ABSTRACT

After a review of the deficiencies of the usual equations of motion for an oscillating ship, two new representations are given. One makes use of the impulse response function and depends only upon the system being linear. The response is given as a convolution integral over the past history of the exciting force with the impulse response function appearing as the kernel. The second representation is based upon a hydrodynamic study, and new forms for the equations of motion are exhibited. The equations resemble the usual equations, with the addition of convolution integrals over the past history of the velocity. However, the coefficients in these new equations are independent of frequency, as are the kernel functions in the convolution integrals. Both representations are quite general and apply to transient motions as well as periodic. The relations between the two representations are given. The treatment considers six degrees of freedom, with linear coupling between the various modes.

The Impulse Response Function and Ship Motions

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Introduction

Just over a decade ago, Weinblum and St. Denis[1]) presented a comprehensive review of the state of knowledge at the end of what we may call the "classical" period in research on seakeeping. Soon after, St. Denis and Pierson[2]) opened the "modern" period (some would prefer to call it the "statistical" period). The studies of the former period were primarily concerned with sinusoidal responses to sinusoidal waves, but the introduction of spectral techniques opened the door for the discussion of responses to random waves, both long and short crested. The construction of the spectral theory on regular wave theory as a foundation delighted us all, as it presented an apparent justification for the admittedly artificial studies of the "classical" period.

The activity during this last decade has been spectacular, with five major and many minor facilities for seakeeping research being opened. Hundreds of models have been tested, many full scale trials have been run, and there has even been some real growth in our knowledge of the subject. In particular, the spectral tool has been sharpened and tempered by the empiricists, and the analysts have made important advances with the rather frightful boundary value problem. In fact, we have all been forging ahead so rapidly that we appear to have forgotten that we are wearing a shoe which doesn't quite fit. The occasional pain from a misplaced toe is ignored in our general enthusiasm for progress.

The "shoe" to which I refer is our mathematical model, the forced representation of the ship response by a system of second order differential equations. The shoe is squeezed on, with no regard for the shape of the foot. The inadequacy of the shoe is evident in the distortions it must take if it is to be worn at all. I am referring, of course, to the frequency dependent coefficients which permit the mathematical model to fit the physical model (if the excitation is purely sinusoidal, that is).

But what happens when we don't have a well defined frequency? The mathematical model becomes almost meaningless. True, a Fourier analysis of the exciting force (or encountered wave) permits the model to be retained, but physical reality is almost lost in the infinity of equations required to represent the motion.

Let us consider this mathematical model briefly, and restrict ourselves to a single degree of freedom. To be completely fair, let us consider a pure, sinusoidal oscillation. The forcing function (if the system is linear) will be sinusoidal, and can be broken into two components, one in phase with the displacement and one 90° out of phase. We further divide the in-phase component into a restoring force, proportional to the displacement, and a remainder. The latter we call the inertial force, and treat it as if it were proportional to the instantaneous acceleration. The out-of-phase component, which provides all the damping, we treat as if it were proportional to the instantaneous velocity.

We can now write an equation, which has the appearance of a differential equation, relating these various quantities:

$$a(\omega)\,\ddot{x} + b(\omega)\,\dot{x} + c(\omega)\,x = F_0 \sin(\omega t + \varepsilon)\,.$$

But a differential equation is supposed to relate the instantaneous values of the functions involved. If the periodic motion continues, this condition is satisfied. Of course, it could just as well be satisfied by the equation

$$b\dot{x} + (c - a\omega^2)\,x = f(t)$$

or more generally

$$(a + d)\,\ddot{x} + b\dot{x} + (c + d\omega^2)\,x = f(t)$$

where d is arbitrary. These are all equally valid models. One of them is to be preferred only if it truly relates the displacement and its first and second derivatives to the excitation in some more general way. But suppose f (t) were to be suddenly doubled. Would the instantaneous acceleration be given by

$$\ddot{x} = \frac{2\,f(t) - b(\omega)\,\dot{x} - c(\omega)\,x}{a(\omega)}\ ?$$

In general, no! Or suppose the amplitude of the oscillation to be suddenly increased. Would the out of phase component of f (t), immediately after the change, be equal to $b\dot{x}$? Again, in general, no. Thus, at best, b (ω) must be considered as a sort of "apparent" damping coefficient, a (ω) as an "apparent" apparent mass, and the physical significance of both is obscure. When the oscillation consists of several coupled modes, the so-called coupling coefficients are equally confused and confusing.

If we restrict ourselves to a phenomenological investigation of how a given ship behaves in a given wave system, these difficulties do not concern us. We simply measure responses to known waves. Most of the work over the past decade has been of this nature, and much of it has been excellent. However, sooner or later, we are required to consider not "what" but "why," and a more analytical technique is demanded. The phenomenological study can tell us the effect of a change in ship loading on seakeeping qualities only after we have measured it; there is no basis for quantitative prediction given the results for one gyradius. And the effect of a change in form is presented as an isolated result, unrelated and unrelatable to the geometric parameters involved. We are driven to the use of the model discussed above in an attempt to clarify the relation of cause and effect. But such a poor mirror of reality is of little value, and in fact can do much harm.

I am not the first to raise this issue. The difficulties are well known and a number of writers have discussed them. In particular, Tick[3]) has vigorously argued against our usual practice and has proposed a model which is very close to the one which will be exhibited here. His case is based solely upon the general characteristics of linear systems, while we shall take ad-

1) References are listed at the end of the paper

vantage of the principles of hydrodynamics to tie the model to the phenomena. More recently, Davis[4]) has proposed a rational approach from the point of view of statistics. This is suggestive, particularly since it was the spectral theory of statistics which first gave weight to the investigation of responses to periodic waves.

Briefly, the specific objectives of this paper are:

1. To exhibit a model which permits the representation of the response of a ship (in six degree of freedom) to an arbitrary forcing function (with excitation in all six modes). The model will not involve frequency dependent parameters.
2. To separate the various factors governing the response into clearly identifiable units, the effect of each to be separately determinable. Thus the effect of gyradius will be separable from added mass. The added mass will be related only to inertial forces and moments. The nature of the damping force will be exhibited. The effect of coupling will be derivable and the effect of "tuning" upon coupling will be determinable.

In this paper we shall not consider the complementary problem of the relation of the exciting force to the incident wave system. This problem is equally basic, and when it has been adequately treated, we will begin to have a satisfactory framework for the interpretation of our empirical studies.

The Impulse Response Function

The basic tool which will be used in this study is an elementary one, widely used in other fields and well known to all engineers: the impulse response function. It is difficult to understand its neglect in our field. Perhaps as Tick suggests, it is because waves look sinusoidal.

For any stable linear system, if R (t), the response to a unit impulse, is known, then the response of the system to an arbitrary force f (t) is

$$x(t) = \int_{-\infty}^{t} R(t-\tau)\, f(\tau)\, d\tau$$

or [1]

$$x(t) = \int_{0}^{\infty} R(\tau)\, f(t-\tau)\, d\tau\,.$$

The only assumption required (aside from convergence) is linearity. In the present context this is, of course, a very strong assumption, and the purists will argue that it implies a thin ship or the equivalent. However all experimental data indicate that the assumption is a good working approximation for small to moderate oscillations of real ship forms. We shall hypothesize that the assumption holds absolutely.

Let x_i, $(i = 1, \ldots, 6)$ be displacements in the six modes of response:

x_1 = surge (positive forward)
x_2 = sway (positive to port)
x_3 = heave (positive upward)
x_4 = roll (positive, deck to starboard)
x_5 = pitch (positive, bow downward)
x_6 = yaw (positive, bow to port)

Let $R_{ij}(t)$ be the response in mode j to a unit impulse at $t = 0$ in mode i. Note that $R_{ij}(\infty)$ does not necessarily equal zero, though in a damped system which is not unstable, it will ordinarily be finite. In modes without a restoring force (sway, surge, and yaw), the impulse response will asymptotically approach some value. For other modes, $R_{ij}(\infty) = 0$.

