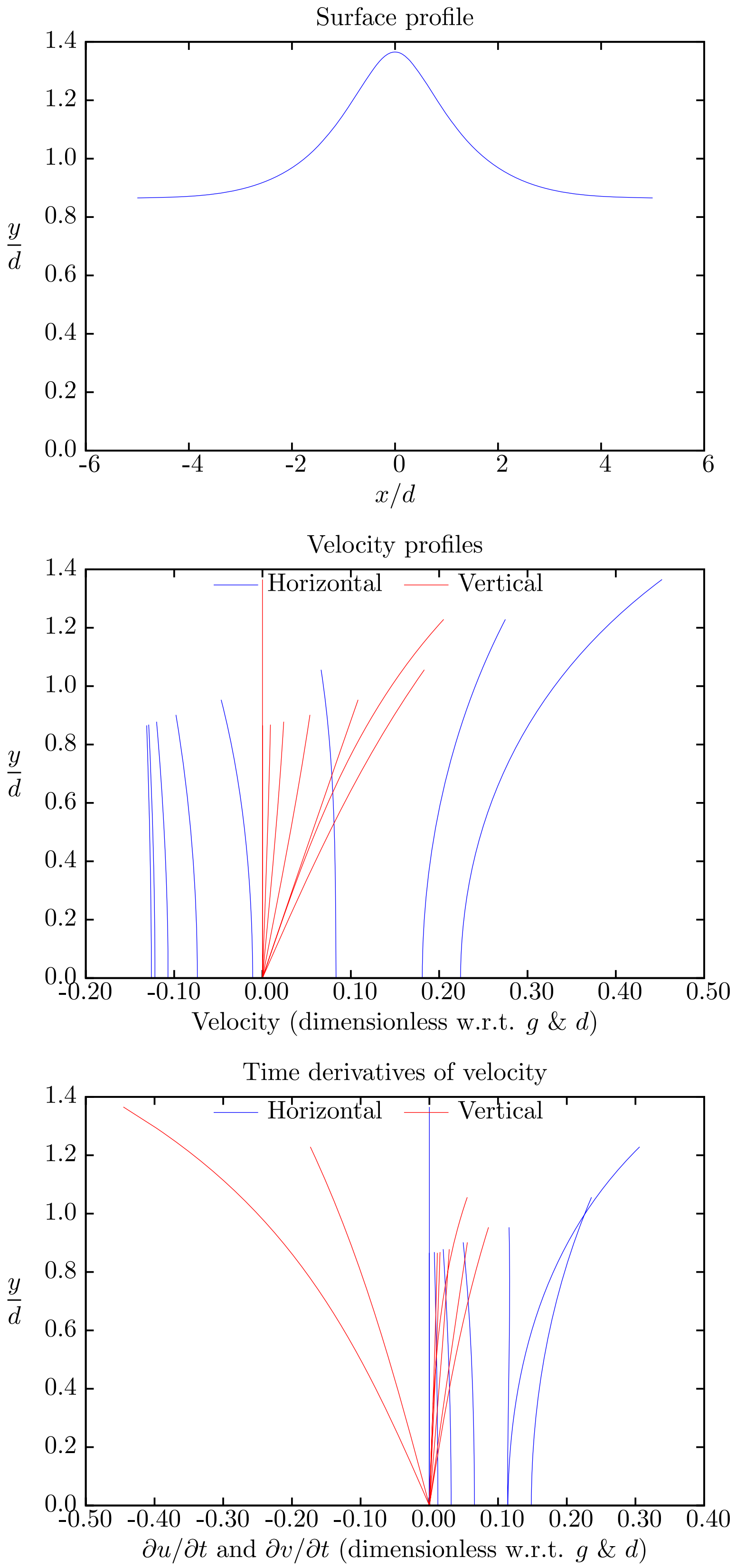

Figure 6-1. Figures obtained by Gnuplot from output files produced by the program for a wave of height $H/d = 0.5$ and length $\lambda/d = 10$

