An Investigation of Stochastic Market Equilibrium in Cognitive Radio Networks

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***Abstract*—In this letter, we model a cognitive radio network as an interference market where each primary user (PU) and secondary user (SU) are price–taking interference offerers and purchasers, respectively. We consider stochastic interference demands of SUs and the existence of random disturbances in PUs' interference observations. Exploiting the fact that the dB value of the uplink inter–cell interference in CDMA networks is modeled as a Gaussian normal distribution, the price–demand decision process in each PU is given by a pair of stochastic differential equations. Then, we investigate an equilibrium of the interference market and its convergence in the context of asymptotic stability in the mean. Numerical results illustrate the convergence of the price–demand decision process to the equilibrium.**

***Index Terms*—Cognitive radio, spectrum market, stochastic market equilibrium, stochastic differential equation, tâtonnement process.**

I. Introduction

IN cognitive radio networks, spectra or interference are often regarded as marketable products. That is, every primary user (PU) offers its resources (i.e., spectra or interference) to secondary users (SUs) with a certain price (per spectrum or per interference); each SU purchases the offered resources with determining the purchasing amount according to the price. As a result, this market–based approach enables PUs to control SUs' resource usage by adjusting the price.

In general market, when demand and supply depend upon current prices only, there exists an equilibrium price vector under certain condition, called *market equilibrium* where the total demand for each good is equal to the corresponding total supply of that good; this market equilibrium can be expressed as a stable solution of price and demand decision processes, i.e., *tâtonnement process* [1]. In this letter, we consider the tâtonnement process between PUs and SUs in CDMA systems: (i) each PU sets the limit of interference inflicted by SUs (i.e., tolerable interference), and sets and announces the price per interference to SUs; (ii) SUs decides collaboratively how much of interference they would like to purchase for the announced price from each PU; (iii) each PU increases the price if it suffers higher interference than its tolerable interference, and decreases otherwise; (iv) SUs adjusts the amount of interference to each PU in inverse proportion to the price; (v) on every PU, if the total interference demand equates to its interference limit with a positive price, the tâtonnement process terminates in equilibrium, i.e., market equilibrium, and the price becomes an *equilibrium price*. Furthermore, we consider the situation where PUs have no deterministic information such as the number of transmitting SUs, their transmitting power and locations, and channel states. Besides, we assume the existence of an additive random disturbance – representing the thermal noise and given as Gaussian white noise – in the interference observation. Considering the pricing for the uplink interference only[1], we can utilize the random interference model for CDMA networks developed in [2], and whose main features are: (i) if the interference is expressed in a dB value, then the random variable of the uplink inter–cell interference follows normal Gaussian distribution, and its dB mean and dB variance can be obtained by solving the two nonlinear equations consisting of the moment generating functions; (ii) the random interference is dependent on the density of the wireless terminals, the target SNR (signal–to–noise ratio) of SUs, the cell radius, and the distance between secondary base stations (SBSs) and PUs. Then the tâtonnement process is given by a pair of stochastic differential equations (SDEs), and we determine the market equilibrium in the context of the stochastic stability.

Recently, a few notable market–based approaches have been made with considering stochastic demand of spectrum or interference [3], [4]. However, they have neither investigated the stability of the price–demand decision process nor considered the existence of random disturbances in PUs' interference–observations.

II. Interference Market Model

Let $\mathcal{L}$ be the set of all PUs, and let $|\mathcal{L}| = L$. We denote the price per interference (in dB) decided by PU $l \in \mathcal{L}$ as $\pi_l(\cdot)$, and the interference (in dB) PU l suffers from as $Y_l(\cdot)$. Then the tâtonnement process for each $l \in \mathcal{L}$ is given by the following ordinary differential equations (ODEs):

$$\dot{\pi}_l(t) = \alpha\left(Y_l(t) + \xi_l(t) - J_l^{tol}\right); \quad (1)$$

$$\dot{Y}_l(t) = -\beta \dot{\pi}_l(t) \quad (2)$$

where $\alpha > 0$ and $\beta > 0$ are the speeds of adjustment for the interference price and the interference demand, respectively, J_l^{tol} is the interference limit tolerable by PU l, and $\xi_l(\cdot)$ is the Gaussian white noise indicating the random perturbation

[1]This work however can be extended to the downlink case if we have an appropriate random interference model for downlink.