If the $\{f_i(t)\}$ are an arbitrary set for forcing functions, the corresponding responses are

$$x_j(t) = \sum_{i=1}^{6} \int_{0}^{\infty} R_{ij}(\tau) f_i(t-\tau)\, d\tau\,. \qquad [2]$$

Thus, the matrix $\{R_{ij}(t)\}$ completely characterizes the response of the ship to an arbitrary excitation.

Before we go on, let us consider the relation of these functions to the usual coefficients. First consider the case where the modes are uncoupled. Let

$$f_i(t) = F_i \cos(\omega t + \varepsilon_i) \qquad [3]$$

where ε_i is a phase angle whose value will be assigned later.

$$\begin{aligned} x_i(t) &= F_i \int_0^\infty R_{ii}(\tau) \cos[\omega(t-\tau) + \varepsilon_i]\, d\tau \\ &= F_i \Big[\cos(\omega t + \varepsilon_i) \int_0^\infty R_{ii} \cos\omega\tau d\tau \\ &\quad + \sin(\omega t + \varepsilon_i) \int_0^\infty R_{ii} \sin\omega\tau d\tau\Big] \\ &= F_i\,[R_{ii}{}^c(\omega)\cos(\omega t + \varepsilon_i) + R_{ii}{}^s(\omega)\sin(\omega t + \varepsilon_i)] \end{aligned} \qquad [4]$$

where

$$R_{ii}{}^c(\omega) = \int_0^\infty R_{ii}(\tau) \cos\omega\tau d\tau \qquad [5a]$$

$$R_{ii}{}^s(\omega) = \int_0^\infty R_{ii}(\tau) \sin\omega\tau d\tau \qquad [5b]$$

are the Fourier cosine and sine transforms of $R_{ii}(t)$. We shall call these transforms the frequency response functions. We make the further reduction

$$\begin{aligned} x_i(t) = F_i\,[&(R_{ii}{}^c \cos\varepsilon_i + R_{ii}{}^s \sin\varepsilon_i)\cos\omega t \\ &+ (R_{ii}{}^s \cos\varepsilon_i - R_{ii}{}^s \cos\varepsilon_i)\sin\omega t]\,. \end{aligned}$$

Taking $\tan\varepsilon_i = R_{ii}{}^s / R_{ii}{}^c$ [6]

we have $x_i(t) = F_i\,[(R_{ii}{}^s)^2 + R_{ii}{}^c)^2]^{1/2} \cos\omega t\,.$ [7]

Also

$$f_i(t) = \frac{F_i\,(R_{ii}{}^c \cos\omega t - R_{ii}{}^s \sin\omega t)}{[(R_{ii}{}^s)^2 + (R_{ii}{}^c)^2]^{1/2}} \qquad [8]$$

Now consider the usual representation

$$a_i \ddot{x}_i + b_i \dot{x}_i + c_i x_i = f_i(t)\,. \qquad [9]$$

Using the x_i and f_i from [7] and [8], it is easily seen that

$$a_i = 1/\omega^2 \left[c_i - \frac{R_{ii}{}^c}{(R_{ii}{}^c)^2 + (R_{ii}{}^s)^2}\right] \qquad [10a]$$

$$b_i = \frac{R_{ii}{}^s}{\omega\,[(R_{ii}{}^c)^2 + (R_{ii}{}^s)^2]} \qquad [10b]$$

A more useful relationship is obtained by setting $\varepsilon_i = 0$ in [4]:

$$\frac{x_i(t)}{F_i} = R_{ii}{}^c(\omega)\cos\omega t + R_{ii}{}^s(\omega)\sin\omega t\,. \qquad [11]$$

Thus $R_{ii}{}^c$ and $R_{ii}{}^s$ are the amplitudes of the in-phase and out-of-phase components of the response to a unit amplitude forcing function of frequency ω. The impulse response function is related to these functions by

$$\begin{aligned} R_{ii}(\tau) &= \frac{2}{\pi} \int_0^\infty R_{ii}{}^c(\omega)\cos\omega\tau\, d\omega \\ &= \frac{2}{\pi} \int_0^\infty R_{ii}{}^s(\omega)\sin\omega\tau\, d\omega \end{aligned} \qquad [12]$$

using the Fourier inversion formulas. Note that $R_{ii}{}^c$ and $R_{ii}{}^s$ are uniquely related. If one is known, then by [12] and [5], the other is determined.

Equation [11] can also be written

$$\frac{x_i(t)}{F_i} = [(R_{ii}{}^c)^2 + (R_{ii}{}^s)^2]^{1/2} \cos[\omega t - \varepsilon_i(\omega)] \qquad [13]$$

where

$$\tan\varepsilon = \frac{R_{ii}{}^s(\omega)}{R_{ii}{}^c(\omega)}\,. \qquad [14]$$

Thus, the response follows the excitation by the phase $\tan^{-1}(R_{ii}{}^s / R_{ii}{}^c)$ and has the amplitude $[(R_{ii}{}^s)^2 + (R_{ii}{}^c)^2]^{1/2}$.

The response for a given frequency, as determined by the pair of functions R_{ii}^s, R_{ii}^c, or alternatively, the pair $[(R_{ii}^s)^2 + (R_{ii}^c)^2]^{1/2}$, $\tan^{-1}(R_{ii}^s/R_{ii}^c)$, is a mapping in the frequency domain of the unit response function, which is defined in the time domain. As equations [4] and [11] permit us to pass from either domain to the other, the two representations are completely equivalent. Viewed in this way, the frequency response function is a meaningful, useful concept. It is only when we try to attribute a deeper meaning to it, by imbedding it in a false time domain model, that we create confusion.

Now consider the more general, coupled system, with excitations in a single mode of the same form as given in equation [3]. Then

$$x_j(t) = F_i[R_{ij}^c \cos(\omega t + \varepsilon_i) + R_{ij}^s \sin(\omega t + \varepsilon_i)]. \quad [15]$$

If we consider the usual representation

$$\sum_{j=1}^{6} (a_{jk}\ddot{x}_j + b_{jk}\dot{x}_j + c_{jk}x_j) = f_k(t)$$

where $f_k(t) = 0$ for $k \neq i$, we can develop a system of equations in the unknowns, a_{jk}, b_{jk}. (The c_{jk} are assumed known from static measurements.) All 72 of these unknowns are present, in principle, except where modes are uncoupled. To determine them, it is necessary to consider the responses to excitations in each of the modes separately. We then have enough equations, if we separate the in-phase and out-of-phase components, to determine the coefficients. We have no need for them here, so we defer further discussion until we face a closely related problem. It is only significant to note that they can, in principle, be determined from the set of impulse response functions, and therefore they contain no information which is not derivable from these functions.

Setting $\varepsilon_i = 0$ in [15], we have the system

$$\frac{x_j(t)}{F_i} = R_{ij}^c \cos\omega t + R_{ij}^s \sin\omega t. \quad [16]$$

Thus, R_{ij}^c and R_{ij}^s are the amplitudes of the in-phase and out-of-phase responses in the j^{th} mode to unit amplitude excitation in the i^{th} mode. As before,

$$R_{ij}(t) = \frac{2}{\pi}\int_0^\infty R_{ij}^c \cos\omega t \, d\omega \quad [17a]$$

$$= \frac{2}{\pi}\int_0^\infty R_{ij}^s \sin\omega t \, d\omega \quad [17b]$$

and

$$\frac{x_j(t)}{F_i} = [(R_{ij}^c)^2 + (R_{ij}^s)^2]^{1/2} \cos(\omega t - \varepsilon_j) \quad [18]$$

where

$$\tan\varepsilon_j = R_{ij}^s / R_{ij}^c. \quad [19]$$

We have passed over the question of convergence of the integrals in equations [2] and [5]. Consistent with our hypothesis of linearity, we shall assume $|f_i(t)|$ is bounded. There will then be no difficulty unless $\int_0^\infty |R_{ij}(\tau)|\, d\tau$ does not exist. Unfortunately, in three modes there are no restoring forces (or else they are negative), and evidently some care is needed in treating these cases. A negative restoring force implies an unstable system, which would be beyond the scope of this analysis. However, the case in which R_{ij} approaches some non-zero but finite limit can be treated. The divergence of the integrals can be overcome if we arbitrarily assign a value to $x_j(0)$. We formally write

$$x_j(t) = \int_{-\infty}^{t} R_{ij}(t-\tau) f_i(\tau)\, d\tau - \int_{-\infty}^{0} R_{ij}(-\tau) f_i(\tau)\, d\tau + x_j(0)$$

or

$$x_j(t) = \int_0^t R_{ij}(\tau) f_i(t-\tau)\, d\tau$$

$$+ \int_0^\infty [R_{ij}(t+\tau) - R_{ij}(\tau)] f_i(-\tau)\, d\tau + x_j(0.) \quad [20]$$