302(1466), 139–188. http://www.jstor.org/stable/36960

Appendix A. Maximum wave height possible for a given length

The range over which periodic solutions for waves can occur is given in Figure A-1, which shows limits to the existence of waves determined by computational studies. The highest waves possible, $H = H_m$, are shown by the thick line, which is the approximation to the results of Williams (1981), presented as equation (32) in Fenton (1990):

$$\frac{H_m}{d} = \frac{0.141063\,\frac{\lambda}{d} + 0.0095721\left(\frac{\lambda}{d}\right)^2 + 0.0077829\left(\frac{\lambda}{d}\right)^3}{1 + 0.0788340\,\frac{\lambda}{d} + 0.0317567\left(\frac{\lambda}{d}\right)^2 + 0.0093407\left(\frac{\lambda}{d}\right)^3}. \tag{A-1}$$

Nelson (1987 and 1994), has shown from many experiments in laboratories and the field, that the maximum wave height achievable in practice is actually only $H_m/d = 0.55$. Further evidence for this conclusion is provided by the results of Le Méhauté, Divoky & Lin (1968), whose maximum wave height tested was $H/d = 0.548$, described as ”just below breaking”. It seems that there may be enough instabilities at work that real waves propagating over a flat bed cannot approach the theoretical limit given by equation (A-1).

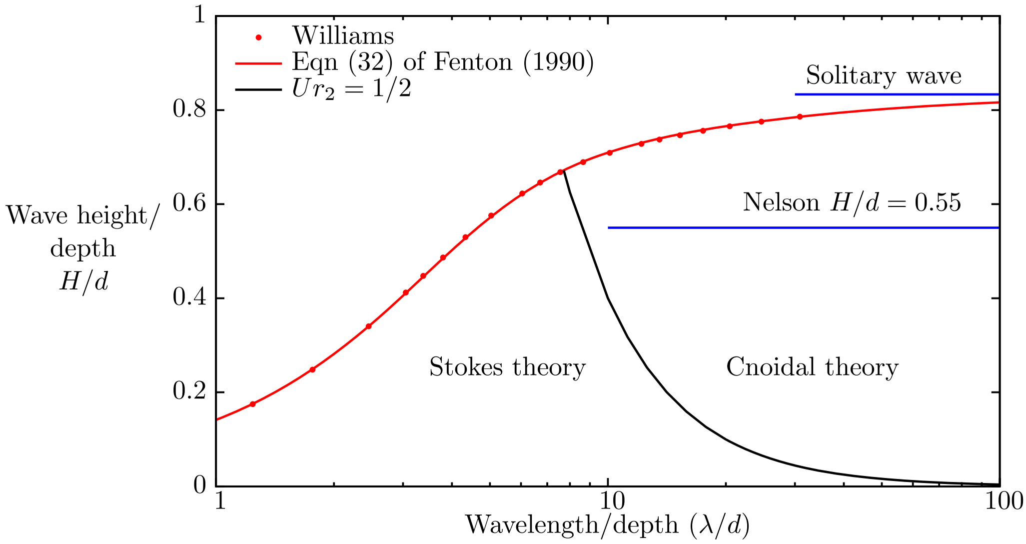

Figure A-1. The region of possible steady waves, showing the theoretical highest waves (Williams) and the fitted equation (1), and the highest long waves in the field (Nelson).

Also shown on the figure, although not so important for applications of the FOURIER program is the boundary between regions where Stokes and cnoidal theories can be applied, as suggested by Hedges (1995):

$$\mathbf{U} = \frac{H\lambda^2}{d^3} = 40, \tag{A-2}$$

where $\mathbf{U}$ is the Ursell number. The FOURIER program can be used over almost the whole region of possible waves, up to within 2% of the boundary given by equation (A-1).