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in PU l's interference-observation[2]. On every time t, each PU l decides its price per interference according to the current amount of interference from which it suffers, and the SUs collaboratively update the amount of interference that will be imposed on each PU l according to the amount of change in the current price decided by PU l.

As shown in (2), it is assumed that the SUs can control the random interference imposed on each PU collaboratively. More strictly, by exploiting the result that Y_l is given as a Gaussian random variable [2], we assume that its statistical parameters, i.e., mean and variance, can be controlled by SUs collaboratively. Actually, it is possible to control the statistical parameters as given by Y_l by controlling the target SNR between SUs and SBS, and which however requires further investigations and is remained as one of our future works. Furthermore, for the simplicity, we assume that there is a separate mechanism of transmission power control among PUs, that is, we don't consider the interference due to other PUs' transmission.

Since the Gaussian white noise is the time derivative of the *Wiener process* denoted $W(\cdot)$, the ODEs (1) and (2) are properly interpreted into SDEs provided by

$$d\pi_l(t) = \alpha\left(Y_l(t) - J_l^{tol}\right)dt + \alpha W_l(t); \tag{3}$$

$$dY_l(t) = -\beta d\pi_l(t). \tag{4}$$

Then the equilibrium solution of the tâtonnement process is given by a pair $\left(\bar{\pi}, \bar{\mathbf{Y}}\right)$, where $\bar{\pi} = (\bar{\pi}_1, \ldots, \bar{\pi}_L)$ and $\bar{\mathbf{Y}} = \left(\bar{Y}_1, \ldots, \bar{Y}_L\right)$, and is defined in pursuit of the context of *asymptotic stability in the mean* [1] as follows:

Definition 1: If $\left(\bar{\pi}, \bar{\mathbf{Y}}\right)$ is a stable solution for the tâtonnement process (3) and (4), then $\left(\bar{\pi}, \bar{\mathbf{Y}}\right)$ is *asymptotically stable in the mean* (relative to the region H_0) if $\lim_{t\to\infty} E\left[\left|\pi(t) - \bar{\pi}\right|\right] = 0$ for $\left|\pi(0) - \bar{\pi}\right| < H_0$, and $\lim_{t\to\infty} E\left[\left|\mathbf{Y}(t) - \bar{\mathbf{Y}}\right|\right] = 0$ for $\left|\mathbf{Y}(0) - \bar{\mathbf{Y}}\right| < H_0$.

Therefore, if $\bar{\mathbf{Y}} = \mathbf{J}^{tol}$ where $\mathbf{J}^{tol} = \left(J_1^{tol}, \ldots, J_L^{tol}\right)$, the equilibrium of the interference market is yielded with the equilibrium price $\bar{\pi}$. In the next section, we give the formal proof that the tâtonnement process converges to the equilibrium.

III. Proof of the Convergence to the Market Equilibrium

Prior to describing the proof, we introduce some important properties in stochastic calculus [5]:

Lemma 2: For all real–valued stochastic process G, we have

$$E\left(\int_0^T G dW\right) = 0; \tag{5}$$

$$E\left(\left(\int_0^T G dW\right)^2\right) = E\left(\int_0^T G^2 dt\right) \tag{6}$$

where W is a Wiener process.

[2]This Gaussian white noise indicates simply an additive perturbation due to the system loss factor not related to propagation, and is distinguished from the thermal noise in wireless channels. The thermal noise is already considered in $Y_l(\cdot)$.