The second integral converges, so this expression provides a usable definition of $x_j(t)$. Now let $f_i(t) = \cos\omega t$. After an integration by parts, we have

$$x_j(t) = 1/\omega \int_0^t \dot{R}_{ij}(\tau) \sin\omega(t-\tau)\, d\tau$$

$$+ \int_0^\infty [R_{ij}(t+\tau) - R_{ij}(\tau)] \cos\omega\tau\, d\tau + x_j(0).$$

Our only concern is with the oscillatory components of x_j. These are easily determined by considering the asymptotic form of the above expression as t becomes large. $R_{ij}(t+\tau) \to R_{ij}(\infty)$, and the second integral becomes constant. If we set

$$x_j(0) = -\lim_{t\to\infty} \int_0^\infty [R_{ij}(t+\tau) - R_{ij}(\tau)] \cos\omega t\, d\tau$$

then

$$x_j(t) = \frac{1}{\omega}(-\dot{R}_{ij}^s \cos\omega t + \dot{R}_{ij}^c \sin\omega t) \quad [21]$$

where $\dot{R}_{ij}^s$ and $\dot{R}_{ij}^c$ are the sine and cosine transforms of $\dot{R}_{ij}(t)$. We know that $x_j(t)$ is sinusoidal, with frequency ω. Therefore, this expression holds not only for large t but for all t.

If we define

$$R_{ij}^c = -\dot{R}_{ij}^s/\omega \quad [22a]$$

$$R_{ij}^s = \dot{R}_{ij}^c/\omega \quad [22b]$$

then [16] still holds. Note however, that R_{ij}^c and R_{ij}^s are no longer transforms of R_{ij} because these do not exist. Nevertheless, an inversion is still possible. Consider

$$\int_0^\infty [R_{ij}(\tau) - R_{ij}(\infty)] \cos\omega\tau\, d\tau$$

$$= \frac{1}{\omega}[R_{ij}(\tau) - R_{ij}(\infty)] \sin\omega\tau \Big|_0^\infty - \frac{1}{\omega}\int_0^\infty \dot{R}_{ij}(\tau) \sin\omega\tau\, d\tau$$

$$= -\dot{R}_{ij}{}^s/\omega = R_{ij}^c.$$

That is, R_{ij}^c is the cosine transform of $[R_{ij}(t) - R_{ij}(\infty)]$ and

$$R_{ij}(t) = R_{ij}(\infty) + \frac{2}{\pi}\int_0^\infty R_{ij}^c \cos\omega t\, d\omega$$

Letting t equal zero,

$$R_{ij}(\infty) = -\frac{2}{\pi}\int_0^\infty R_{ij}^c\, d\omega \quad [23]$$

so

$$R_{ij}(t) = \frac{2}{\pi}\int_0^\infty R_{ij}^c(\cos\omega t - 1)\, d\omega. \quad [24]$$

When $R_{ij}(\infty) = 0$, this reduces to [17a].

Similarly,

$$\int_0^\infty [R_{ij}(\tau) - R_{ij}(\infty)] \sin\omega\tau\, d\tau$$

$$= -R_{ij}(\infty)/\omega + \frac{1}{\omega}\int_0^\infty \dot{R}_{ij} \cos\omega\tau\, d\tau$$

$$= [\dot{R}_{ij}^c - R_{ij}(\infty)]/\omega$$

and

$$R_{ij}(t) = R_{ij}(\infty) + \frac{2}{\pi}\int_0^\infty \left[R_{ij}^s - \frac{R_{ij}(\infty)}{\omega}\right] \sin\omega\tau\, d\omega$$

$$= R_{ij}(\infty)\left[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\tau}{\omega}\,d\omega\right] + \frac{2}{\pi}\int_0^\infty R_{ij}{}^s \sin\omega\tau\,d\omega$$

$$= \frac{2}{\pi}\int_0^\infty R_{ij}{}^s \sin\omega t\,d\omega$$

since

$$\int_0^\infty \frac{\sin\omega t}{\omega}\,d\omega = \frac{\pi}{2}\ .$$

Therefore, [17b] holds even when $R_{ij}(\infty) \neq 0$.
If $R_{ij}{}^c$ and $R_{ij}{}^s$ are known, it is not difficult to determine whether or not $R_{ij}(\infty) = 0$. Equation [23] gives $R_{ij}(\infty)$ in terms of $R_{ij}{}^c$.
Also

$$R_{ij}(\infty) = \lim_{t\to\infty} \frac{2}{\pi}\int_0^\infty R_{ij}{}^s \sin\omega t\,d\omega$$

$$= \lim_{t\to\infty} \frac{2}{\pi}\int_0^\infty \dot{R}_{ij}{}^c \frac{\sin\omega t}{\omega}\,d\omega$$

$$= \dot{R}_{ij}{}^c(0) = \lim_{\omega\to 0} \omega R_{ij}{}^s \qquad [25]$$

using a well known theorem in Fourier transform theory (Reference 5, page 12).

When the matrix of impulse response functions is known, our first objective of finding a representation of the ship response which is free of frequency dependence is achieved. These functions, which we shall collectively call the impulse response matrix, can in principle be determined experimentally.

Equations of Motion

The transient response of a ship has been considered by Haskind[6]), who attempted an explicit solution of the boundary value problem. This, we shall not try, as we are concerned only with finding an appropriate form for the equations of motion to use as a basis for the interpretation of experimental results. We do not agree with certain of Haskind's hypotheses, and our resulting equations differ from his in several important respects.

Golovato[7]) carried out an experimental investigation of the declining oscillation motion in pitch. However, Golovato was not aware of the equivalence between the transient and steady state responses which we have just discussed, so he attempted only to match the coefficients derived from the transient experiment, at the frequency of the declining oscillation, with those from a forced oscillation experiment at this frequency. He was handicapped because of the anomalous behavior of the curve of declining amplitudes. For a simple harmonic oscillator, this curve is a straight line when plotted on semi-logarithmic paper. His curves departed radically from such a pattern. He recognized that this implied that the mathematical model was faulty, and attempted, with some success, to fit his results with forms based on Haskind's study.

More recently, Tasai[8]) has performed declining oscillation experiments in heave, using two dimensional forms. His results are not significantly different from those of Golovato. He matched his results at the measured frequency with Ursell's theoretical results for forced oscillation. The agreement is quite good.

Case 1 — No Forward Speed

Let the ship be floating at rest in still water. We use a system of coordinates $(\zeta_1, \zeta_2, \zeta_3)$, fixed in space, with origin in the free surface above the center of gravity of the ship.

At time $t = 0$, we suppose the ship to be given an impulsive displacement Δx_j in the j^{th} mode. The time history of this impulse is not significant, but for purposes of visualization, it may be considered to consist of a movement at a large, uniform, velocity v_j for a small time Δt, with the motion terminated abruptly at the end of this time interval. Then

$$\Delta x_j = v_j \Delta t\ .$$

During the impulse, the flow will have a velocity potential which is proportional to the instantaneous impulsive velocity of the ship. It may, therefore, be written $v_j\psi_j$, where ψ_j is a normalized potential for impulsive flow. ψ_j will satisfy the conditions:

$$\psi_j = 0 \quad \text{on} \quad \zeta_3 = 0 \qquad [26]$$

$$-\partial\psi_j/\partial n = s_j \quad \text{on } S \qquad [27]$$

where

$$\left.\begin{aligned} s_j &= \vec{n}\cdot\vec{i}_j & j &= 1,2,3 \\ &= \vec{r}\times\vec{n}\cdot\vec{i}_{j-3} & j &= 4,5,6 \end{aligned}\right. \qquad [28]$$

S = surface of the ship

$\vec{n}$ = outwardly directed unit normal

$\vec{i}_j$ = unit vector in j^{th} direction

$\vec{r}$ = position vector with respect to c. g. of ship.