Appendix B. Theory

Here we present an outline of the theory. If the fluid is incompressible, in two dimensions a stream function $\psi(X,Y)$ exists such that the velocity components are given by

$$U = \partial\psi/\partial Y, \quad \text{and} \quad V = -\partial\psi/\partial X.$$

If motion is irrotational, then $\nabla \times \mathbf{u} = \mathbf{0}$ and it follows that ψ satisfies Laplace's equation throughout the fluid:

$$\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} = 0. \tag{B-1}$$

The kinematic boundary conditions to be satisfied are

$$\psi(X, 0) = 0 \quad \text{on the bottom, and} \tag{B-2}$$
$$\psi(X, d + \eta(X)) = -Q \quad \text{on the free surface,} \tag{B-3}$$

where $Y = d + \eta(X)$ on the free surface and Q is a positive constant denoting the volume rate of flow per unit length normal to the flow underneath the stationary wave in the (X, Y) co-ordinates. In these co-ordinates the apparent flow is in the negative X direction. The dynamic boundary condition to be satisfied is that pressure is zero on the surface so that Bernoulli's equation becomes

$$\frac{1}{2}\left(\left(\frac{\partial \psi}{\partial X}\right)^2 + \left(\frac{\partial \psi}{\partial Y}\right)^2\right) + g\,(d + \eta) = R \quad \text{on the free surface,} \tag{B-4}$$

where R is a constant.

The basis of the method is to write the analytical solution for ψ in separated variables form

$$\psi(X, Y) = -\bar{U}\,Y + \sqrt{\frac{g}{k^3}} \sum_{j=1}^{N} B_j \frac{\sinh jkY}{\cosh jkd} \cos jkX, \tag{B-5}$$

where $\bar{U}$ is the mean fluid speed on any horizontal line underneath the stationary waves, the minus sign showing that in this frame the apparent dominant flow is in the negative x direction. The $B_{1,} \ldots, B_N$ are dimensionless constants for a particular wave, and N is a finite integer. The truncation of the series for finite N is the only mathematical or numerical approximation in this formulation. The quantity k is the wavenumber $k = 2\pi/\lambda$ where λ is the wavelength, which may or may not be known initially, and d is the mean depth as shown on Figure 3-1. Each term of this expression satisfies the field equation (B-1) and the bottom boundary condition (B-2) identically. The use of the denominator $\cosh jkd$ is such that for large j the B_j do not have to decay exponentially, thereby making solution rather more robust. For points on the free surface, where $Y = d + \eta$, for large jkd

$$\frac{\sinh jk\,(d + \eta)}{\cosh jkd} \sim e^{jk\eta},$$

not nearly as large as the numerator and denominator would be.

If one were proceeding to an analytical solution, the coefficients B_j would be found by using a perturbation expansion in wave height. Here they are found numerically by satisfying the two nonlinear equations (B-3) and (B-4) from the surface boundary conditions. Substituting $Y = d + \eta(X)$, into the surface equations and introducing the variables $q = \bar{U}d - Q$, the volume flux due to the waves which is actually a positive quantity, and $r = R - gd$, the energy per unit mass with datum at the mean water level, and dividing through to make the equations dimensionless:

$$\sum_{j=1}^{N} B_j \left[\frac{\sinh jk\,(d + \eta)}{\cosh jkd}\right] \cos jkX - \bar{U}\sqrt{k/g}\,k\eta - q\sqrt{\frac{k^3}{g}} = 0, \quad \text{and} \tag{B-6}$$

$$\frac{1}{2}\left(-\bar{U}\sqrt{k/g} + \sum_{j=1}^{N} jB_j \left[\frac{\cosh jk\,(d + \eta)}{\cosh jkd}\right] \cos jkX\right)^2 + \frac{1}{2}\left(\sum_{j=1}^{N} jB_j \left[\frac{\sinh jk\,(d + \eta)}{\cosh jkd}\right] \sin jkX\right)^2 + k\eta - rk/g = 0, \tag{B-7}$$

both to be satisfied for all x. In both equations, we will never evaluate the terms in square brackets as they are written, for both numerators and denominators can become very large. Instead we re-write them and evaluate them in the forms

$$C_j(kd, k\eta) = \frac{\cosh jk\,(d+\eta)}{\cosh jkd} = \cosh jk\eta + \tanh jkd\,\sinh jk\eta, \tag{B-8a}$$

$$S_j(kd, k\eta) = \frac{\sinh jk\,(d+\eta)}{\cosh jkd} = \sinh jk\eta + \tanh jkd\,\cosh jk\eta. \tag{B-8b}$$

The two functions of $jk\eta$ here are much smaller than $\cosh jk\,(d+\eta)$, while the $\tanh jkd$ function is not a problem, as it simply goes to 1 for large arguments.

To solve the problem numerically these two equations are to be satisfied at a sufficient number of discrete points so that we have enough equations for solution. If we evaluate the equations at $N+1$ discrete points over one half wave from the crest to the trough for $m = 0, 1, \ldots, N$, such that $X_m = m\lambda/2N$ and $kX_m = m\pi/N$, and where $\eta_m = \eta(X_m)$, then (B-6) and (B-7) provide $2N+2$ nonlinear equations in the $2N+5$ dimensionless variables: $k\eta_m$ for $m = 0, 1, \ldots, N$; B_j for $j = 1, 2, \ldots, N$; $\bar{U}\sqrt{k/g}$; kd; $q\sqrt{k^3/g}$; and rk/g. We now consider more equations and variables.

An extra equation is the expression requiring that the mean of the dimensionless depths $k\eta_m$ be zero, simply using the trapezoidal rule:

$$\frac{1}{N}\left(\frac{1}{2}\left(k\eta_0 + k\eta_N\right) + \sum_{m=1}^{N-1} k\eta_m\right) = 0. \tag{B-9}$$

For quantities which are periodic such as here, the trapezoidal rule is very much more accurate than usually believed. It can be shown that the error is of the order of the last (Nth) coefficient of the Fourier series of the function being integrated. As that is essentially the approximation used throughout this work, where it is assumed that the series can be truncated at a finite value of N, this is in keeping with the overall accuracy.

In practice the physical dimensions of mean water depth d and wave height H are known giving a numerical value of H/d for which an equation can be provided connecting the crest and trough heights $k\eta_0$ and $k\eta_N$ respectively: $H = \eta_0 - \eta_N$,which we write in terms of our dimensionless variables as

$$k\eta_0 - k\eta_N - kd\,\frac{H}{d} = 0. \tag{B-10}$$

B.1 Specification of wavelength

In some problems we know the wavelength λ and so we have a numerical value for kd, that we write here as an equation

$$kd - 2\pi\frac{d}{\lambda} = 0, \tag{B-11}$$

and the problem is now closed.

B.2 Specification of wave period and current

In many problems it is not the wavelength λ which is known but the wave period τ as measured in a stationary frame. The two are connected by the simple relationship

$$c = \frac{\lambda}{\tau}, \tag{B-12}$$

where c is the wave speed, however it is not known *a priori*, and in fact depends on the current on which the waves are travelling. In the frame travelling with the waves at velocity c the mean horizontal fluid velocity at any level is $-\bar{U}$, hence in the stationary frame the time-mean horizontal fluid velocity at any point denoted by $\bar{u}_1$, the mean current which a stationary meter would measure, is given by

$$\bar{u}_1 = c - \bar{U}. \tag{B-13}$$

In the special case of no mean current at any point, $\bar{u}_1 = 0$ and $c = \bar{U}$, which is Stokes' first approximation to the wave speed, usually incorrectly referred to as his "first definition of wave speed", and is that relative to a frame in which the current is zero. Most wave theories have presented an expression for $\bar{U}$, obtained from its definition as a mean fluid speed. It has often been referred to, incorrectly, as "the wave speed".

A second type of mean fluid speed or current is the depth-integrated mean speed of the fluid under the waves in the frame in which motion is steady. If Q is the volume flow rate per unit span underneath the waves in the (X, Y) frame, the depth-averaged mean fluid velocity is $-Q/d$, where d is the mean depth. In the physical (x, y) frame, the depth-averaged mean fluid velocity, the "mass-transport velocity", is $\bar{u}_2$, given by

$$\bar{u}_2 = c - Q/d. \tag{B-14}$$

If there is no mass transport, such as in a closed wave tank, $\bar{u}_2 = 0$, and Stokes' second approximation to the wave speed is obtained: $c = Q/d$. In general, neither of Stokes' first or second approximations is the actual wave speed, and the waves can travel at any speed. Usually the overall physical problem will impose a certain value of current on the wave field, thus determining the wave speed. To apply the methods of this section, where wave period is known, to obtain a unique solution it is also necessary to specify the magnitude and nature of that current.