It is well known[9]) that the above problem is equivalent to that obtained by reflecting S in $\zeta_3 = 0$ and taking the surface condition over the reflection to be the negative of that over S. The solution to the Neumann problem for the flow outside this composite surface is also the solution to the given problem in the lower half-space. For non-pathological surfaces, the solution exists, and in fact can be computed by means of modern, high-speed equipment.[10])

During the impulse, the free surface will be elevated by an amount

$$\Delta\eta_j = -v_j\frac{\partial\psi_j}{\partial\zeta_3}\Delta t = -\frac{\partial\psi_j}{\partial\zeta_3}\Delta x_j\ . \qquad [29]$$

After the impulse, this elevation will dissipate in a radiating disturbance of the free surface, until ultimately the fluid is again at rest in the neighborhood of the ship. Let the velocity potential of this decaying wave motion be $\varphi_j(t)\Delta x_j$. It must satisfy the initial conditions

$$\varphi_j(\zeta_1, \zeta_2, \zeta_3, 0) = 0 \qquad [30]$$

and

$$\Delta x_j\frac{\partial\varphi_j}{\partial t} = g\Delta\eta_j = -g\frac{\partial\psi_j}{\partial\zeta_3}\Delta x_j \text{ on } \zeta_3 = 0$$

or

$$\frac{\partial\varphi_j(\zeta_1, \zeta_2, 0)}{\partial t} = -g\frac{\partial\psi_j(\zeta_1, \zeta_2, 0)}{\partial\zeta_3} \qquad [31]$$

Afterward, it satisfies the usual free surface condition

$$\frac{\partial^2\varphi_j}{\partial t^2} + g\frac{\partial\varphi_j}{\partial\zeta_3} = 0\cdot \qquad [32]$$

and the boundary condition on S

$$\frac{\partial\varphi_j}{\partial n} = 0\,. \qquad [33]$$

We may take this to hold on the original position of S, only introducing errors of higher order in Δx_j. This is a classical problem of the Cauchy-Poisson type, and there exists an extensive literature on the subject. With condition [33], it is more difficult, by an order of magnitude, than the Neumann problem. We assume that it has a solution.

Now let the ship undergo an arbitrary small motion in the j^{th} mode, $x_j(t)$. To the first order, the velocity potential of the resulting flow will be simply

$$\Theta = \dot{x}\psi_j + \int_{-\infty}^{t} \varphi_j(t-\tau)\,\dot{x}_j(\tau)\,d\tau\,. \qquad [34]$$

It is evident that the boundary condition on S is satisfied on the equilibrium position of S, as the first term provides the proper normal velocity and $\partial\varphi_j/\partial n = 0$ on this boundary. But also, the value of $\partial\Theta/\partial n$ on the actual position of S will only differ from its value on S by terms of second and higher order in x_j and its derivatives, so we may consider that [34] holds on the actual position of the hull.

To verify that the free surface condition is satisfied, first note that

$$\frac{\partial^2\Theta}{\partial t^2} = \frac{d^2\dot{x}_j}{dt^2}\,\psi_j + \varphi_j(0)\,\frac{d\dot{x}_j}{dt} + \frac{\partial\varphi_j(0)}{\partial t}\,\dot{x}_j + \int_{-\infty}^{t} \frac{\partial^2\varphi_j(t-\tau)}{\partial t^2}\,\dot{x}_j(\tau)\,d\tau\,.$$

By [26] and [30], on $\zeta_3 = 0$ this reduces to

$$\frac{\partial^2\Theta}{\partial t^2} = \dot{x}_j\,\frac{\partial\varphi_j(0)}{\partial t} + \int_{-\infty}^{t} \frac{\partial^2\varphi_j(t-\tau)}{\partial t^2}\,\dot{x}_j(\tau)\,d\tau\,.$$

Also, $$\frac{\partial\Theta}{\partial\zeta_3} = \dot{x}_j\,\frac{\partial\psi_j}{\partial\zeta_3} + \int_{-\infty}^{t} \frac{\partial\varphi_j(t-\tau)}{\partial\zeta_3}\,\dot{x}_j(\tau)\,d\tau\,.$$

Substituting these in the free surface condition

$$\frac{\partial^2\Theta}{\partial t^2} + g\frac{\partial\Theta}{\partial\zeta_3} = \dot{x}_j\left(\frac{\partial\varphi_j(0)}{\partial t} + g\frac{\partial\psi_j}{\partial\zeta_3}\right) + \int_{-\infty}^{t}\left(\frac{\partial^2\varphi_j}{\partial t^2} + g\frac{\partial\varphi_j}{\partial\zeta_3}\right)\dot{x}(\tau)\,d\tau = 0 \qquad [35]$$

by [31] and [32]. Thus, this condition is also satisfied, and Θ is the required potential.

The formula [34] is a hydrodynamic analog of [1]. It is quite general, and can, for instance, be used to find the velocity potential due to a sinusoidal oscillation with arbitrary frequency. It is, of course, necessary to know the function $\varphi_j(t)$, and this presents unpleasant difficulties. In this study we are content that $\varphi_j(t)$ exists, and these difficulties do not concern us.

Of more importance than the velocity potential is the force acting on the body. The dynamic pressure in our linearized model is simply

$$p = \varrho\frac{\partial\Theta}{\partial t}$$

or

$$\frac{p}{\varrho} = \ddot{x}_j\psi_j + \varphi_j(0)\,\dot{x}_j + \int_{-\infty}^{t} \frac{\partial\varphi_j(t-\tau)}{\partial t}\,\dot{x}_j(\tau)\,d\tau$$

$$= \ddot{x}_j\psi_j + \int_{-\infty}^{t} \frac{\partial\varphi_j(t-\tau)}{\partial t}\,\dot{x}_j(\tau)\,d\tau\,. \qquad [36]$$

The net hydrodynamic force (or moment) acting on the hull in the k^{th} mode is then given by

$$-F_{jk} = \int_S p\,s_k\,d\sigma$$

$$= \ddot{x}_j\,\varrho\int_S \psi_j\,s_k\,d\sigma + \varrho\int_S s_k\,d\sigma\int_{-\infty}^{t} \frac{\partial\varphi_j(t-\tau)}{\partial t}\,\dot{x}_j(\tau)\,d\tau$$

$$= \ddot{x}_j m_{jk} + \varrho\int_{-\infty}^{t} \dot{x}_j(\tau)\,d\tau\int_S \frac{\partial\varphi_j(t-\tau)}{\partial t}\,s_k\,d\sigma$$

$$= \ddot{x}_j\,m_{jk} + \int_{-\infty}^{t} K_{jk}(t-\tau)\,\dot{x}_j(\tau)\,d\tau \qquad [37]$$

where $$m_{jk} = \varrho\int_S \psi_j\,s_k\,d\sigma \qquad [38]$$

$$K_{jk}(\tau) = \varrho\int_S \frac{\partial\varphi_j(\tau)}{\partial t}\,s_k\,d\sigma\,. \qquad [39]$$

We can now write the equations of motion of the ship which is subjected to an arbitrary set of exciting forces, $\{f_k(t)\}$. These will be

$$\sum_{j=1}^{6}\left[(m_j\,\delta_{jk} + m_{jk})\,\ddot{x}_j + c_{jk}\,x_j + \int_{-\infty}^{t} K_{jk}(t-\tau)\,\dot{x}_j(\tau)\,d\tau\right] = f_k(t) \qquad [40]$$

where

m_j = inertia of the ship in the j^{th} mode

$c_{jk}\,x_j$ = hydrostatic force in the k^{th} mode, due to displacement x_j in the j^{th} mode

δ_{jk} = Kronecker delta ($\delta_{jk} = 1$ if $j = k$, $= 0$ if $j \neq k$).