Appendix C. Program details

C.1 Initial solution

We calculate the initial values from linear wave theory, assuming zero current. The well-known solution for angular frequency $\sigma = 2\pi/\tau$ in terms of kd is

$$\sigma^2 = gk \tanh kd. \tag{C-1}$$

If the wave period and hence σ is known, it is necessary to solve for kd. The equation could be solved using standard methods for solution of a single nonlinear equation, however Fenton & McKee (1990) have given an approximate explicit solution:

$$kd \approx \frac{\sigma^2 d}{g} \left(\coth \left(\sigma \sqrt{d/g} \right)^{3/2} \right)^{2/3}. \tag{C-2}$$

This expression is an exact solution of (C-1) in the limits of long and short waves, and between those limits its greatest error is 1.5%. Such accuracy is adequate for the present approximate purposes. Having solved for kd linear theory can be applied for an assumption of zero current in relating wavelength and period. We set $q = 0$, $r = \bar{U}^2/2$ (noting that it would have been nicer if Fenton (1988) had defined r as being solely due to the wave motion $r = R - gd - \bar{U}^2/2$, so that it would be zero in the small wave limit). We assume $\eta = \frac{1}{2} H \cos kx$ and substitute into the surface equations with B_1 non-zero and all higher terms in the series zero. The kinematic boundary condition (B-6) gives, taking the terms in the square brackets to be unity, thereby linearising:

$$B_1 \tanh kd \cos kX - \bar{U} \sqrt{k/g}\, \frac{kH}{2} \cos kx = 0, \tag{C-3}$$

and repeating for the dynamic boundary condition (B-7), expanding the terms in brackets and considering only linear terms:

$$-\bar{U} \sqrt{k/g} B_1 \cos kX + \frac{kH}{2} \cos kx = 0, \tag{C-4}$$

with solutions $\bar{U}\sqrt{k/g} = \sqrt{\tanh kd}$, that we could have written down from equation (C-1), making the zero current aproximation $\bar{U} = c$. What is more useful here is the other solution obtained from the pair of equations, $B_1 = kH/2/\sqrt{\tanh kd}$. Hence we have the linear solution, and in view of the common occurrence, we use the symbol $C_0 = \sqrt{\tanh kd}$ borrowed from Fenton (1985).

$$\begin{aligned}
k\eta_m &= \frac{1}{2} kH \cos \frac{m\pi}{N}, \quad \text{for } m = 0, \ldots, N, \\
\bar{U}\sqrt{k/g} &= c\sqrt{k/g} = C_0, \\
B_1 &= \frac{1}{2} \frac{kH}{C_0}, \quad B_j = 0 \text{ for } j = 2, \ldots, N, \\
q &= 0, \\
rk/g &= \tfrac{1}{2} \frac{\bar{U}^2 k}{g} = \tfrac{1}{2} C_0^2.
\end{aligned}$$

For the currents, we use $\bar{u}_1$ or $\bar{u}_2$, if we have a value. Otherwise we assume zero.

C.2 Dimensionless variables

Here we set out and number all the variables above, made dimensionless with respect to ρ, g, and wavenumber k, and in the last column add the initial linear solution. Once an initial value of kd has been calculated, all other quantities can be calculated sequentially.

Variable reference number j	Dimensionless variable z_j	Physical quantity	Initial value from linear theory
1	kd	Depth	Known kd or eqn (C-2)
2	kH	Wave height	$kd \times (H/d)$
3	$\tau\sqrt{gk}$	Period	$2\pi/C_0$
4	$c\sqrt{k/g}$	Wave speed	C_0
5	$\bar{u}_1\sqrt{k/g}$	Mean Eulerian current	0 or $\sqrt{kH} \times \bar{u}_1/\sqrt{gH}$
6	$\bar{u}_2\sqrt{k/g}$	Mean Stokes current	$\sqrt{kH} \times \bar{u}_2/\sqrt{gH}$ or 0
7	$\bar{U}\sqrt{k/g}$	Mean fluid speed in frame of wave	C_0
8	$q\sqrt{k^3/g}$	Discharge due to waves	0
9	rk/g	Energy due to waves	$\frac{1}{2}C_0$
$10..N+10$	$k\eta_m$, $m = 0..N$	$N+1$ surface elevations	$\frac{1}{2}kH\cos\frac{m\pi}{N}$
$N+11..2N+10$	B_m, $m = 1..N$	N Fourier coefficients	$B_1 = \frac{1}{2}kH/C_0$, $B_2 = 0$, $\ldots$

Table C-1. List of dimensionless variables to be determined

C.3 Equations

In all the following it is assumed that values of H and d are known, plus a value of either λ or values of τ and either $\bar{u}_1$ or $\bar{u}_2$.

Equation 1 – Wave height in terms of H/d

$$kH - kd \times (H/d) = 0$$

Equation 2 – Wave height in terms of wavelength or period, whichever is known

$$kH - (H/\lambda) \times 2\pi = 0 \quad \text{or}$$
$$kH - \left(H/g\tau^2\right) \times \left(\tau\sqrt{gk}\right)^2 = 0.$$

Equation 3 – Definition of wave speed $c = \lambda/\tau$, equation (B-12) in dimensionless terms,

$$c\sqrt{k/g} \times \tau\sqrt{gk} - 2\pi = 0.$$

Equation 4 – Mean Eulerian current, equation (B-13)

$$\bar{u}_1\sqrt{k/g} + \bar{U}\sqrt{k/g} - c\sqrt{k/g} = 0.$$

Equation 5 – Mean mass-transport current, equation (B-14) converted to use q

$$\bar{u}_2\sqrt{k/g} + \bar{U}\sqrt{k/g} - c\sqrt{k/g} - \frac{q\sqrt{k^3/g}}{kd} = 0.$$

Equation 6 – From known or assumed numerical value of one current or the other $\bar{u}_1/\sqrt{gH}$ or $\bar{u}_2/\sqrt{gH}$

$$\bar{u}_m\sqrt{k/g} - \frac{\bar{u}_m}{\sqrt{gH}} \times \sqrt{kH} = 0, \quad \text{for } m = 1 \text{ or } 2.$$

Equation 7 – Mean value of η is zero, equation (B-9)

$$\frac{1}{2}\left(k\eta_0 + k\eta_N\right) + \sum_{m=1}^{N-1} k\eta_m = 0.$$

Equation 8 – Definition of wave height, equation (B-10)

$$k\eta_0 - k\eta_N - kH = 0.$$

Equations 9 to $N+9$ – Kinematic free surface boundary condition (B-6): using equation (B-8b) and with $kX_m = m\pi/N$ for $m = 0..N$:

$$\sum_{j=1}^{N} B_j S_j(kd, k\eta_m)\cos\frac{jm\pi}{N} - \bar{U}\sqrt{k/g}\,k\eta_m - q\sqrt{\frac{k^3}{g}} = 0.$$

Equations $N+10$ to $2N+10$ – Dynamic free surface boundary condition (B-7): using equations (B-8) and with $kX_m = m\pi/N$ for $m = 0..N$:

$$\frac{1}{2}\left(-\bar{U}\sqrt{k/g} + \sum_{j=1}^{N} jB_jC_j(kd, k\eta_m)\cos\frac{jm\pi}{N}\right)^2 + \frac{1}{2}\left(\sum_{j=1}^{N} jB_jS_j(kd, k\eta_m)\sin\frac{jm\pi}{N}\right)^2 + k\eta_m - rk/g = 0.$$

C.4 Enumeration of variables and equations

From Table C-1 it can be seen that there are $2N+10$ variables and here we have written out $2N+10$ equations. Some formulations of the problem (*e.g.* Dean, 1965) allow more surface collocation points and the equations are solved in a least-squares sense. This is a good idea and in general would be thought to be desirable, but in practice seems not to make much difference, and here the procedure of Rienecker & Fenton (1981) and Fenton (1988) is followed, where the same numbers of equations as unknowns is used. In the computer program the numbering of variables follows that of Fenton (1988)