Case II — Ship Underway

The case of the ship experiencing small oscillations about a reference position of mean uniform velocity is much more complex. A pair of functions, ψ_j and φ_j, no longer suffices, although the pattern of our analysis will be similar to that followed in Case I.

We use a fixed reference system, with $\zeta_3 = 0$ on the free surface and with the c. g. of the ship at $\zeta_1 = 0$ at time $t = 0$. We suppose the ship to be moving with a uniform velocity V in the ζ_1 direction.

Consider the Cauchy-Poisson problem defined by [30], [31], [32], and [33], except that now [33] is to hold on the moving surface S. This problem has a solution $\varphi_j(\zeta_1, \zeta_2, \zeta_3, t, V)$ which is, of course, identical with the φ_j of Case I when $V = 0$. Using this φ_j and the ψ_j obtained in Case I, we may write the velocity potential for steady motion,

$$\Theta = V\left[\psi_1(\zeta_1 - Vt, \zeta_2, \zeta_3) + \int_{-\infty}^{t} \varphi_1(\tau, t-\tau)\,d\tau\right] \qquad [41]$$

where

$$\varphi_1(\tau, t-\tau) = \varphi_1(\zeta - V\tau, \zeta_2, \zeta_3, t-\tau, V)\,.$$

That this satisfies the boundary condition on S is evident, as $V\psi_1$ provides the necessary instantaneous normal velocities, and $\partial\varphi_1/\partial n = 0$ on S for all τ. The free surface condition is also satisfied, as may be verified by direct evaluation, as in Case I.

The velocity potential for the flow generated by the ship moving with constant velocity, after an impulsive start at time zero, is

$$\Theta = V\left[\psi_1(\zeta_1 - Vt, \zeta_2, \zeta_3) + \int_{0}^{t} \varphi_1(\tau, t-\tau)\,d\tau\right] \qquad [42]$$

The free surface and ship surface conditions are satisfied as before. The surface elevation at $t = 0$ is

$$\left[\frac{1}{g}\,\frac{\partial\Theta}{\partial t}\right]_{\substack{\zeta_3=0\\ t=0}} = \frac{V}{g}\left[-V\frac{\partial\psi_1}{\partial\zeta_1} + \varphi_1(0,0)\right] = 0$$

as required, and the initial conditions are met. Therefore, this must be the stated potential.

We shall need the steady motion velocity potential for the case in which the ship is displaced by Δx_j from its reference position. We could, of course, consider the displaced ship as a completely new hull and write down a potential similar to [41], with new functions ψ_1 and φ_1. Instead, we determine the corrections to the ψ_1 and φ_1, discussed above, which are necessary to satisfy the new boundary conditions. We wish a ψ_{1j} such that

$$\psi_1 + \Delta x_j \psi_{1j} = 0 \text{ on } \zeta_3 = 0 \tag{43}$$

which implies that

$$\psi_{1j} = 0 \quad \text{on } \zeta_3 = 0 \tag{43a}$$

Also

$$\frac{\partial}{\partial n}(\psi_1 + \Delta x_j \psi_{1j}) = \vec{n}\cdot\vec{i}_1 \qquad \text{on S (displaced)} \tag{44}$$

or

$$\Delta x_j \frac{\partial \psi_{ij}}{\partial n} = \vec{n}\cdot\vec{i}_1 - \frac{\partial \psi_1}{\partial n} \quad \text{on S (displaced)} \tag{44a}$$

In three cases, solutions are immediately available. If $j = 1$

$$\psi_1(\zeta_1 - \Delta x_1, \psi_2, \psi_3) = \psi_1 - \Delta x_1 \frac{\partial \psi_1}{\partial \zeta_1} + o(\Delta x_1)$$

is a solution of [43] and [44] since in this case we have simple translation. Therefore

$$\psi_{11} = -\frac{\partial \psi_1}{\partial \zeta_1}. \tag{45a}$$

Similarly

$$\psi_{12} = -\frac{\partial \psi_1}{\partial \zeta_2}. \tag{45b}$$

For $j = 3$, there is no such simple solution. Noting that the right side of [44a] is zero on S (original), it is only necessary to find its change when S is displaced. Then

$$\Delta x_3 \frac{\partial \psi_{13}}{\partial n} = -\Delta x_3 \frac{\partial^2 \psi_1}{\partial n \partial \zeta_3}$$

or

$$\frac{\partial \psi_{13}}{\partial n} = -\frac{\partial^2 \psi_1}{\partial n \partial \zeta_3}. \tag{46}$$

If $j = 6$, the displacement is simply a rotation in yaw. The translation of a yawed body is equivalent to simultaneous translations parallel and perpendicular to the body axis. Therefore, the solution to [43] and [44] is

$$\begin{aligned}&\psi_1(\zeta_1 + \zeta_2 \Delta x_6, \zeta_2 - \zeta_1 \Delta x_6, \zeta_3) - \\ &\Delta x_6 \psi_2(\zeta_1 + \zeta_2 \Delta x_6, \zeta_2 - \zeta_1 \Delta x_6, \psi_3) \\ &= \psi_1 + \Delta x_6 \left(\zeta_2 \frac{\partial \psi_1}{\partial \zeta_1} - \zeta_1 \frac{\partial \psi_1}{\partial \zeta_2} + \psi_2 \right)\end{aligned}$$

so

$$\psi_{16} = \zeta_2 \frac{\zeta \psi_1}{\partial \zeta_1} - \zeta_1 \frac{\partial \psi_1}{\partial \zeta_2} + \psi_2. \tag{47}$$

If $j = 4$, or 5, the first term of [44a] becomes

$$[\vec{n} + \Delta x_j (\vec{i}_{j-3} \vec{x} \vec{n})] \cdot \vec{i}_1.$$

The second term is

$$-\frac{\partial \psi_1}{\partial n} = \vec{n}\cdot \nabla \psi_1 \quad \text{on S (displaced)}$$

which may be written, using values of $\vec{n}$ and $\nabla\psi$ evaluted on S (original),

$$-[\vec{n} + \Delta x_j (\vec{i}_{j-3} x \vec{n})] \cdot [\nabla \psi_1 + \Delta x_j (\vec{i}_{j-3} x \vec{r} \cdot \nabla) \nabla \psi_1]$$

If we drop terms of higher order in Δx_j and use [27], condition [44a] reduces to

$$\frac{\partial \psi_{1j}}{\partial n} = \vec{i}_{j-3} x \vec{n} \cdot \vec{i}_1 - [\vec{i}_{j-3} x \vec{n} \cdot \nabla \psi_1 + \vec{n} \cdot (\vec{i}_{j-3} x \vec{r} \cdot \nabla) \nabla \psi_1]$$

or

$$\frac{\partial \psi_{14}}{\partial n} = -\left[\vec{n}\cdot\left(\vec{i}_2 \frac{\partial \psi_1}{\partial \zeta_3} - \vec{i}_3 \frac{\partial \psi_1}{\partial \zeta_2}\right) + \zeta_2 \frac{\partial^2 \psi_1}{\partial n \partial \zeta_3} - \zeta_3 \frac{\partial^2 \psi_1}{\partial n \partial \zeta_2}\right] \tag{48a}$$

and

$$\frac{\partial \psi_{15}}{\partial n} = s_3 - \left[\vec{n}\cdot\left(\vec{i}_3 \frac{\partial \psi_1}{\partial \zeta_1} - \vec{i}_1 \frac{\partial \psi_1}{\partial \zeta_3}\right) + \zeta_3 \frac{\partial^2 \psi_1}{\partial n \partial \zeta_1} - \zeta_1 \frac{\partial^2 \psi_1}{\partial n \partial \zeta_3}\right] \tag{48b}$$

Conditions [43a] und [44a] are sufficient to determine ψ_{1j}. Strictly, [44a] holds on S (displaced), but we only introduce errors of order $(\Delta x_j)^2$ if we take the ship surface condition to hold on the original S. Similarly, [46] and [48] can be applied on the reference position of S.

To ψ_{1j} corresponds a φ_{1j}, with

$$\frac{\partial \varphi_{1j}}{\partial t} = -g \frac{\partial \psi_{1j}}{\partial \zeta_3} \qquad \text{for } \zeta_3 = 0, t = 0 \tag{49}$$

and with conditions corresponding to [30], [32], and [33] holding. Again we take the ship surface condition to hold on the reference position of S.

We need yet one more pair of functions. The normal derivative $\partial\varphi_1/\partial n$ will differ from zero on S (displaced) to the first order in Δx_j. To correct it, we define a function which satisfies the conditions

$$\Delta x_j \frac{\partial \psi_{oj}}{\partial n} = -\frac{\partial}{\partial n} \int_{-\infty}^{t} \varphi_1(\tau, t-\tau)\, d\tau \text{ on S (displaced)} \tag{50}$$

and

$$\psi_{oj} = 0 \text{ on } \zeta_3 = 0 \tag{51}$$

As we do not intend to exhibit solutions for ψ_{oj}, we shall not reduce the right side. We also need a φ_{oj}, with

$$\frac{\partial \varphi_{oj}}{\partial t} = -g \frac{\partial \psi_{oj}}{\partial \zeta_3} \tag{52}$$

and the other appropriate conditions also holding.

We now have all the pieces needed to write the velocity potential for the flow about the ship when displaced by Δx_j from its reference position. It will be

$$\begin{aligned}\Theta = V \{&[\psi_1 + \int_{-\infty}^{t} \varphi_1(\tau, t-\tau)\, d\tau] \\ &+ \Delta x_j [\psi_{1j} + \int_{-\infty}^{t} \varphi_{1j}(\tau, t-\tau)\, d\tau] \\ &+ \Delta x_j [\psi_{oj} + \int_{-\infty}^{t} \varphi_{oj}(\tau, t-\tau)\, d\tau]\}\end{aligned} \tag{53}$$

The terms $\qquad V(\psi_1 + \Delta x_j \psi_{1j})$

provide the necessary normal velocity in the displaced position. The normal velocities due to

$$V \Delta x_j \psi_{oj} \text{ and } V \int_{-\infty}^{t} \varphi_1(\tau, t-\tau)\, d\tau$$

cancel, and none of the other terms contributes normal velocities of first order in Δx_j. Therefore, the ship surface condition is satisfied in the displaced position. Further, each pair of terms in brackets satisfies the free surface condition, as may be verified by direct evaluation.

We also have all the pieces needed to assemble the potential for the flow generated by a ship experiencing small oscillations $\{x_j(t)\}$. It will be

$$\begin{aligned}\Theta = V\{&[\psi_1 + \int_{-\infty}^{t} \varphi_1(\tau, t-\tau)\, d\tau] \\ &+ \sum_{j=1}^{6} [x_j \psi_{1j} + \int_{-\infty}^{t} \varphi_{1j}(\tau, t-\tau)\, x_j(\tau)\, d\tau]\end{aligned}$$

$$+\sum_{j=1}^{6}[x_j\,\psi_{oj}+\int_{-\infty}^{t}\varphi_{oj}(\tau,t-\tau)\,x_j(\tau)\,d\tau]\}$$

$$+\sum_{j=1}^{6}[\dot{x}_j\,\psi_j+\int_{-\infty}^{t}\varphi_j(\tau,t-\tau)\,\dot{x}_j(\tau)\,d\tau] \qquad [54]$$

The ship surface condition is satisfied as before, except now the term $\Sigma\dot{x}_j\,\psi_j$ provides the additional components required for the oscillatory velocities. And again, the bracketed pairs of terms satisfy the free surface condition.

The dynamic pressure at any point in the fluid is given by

$$\frac{p}{\varrho}=\frac{\partial\Theta}{\partial t}=\sum_{j=1}^{6}\left\{V\left[\dot{x}_j(\psi_{1j}+\psi_{oj})-x_j\,V\left(\frac{\partial\psi_{1j}}{\partial\zeta_1}+\frac{\partial\psi_{oj}}{\partial\zeta_1}\right)\right]\right.$$

$$+\ddot{x}_j\,\psi_j-\dot{x}V\frac{\partial\psi_j}{\partial\zeta_1}$$

$$+V\int_{-\infty}^{t}\left[\frac{\partial\varphi_{1j}(\tau,t-\tau)}{\partial t}+\frac{\partial\varphi_{oj}(\tau,t-\tau)}{\partial t}\right]x_j(\tau)\,d\tau$$

$$\left.+\int_{-\infty}^{t}\frac{\partial\varphi_j(\tau,t-\tau)}{\partial t}\,\dot{x}_j(\tau)\,d\tau\right\}$$

$$-V^2\frac{\partial\psi_1}{\partial\zeta_1}+V\int_{-\infty}^{t}\frac{\partial\varphi_1(\tau,t-\tau)}{\partial t}\,d\tau \qquad [55]$$

There are two convolution integrals in [55], one involving the oscillatory displacement and one involving the oscillatory velocity. These may be reduced to one by means of an integration by parts. We can go either way, but there is some advantage in defining

$$\int_{-\infty}^{\tau_0}[\varphi_{1j}(\tau,t-\tau)+\varphi_{oj}(\tau,t-\tau)]\,d\tau=\Phi_j(t-\tau_o) \qquad [56]$$

so that

$$\int_{-\infty}^{\tau_0}\left[\frac{\partial\varphi_{1j}}{\partial t}+\frac{\partial\varphi_{oj}}{\partial t}\right]d\tau=\frac{\partial\Phi_j}{\partial t} \qquad [57]$$

and

$$\int_{-\infty}^{t}\left[\frac{\partial\varphi_{1j}}{\partial t}+\frac{\partial\varphi_{oj}}{\partial t}\right]x_j(\tau)\,d\tau=$$

$$=\frac{\partial\Phi_j(0)}{\partial t}\,x_j(t)-\int_{-\infty}^{t}\frac{\partial\Phi_j(t-\tau)}{\partial t}\,\dot{x}_j(\tau)\,d\tau$$

$$=-Vx_j\frac{\partial\Phi_j(0)}{\partial\zeta_1}-\int_{-\infty}^{t}\frac{\partial\Phi_j(t-\tau)}{\partial t}\,\dot{x}_j(\tau)\,d\tau\,. \qquad [58]$$

The significance of this function Φ_j can be seen by rewriting the potential for the uniform flow with the body deflected (Equation [53]). It becomes

$$V\left\{\psi_1+\int_{-\infty}^{t}\varphi_1(\tau,t-\tau)\,d\tau+\Delta x_j\,[(\psi_{1j}+\psi_{oj})+\Phi_j(0)]\right\}. \quad [59]$$

Equation [55] now reduces to

$$\frac{p}{\varrho}=\sum_{j=1}^{6}\left\{\ddot{x}_j\,\psi_j+\dot{x}_j\,V\left(\psi_{1j}+\psi_{oj}-\frac{\partial\psi_j}{\partial\zeta_1}\right)\right.$$

$$-x_j\,V^2\frac{\partial}{\partial\zeta_1}[\psi_{1j}+\psi_{oj}+\Phi_j(0)]$$

$$\left.+\int_{-\infty}^{t}\left(\frac{\partial\varphi_j(\tau,t-\tau)}{\partial t}-V\frac{\partial\Phi_j(t-\tau)}{\partial t}\right)\dot{x}_j(\tau)\,d\tau\right\}$$

$$-V^2\frac{\partial\psi_1}{\partial\zeta_1}+V\int_{-\infty}^{t}\frac{\partial\varphi_1(\tau,t-\tau)}{\partial t}\,d\tau\,. \qquad [60]$$

We are concerned with the oscillatory value of the hydrodynamic force, but not steady components. The last term [59] does not involve the $\{x_j\}$. However, when we integrate the pressure over S, the fact that S is changing its position in a steady flow field implies that even this term contributes to the oscillatory pressure. These pressures will be functions of the displacement only.