C.5 Computational method

The system of nonlinear equations can be iteratively solved using Newton's method. If we write the system of equations as

$$F_i(\mathbf{x}) = 0, \quad \text{for } i = 1, \ldots, 2N+10,$$

where F_i represents equation i and $\mathbf{x} = \{x_j, j = 1, \ldots, 2N+10\}$, the vector of variables x_j (there should be no confusion with that same symbol as a space variable), then if we have an approximate solution $\mathbf{x}^{(n)}$ after n iterations, writing a multi-dimensional Taylor expansion for the left side of equation i obtained by varying each of the $x_j^{(n)}$ by some increment $\delta x_j^{(n)}$:

$$F_i\left(\mathbf{x}^{(n+1)}\right) \approx F_i\left(\mathbf{x}^{(n)}\right) + \sum_{j=1}^{2N+5}\left(\frac{\partial F_i}{\partial x_j}\right)^{(n)} \delta x_j^{(n)}.$$

If we choose the $\delta x_j^{(n)}$ such that the equations would be satisfied by this procedure such that $F_i\left(\mathbf{x}^{(n+1)}\right) = 0$, then we have the set of linear equations for the $\delta x_j^{(n)}$:

$$\sum_{j=1}^{2N+5}\left(\frac{\partial F_i}{\partial x_j}\right)^{(n)}\delta x_j^{(n)} = -F_i\left(\mathbf{x}^{(n)}\right) \quad \text{for } i = 1,\ldots,2N+10,$$

which is a set of equations linear in the unknowns $\delta x_j^{(n)}$ and can be solved by standard methods for systems of linear equations. Having solved for the increments, the updated values of all the variables are then computed for $x_j^{(n+1)} = x_j^{(n)} + \delta x_j^{(n)}$ for all the j. As the original system is nonlinear, this will in general not yet be the required solution and the procedure is repeated until it is.

It is possible to obtain the array of derivatives of every equation with respect to every variable, $\partial F_i/\partial x_j$ by performing the analytical differentiations, however as done in Fenton (1988), it is rather simpler to obtain them numerically. That is, if variable x_j is changed by an amount ε_j, then on numerical evaluation of equation i before and after the increment (after which it is reset to its initial value), we have the numerical derivative

$$\frac{\partial F_i}{\partial x_j} \approx \frac{F(x_1,\ldots,x_j+\varepsilon_j,\ldots,x_{2N+5}) - F(x_1,\ldots,x_j,\ldots,x_{2N+5})}{\varepsilon_j}.$$

The complete array is found by repeating this for each of the $2N+10$ equations for each of the $2N+10$ variables. Compared with the solution procedure, which is $O(N^3)$, this is not a problem, and gives a rather simpler program.

C.6 Post-processing to obtain quantities for practical use

Once the solution has been obtained, quantities rather more useful for physical calculations can be evaluated, notably surface elevations and velocities and accelerations.

It can be shown from (B-5) and the Cauchy-Riemann equations

$$\frac{\partial \Phi}{\partial X} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \psi}{\partial X},$$

where Φ is the velocity potential in the frame moving with the wave, and $X = x - ct$, such that

$$\Phi(X,y) = -\bar{U}\,X + \sqrt{\frac{g}{k^3}}\sum_{j=1}^{N} B_j \frac{\cosh jky}{\cosh jkd}\sin jkX.$$

In the physical frame, the now unsteady velocity potential $\phi(x,y,t)$ is written

$$\phi(x,y,t) = \Phi(x-ct,y) + c\,(x-ct)$$

such that the horizontal velocities in the two systems are related by

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \Phi}{\partial x} + c = U + c,$$

which result would have been obtained by just adding cx to Φ, but it is slightly simpler to have ϕ expressed also as a function of $x - ct$. The additional function of time does not affect the dynamics, it will merely affect the manner in which we subsequently write the unsteady Bernoulli equation. Now we have

$$\phi(x,y,t) = \left(c-\bar{U}\right)(x-ct) + \sqrt{\frac{g}{k^3}}\sum_{j=1}^{N} B_j \frac{\cosh jky}{\cosh jkd}\sin jk\,(x-ct)\,. \tag{C-5}$$