Integrating the pressure over the surface of the ship, we can write the equations of motion

$$\sum_{j=1}^{6}[(m_j\,\delta_{jk}+m_{jk})\,\ddot{x}_j+b_{jk}\,\dot{x}_j+c_{jk}\,x_j$$

$$+\int_{-\infty}^{t}K_{jk}(t-\tau)\,\dot{x}_j(\tau)\,d\tau]=f_k(t) \qquad [61]$$

where m_j and m_{jk} are as defined in [40] and [38], and

$$b_{jk}=\varrho V\int_S\left(\psi_{1j}+\psi_{oj}-\frac{\partial\psi_j}{\partial\zeta_1}\right)s_k\,d\sigma \qquad [62]$$

$c_{jk}\,x_j$ = Total hydrodynamic and hydrostatic force in the k^{th} mode, due to displacement x_j in the j^{th} mode.

$$K_{jk}(t-\tau)=\int_S\left(\frac{\partial\varphi_j(\tau,t-\tau)}{\partial t}-V\frac{\partial\Phi_j(t-\tau)}{\partial t}\right)s_k\,d\sigma\,. \quad [63]$$

There are symmetries which reduce the number of coefficients. For instance

$$m_{jk}=\varrho\int_S\psi_j\,s_k\,d\sigma$$

$$=-\varrho\int_S\psi_j\frac{\partial\psi_k}{\partial n}\,d\sigma\,.$$

If we consider the space enclosed by S, the free surface, and an infinite hemisphere, we can apply Green's theorem, and we find

$$m_{jk}=-\varrho\int_S\psi_k\frac{\partial\psi_j}{\partial n}\,d\sigma=m_{kj}\,. \qquad [64]$$

Further, if we consider the transverse symmetry of the ship, the matrix $\{m_{jk}\}$ reduces to

$$\{m_{jk}\}=\begin{bmatrix} m_{11} & o & m_{13} & o & m_{15} & o\\ o & m_{22} & o & m_{24} & o & m_{26}\\ m_{31} & o & m_{33} & o & m_{35} & o\\ o & m_{42} & o & m_{44} & o & m_{46}\\ m_{51} & o & m_{53} & o & m_{55} & o\\ o & m_{62} & o & m_{64} & o & m_{66}\end{bmatrix} \qquad [65]$$

Evidently, the matrix $\{b_{jk}\}$ is of the same form, except that in general $b_{jk}\neq b_{kj}$. The matrix c_{jk} is even simpler as surge and sway displacements provide no restoring forces, hydrostatic or hydrodynamic.

Therefore

$$\{c_{jk}\}=\begin{bmatrix} o & o & o & o & o & o\\ o & o & o & o & o & o\\ c_{31} & o & c_{33} & o & c_{35} & o\\ o & c_{42} & o & c_{44} & o & c_{46}\\ c_{51} & o & c_{53} & o & c_{55} & o\\ o & c_{62} & o & c_{64} & o & c_{66}\end{bmatrix} \qquad [66]$$

The matrix $\{K_{jk}(t)\}$ is of the same form as $\{b_{jk}\}$.

Equations [61], though similar in form to those developed by Haskind, differ from his in several essentials. Haskind found no hydrodynamic force proportional to the displacement, nor did he find the components of b_j due to ψ_j and ψ_{oj}. He also found that $b_{33} = b_{55} = 0$, and $b_{35} = -b_{53}$. The presence of ψ_{oj} in the definition of b_{jk} makes it unlikely that such relations hold here. Further, his kernel in the convolution integral must differ from that found here. The reason for these differences is that Haskind neglected terms in satisfying the boundary condition on the displaced S which are of first order in x_j.

With equation [61], we have advanced a long way toward the second objective of this paper. The dynamics of the body have been separated from the dynamics of the fluid. Further, the hydrodynamic effects have been separated into well defined components, each of which can be found (in principle) from the solution of a Neumann problem or a Cauchy-Poisson problem. Specifically, we draw the conclusions:

1. The equations of motion are universally valid within the range of validity of our assumption of linearity. That is, any excitation, periodic or non-periodic, continuous or discontinuous, is permissible, just so it results in small displacements from a condition of uniform forward velocity. The case of motion with a negative restoring force, or at least the early history of such motion, is not excluded.
2. The inertial properties of the fluid are reflected in the products $m_{jk}\ddot{x}_j$. The coefficients are independent of frequency and of the past history of the motion, so they are legitimate added masses. Further, they are independent of forward velocity.
3. There is an effect proportional to $\dot{x}_i$ which accounts for some of the damping. This effect vanishes when the mean forward speed is zero.
4. There is a hydrodynamic "restoring" force (it may be negative). It is equal to the difference between the hydrodynamic forces acting on the ship due to the steady flow in the equilibruim position and in the deflected position.
5. The effect of past history is embedded in a convolution integral over $\dot{x}_i(t)$. For sinusoidal motions, this integral will ordinarily have components both in phase with the motion and 90° out of phase. The latter component contributes to the damping.

Hydrodynamics of the Impulse Response Function

We now have two systems of relations between the excitation and the response of the ship: the impulse response relations, [2], and the equations of motion, [61]. The former are of greater value in describing the response to a given excitation, while the latter are useful in analyzing the nature of the response. Both systems hold for small oscillatory motions, so there are relations between them. We shall examine these.

First, let us start with the equations of motion and derive the functions $\{R_{ij}(t)\}$. Suppose a ship, moving at constant forward velocity, to be subjected to a unit impulse in the i^{th} mode at time $t = 0$. During the impulse, the equations of motion reduce to

$$m_k \ddot{x}_k + \sum_{j=1}^{6} m_{jk}\ddot{x}_j = f_i \delta_{ik}$$

where δ_{ik} is the Kronecker delta. Suppose the impulse acts during time Δt.

Then, since $\ddot{x}_j \Delta t = \Delta \dot{x}_j = R_{ij}(+0)$

we have
$$m_k \dot{R}_{ik}(+0) + \sum_{j=1}^{6} m_{jk}\dot{R}_{ij}(+0) = \delta_{ik}. \qquad [67]$$

As i and j range independently from 1 to 6, we have 36 equations relating the two sets, $\{m_{ij}\}$ and $\{\dot{R}_{ij}(0)\}$. If the equations of motion are known, equations [67] fix the initial conditions from which the impulse response functions can be determined. Conversely, if the impulse response functions are known, these equations yield the apparent masses.

Immediately after the impulse, we have

$$x_j = 0(t)$$
$$\dot{x}_j = \dot{x}_j(0) + 0(t)$$
$$\ddot{x}_j = \ddot{x}_j(0) + 0(t)$$
$$\int_0^\infty K_{ij}(\tau)\dot{x}_i(t-\tau)\,d\tau = 0(t).$$

Therefore, considering only zero order terms in t, the equations of motion yield:

$$m_k \ddot{R}_{ik}(+0) + \sum_{j=1}^{6}[m_{jk}\ddot{R}_{ij}(+0) + b_{jk}\dot{R}_{ij}(+0)] = 0 \qquad [68]$$

which relates the coefficients $\{b_{jk}\}$ to the accelerations $\{\ddot{R}_{ij}(+0)\}$.

Now suppose the ship to be acted upon by a constant unit force in the i^{th} mode (we assume a positive restoring force to exist in this mode). Then, after equilibruim is reached,

$$\sum_{j=1}^{6} c_{jk} x_j = \delta_{ik}$$

and
$$x_j = \int_0^\infty R_{ij}(\tau)\,d\tau$$

or
$$\sum_{j=1}^{6} c_{jk}\int_0^\infty R_{ij}(\tau)\,d\tau = \delta_{ik} \qquad [69]$$

In modes without a positive restoring force there is difficulty as there is no guarantee that all of the coupling coefficients are necessarily zero. Thus, $c_{62}x_6$, the sway force due to a yaw angle x_6, will not ordinarily be zero, or even negligible. We shall return to this point a little later.