The velocity components anywhere in the fluid are given by $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$:

$$u = c - \bar{U} + \sqrt{\frac{g}{k}}\sum_{j=1}^{N} jB_j \frac{\cosh jky}{\cosh jkd}\cos jk\,(x-ct)\,, \tag{C-6}$$

$$v = \sqrt{\frac{g}{k}}\sum_{j=1}^{N} jB_j \frac{\sinh jky}{\cosh jkd}\sin jk\,(x-ct)\,, \tag{C-7}$$

and as ϕ is a function of $x - ct$ we have simply

$$\frac{\partial \phi}{\partial t} = -c\frac{\partial \phi}{\partial x} = -cu. \tag{C-8}$$

Acceleration components can be obtained simply from these expressions by differentiation, and from the Cauchy-Riemann equations, and are given by.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -c \times \frac{\partial u}{\partial x}, \quad \text{where} \quad \frac{\partial u}{\partial x} = -\sqrt{gk}\sum_{j=1}^{N} j^2 B_j \frac{\cosh jky}{\cosh jkd} \sin jk\left(x - ct\right), \\
\frac{\partial v}{\partial t} &= -c \times \frac{\partial v}{\partial x}, \quad \text{where} \quad \frac{\partial v}{\partial x} = \sqrt{gk}\sum_{j=1}^{N} j^2 B_j \frac{\sinh jky}{\cosh jkd} \cos jk\left(x - ct\right), \\
\frac{\partial u}{\partial y} &= \frac{\partial v}{\partial x}, \\
\frac{\partial v}{\partial y} &= -\frac{\partial u}{\partial x}.
\end{aligned}$$

The total material accelerations of a fluid particle are then

$$\begin{aligned}
\frac{Du}{Dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}, \quad \text{and} \\
\frac{Dv}{Dt} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}.
\end{aligned}$$

The free surface elevation at an arbitrary point requires another step, as we only have it at discrete points $k\eta_m$, obtained as part of the solution. We take the cosine transform of the $N + 1$ surface elevations:

$$E_j = {\sum_{m=0}^{N}}'' k\eta_m \cos\frac{jm\pi}{N} \quad \text{for } j = 1, \ldots, N,$$

where Σ'' means that it is a trapezoidal-type summation, with factors of 1/2 multiplying the first and last contributions. The cosine transform could be performed using fast Fourier methods, but as N is not large, simple evaluation of the series is reasonable. The interpolating cosine series for the surface elevation is then

$$\eta(x, t) = \frac{2}{N}{\sum_{j=0}^{N}}'' E_j \cos jk\left(x - ct\right), \tag{C-9}$$

which can be evaluated for any x and t.

The pressure at any point can be evaluated using Bernoulli's theorem, but most simply in the form from the steady flow, but using the velocities as computed from (C-6) and (C-7):

$$\frac{p}{\rho} = R - gy - \frac{1}{2}\left((u - c)^2 + v^2\right). \tag{C-10}$$

We consider how this relates to the unsteady Bernoulli equation

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gy + \frac{1}{2}\left(u^2 + v^2\right) = f(t), \tag{C-11}$$

where $f(t)$ is an arbitrary function of time, determined by boundary conditions. Substituting equation (C-10) for pressure into this gives

$$\frac{\partial \phi}{\partial t} + R - f(t) + cu - \frac{1}{2}c^2 = 0.$$

From equation advect we have $\partial\phi/\partial t + cu = 0$, giving the expression for f which is, in fact, a constant:

$$f = R - \frac{1}{2}c^2,$$

so we have the unsteady Bernoulli equation in the form

$$\frac{\partial \phi}{\partial t}+\frac{p}{\rho}+gy+\frac{1}{2}\left(u^2+v^2\right)-\left(R-\frac{1}{2}c^2\right)=0. \tag{C-12}$$

As a partial check on the subroutine POINT included in the C++ program, where $\partial\phi/\partial t$, velocities u and v (and accelerations) are calculated, it also calculates the value of the left side of this equation.