If we rewrite [61] in the form

$$\sum_{j=1}^{6}\int_0^t K_{jk}(\tau)\dot{R}_{ij}(t-\tau)\,d\tau = -\sum_{j=1}^{6}[(m_j\delta_{jk} + m_{jk})\ddot{R}_{ij}(t) + b_{jk}\dot{R}_{ij}(t) + c_{jk}R_{ij}(t)] \qquad [70]$$

we have a set of 36 equations which can either be regarded as a set of simultaneous integral equations for the kernels $\{K_{jk}(\tau)\}$, or a set of simultaneous integro-differential equations for the impulse response functions $\{R_{ij}(t)\}$.

We have already seen (equation [16]) that if

$$f_i(t) = \cos\omega t$$

then

$$x_j(t) = R_{ij}{}^c \cos\omega t + R_{ij}{}^s \sin\omega t$$

Substituting these values in the equations of motion, we get

$$-\sum_{j=1}^{6}\{[(m_j\delta_{jk} + m_{jk})\omega^2 R_{ij}{}^c - b_{jk}\omega R_{ij}{}^s - c_{jk}R_{ij}{}^c - \omega(R_{ij}{}^s K_{jk}{}^c + R_{ij}{}^c K_{jk}{}^s)]\cos\omega t + [(m_j\delta_{jk} + m_{jk})\omega^2 R_{ij}{}^s - b_{jk}\omega R_{ij}{}^c - c_{jk}R_{ij}{}^s - \omega(R_{ij}{}^s K_{jk}{}^s - R_{ij}{}^c K_{jk}{}^c)]\sin\omega t\} = \delta_{ik}\cos\omega t$$

For any given frequency, this is an identity, so the net coefficients of $\cos\omega t$ and $\sin\omega t$ must be zero. This gives us 72 equations relating the transforms $\{R_{ij}{}^c, R_{ij}{}^s\}$ with the transforms $\{K_{ij}{}^c, K_{jk}{}^s\}$.

We have
$$\omega\sum_{j=1}^{6}(R_{ij}{}^s K_{jk}{}^c + R_{ij}{}^c K_{jk}{}^s) = \delta_{ik} + \sum_{j=1}^{6}[(m_j\delta_{jk} + m_{jk})\omega^2 R_{ij}{}^c - b_{jk}\omega R_{ij}{}^s - c_{jk}R_{ij}{}^c] \qquad [71a]$$

and
$$-\omega\sum_{j=1}^{6}(R_{ij}{}^c K_{jk}{}^c - R_{ij}{}^s K_{jk}{}^s) = \omega\sum_{j=1}^{6}[(m_j\delta_{jk} + m_{jk})\omega^2 R_{ij}{}^s + b_{jk}\omega R_{ij}{}^c - c_{jk}R_{ij}{}^s] \qquad [71b]$$

or, equivalently

$$\sum_{j=1}^{6} \{[(m_j \delta_{jk} + m_{jk}) \omega^2 - c_{jk} - \omega K_{jk}^s] R_{ij}^c - (b_{jk} + K_{jk}^c) \omega R_{ij}^s\} = -\delta_{ik} \quad [72a]$$

$$\sum \{(b_{jk} + K_{jk}^c) \omega R_{ij}^c + [(m_j \delta_{jk} + m_{jk}) \omega^2 - c_{jk} - \omega K_{jk}^s] R_{ij}^s\} = 0 \quad [72b]$$

Thus, instead of the integral and integro-differential equations relating $\{R_{ij}\}$ with $\{K_{jk}\}$, Equation [70], we have systems of linear equations relating their transforms.

Equations [72] are particularly revealing. If we were to arbitrarily set the K_{jk}^c and K_{jk}^s to be zero, these are precisely the equations we would get between the frequency response functions, R_{ij}^c and R_{ij}^s and the usual frequency dependent coefficients. Thus, it is clear how frequency dependency of the K_{jk}^c and K_{jk}^s is forced onto these coefficients in the conventional representation.

The transforms of $\{R_{ij}\}$ also yield useful variants of the relations already given. For instance, if we let $\omega = 0$, we have

$$\sum_{j=1}^{6} c_{jk} R_{ij}^c(0) = \delta_{ik} \quad [73]$$

a more general form of [69].

Also, noting that $\dot{R}_{ij}(t) = \frac{2}{\pi} \int_0^\infty \omega R_{ij}^s(\omega) \cos \omega t \, d\omega$

and $\ddot{R}_{ij}(t) = -\frac{2}{\pi} \int_0^\infty \omega^2 R_{ij}^c(\omega) \cos \omega t \, d\omega$

we have $\dot{R}_{ij}(0) = \frac{2}{\pi} \int_0^\infty \omega R_{ij}^s \, d\omega \quad [74a]$

$$\ddot{R}_{ij}(0) = -\frac{2}{\pi} \int_0^\infty \omega^2 R_{ij}^c \, d\omega \quad [74b]$$

Therefore, [67] and [68] may be written

$$\sum_{j=1}^{6} [(m_j \delta_{jk} + m_{jk}) \int_0^\infty \omega R_{ij}^s \, d\omega] = \frac{\pi}{2} \delta_{ik} \quad [75]$$

and

$$\sum_{j=1}^{6} [(m_j \delta_{jk} + m_{jk}) \int_0^\infty \omega^2 R_{ij}^c \, d\omega - b_{jk} \int_0^\infty \omega R_{ij}^s \, d\omega] = 0 \quad [76]$$

Conclusion

In the foregoing, we have presented two mathematical models for representing the response characteristics of a ship. The equations of motion are more general, as they apply to the initial stages of an unstable motion. Where the two systems are equally valid, we have relations which permit us to pass (at least in principle) from either system to the other.

The impulse response function is certainly the better representation for computing responses. It integrates all factors, mechanical, hydrostatic, and hydrodynamic, in the most efficient manner possible for computation. However, for this very reason, it is a poor analytical tool for explaining why the ship responds the way it does or how the response will be affected if any change in conditions occurs. For instance, models are ordinarily tested with restraints in certain modes. A restraint in any mode will affect the impulse response function in any coupled mode. Since the ship is free in all modes, it is evidently improper to use these response functions to predict full-scale behavior unless they are corrected for the effect of such restraints.

The hydrodynamic equations do not suffer from this disadvantage. Known restraints are readily includable and their effects determinable. Or a change in mass distribution can be treated independently of the hydrodynamics. It is not uncommon in model testing to have "incompatible" parasitic inertias in the different modes. Thus, the towing gear may contribute a different mass in surge from that in heave. By means of the equations of motion, the effect of these inertias upon the motions can be analyzed. Thus, the equations of motion provide a more powerful analytic tool for studying the relationship of the response to the parameters governing that response.

We can conclude, then, that these two representations complement each other; the one for response calculation, the other for response analysis. In fact, if it is truly practicable to pass from one representation to the other, several possibilities present themselves:

a) Model experiments may be designed to obtain maximum accuracy rather than maximum realism. Hydrodynamic effects should be emphasized in the design since other effects are separately determinable. Thus, one should test at small gyradius in order that the effect of the inertial properties of the body itself will be minimized.
b) Restraints are permissible if their character is fully known. Thus, rather than directly find the impulse response matrix, in its complete generality, more elementary experiments may be conducted to determine specific terms in the equations of motion. We may restrict ourselves to one, two, or three degrees of freedom and obtain results which are completely valid when interpreted by means of the equations of motion.
c) The recurring difficulty of handling modes in which the restoring force is zero or negative can be easily overcome. It is clear that an accurate experimental investigation of these modes would uncover practical difficulties analogous to the theoretical ones we have discussed. However, the problem can easily be solved by imposing known restraints (i. e. springs) which will restore positive stability. The effect of these restraints is readily includable in the equations of motion, it can be removed by calculation, and the correct impulse response, free of restraint, can be determined.

(Vorgetragen am 25. Januar 1962)

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