A. Asymptotic Stability in the Mean of Y

By showing $\lim_{t\to\infty} E\left[\left|\mathbf{Y}(t) - \mathbf{J}^{tol}\right|\right] = 0$, we can verify the asymptotic stability in the mean of Y. To this end, we first substitute (3) for $d\pi_l(t)$ in (4):

$$dY_l(t) = \alpha\beta\left(J_l^{tol} - Y_l(t)\right)dt - \alpha\beta W_l(t). \tag{7}$$

which is a *linear SDE in narrow sense* [5] whose solution is given by[3]

$$\begin{aligned} Y(t) &= e^{-\alpha\beta t}Y(0) + \int_0^t \alpha\beta J e^{-\alpha\beta(t-s)}ds \\ &+ \int_0^t -\alpha\beta e^{-\alpha\beta(t-s)}dW \\ &= e^{-\alpha\beta t}Y(0) + J\left(1 - e^{-\alpha\beta t}\right) \\ &+ \int_0^t -\alpha\beta e^{-\alpha\beta(t-s)}dW. \end{aligned} \tag{8}$$

Then, the mean and variance are, respectively,

$$\begin{aligned} E[Y(t)] &= e^{-\alpha\beta t}E[Y(0)] + J\left(1 - e^{-\alpha\beta t}\right) \\ &+ \underbrace{E\left[\int_0^t -\alpha\beta e^{-\alpha\beta(t-s)}dW\right]}_{=0 \text{ by } (5)} \\ &= e^{-\alpha\beta t}E[Y(0)] + J\left(1 - e^{-\alpha\beta t}\right), \end{aligned} \tag{9}$$

and

$$\begin{aligned} V[Y(t)] &= E\left[Y^2(t)\right] - E[Y(t)]^2 \\ &= e^{-2\alpha\beta t}E\left[Y^2(0)\right] - e^{-2\alpha\beta t}E[Y(0)]^2 \\ &+ \underbrace{E\left[\left(\int_0^t -\alpha\beta e^{-\alpha\beta(t-s)}dW\right)^2\right]}_{=E\left[\int_0^t (\alpha\beta)^2 e^{-2\alpha\beta(t-s)}ds\right] \text{ by } (6)} \\ &= e^{-2\alpha\beta t}V[Y(0)] + \frac{\alpha\beta}{2}\left(1 - e^{-2\alpha\beta t}\right). \end{aligned} \tag{10}$$

If $Y(0) \sim \mathcal{N}(\mu(0), \sigma^2(0))$, the above SDE is turned out to be the *Langevine equation* that is the one of the models of the motion of a Brownian particle, and it is shown that the distribution of $Y(t)$ is yielded for $t \geq 0$ [5]

$$\mathcal{N}(E[Y(t)], V[Y(t)]). \tag{11}$$

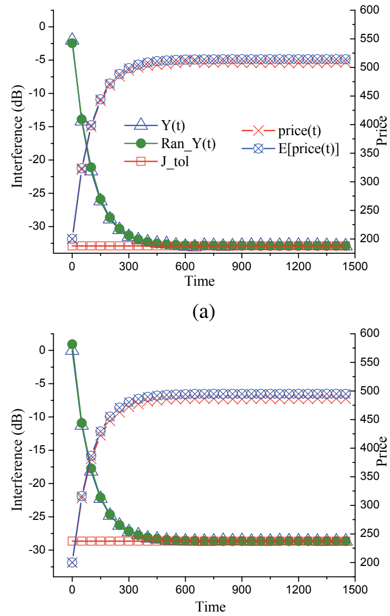
Therefore, we see that the tâtonnement process is asymptotically stable in the mean of Y (relative to the region H_0) since $\lim_{t\to\infty} E\left[\left|Y(t) - J\right|\right] = 0$ for $\left|Y(0) - J\right| < H_0$.

B. Asymptotic Stability in the Mean of π

In order to examine the stability of π, substituting (8) for $Y(t)$ in (3), we have

$$\begin{aligned} d\pi(t) &= \alpha(e^{-\alpha\beta t}Y(0) - Je^{-\alpha\beta t} \\ &+ \int_0^t -\alpha\beta e^{-\alpha\beta(t-s)}dW)dt + \alpha dW, \end{aligned} \tag{12}$$

[3]Henceforth, we omit the index l, and let $J_l^{tol} \equiv J$ for simpler notations in the equations



(a)

(b)

Fig. 1. The transition of the random interference and the price in two sampled PUs: ‘Y(t)’, ‘Ran_Y(t)’, and ‘J_tol’ are correspondent to left Y–axis, and ‘price(t)’ and ‘E[price(t)]’ are correspondent to right Y–axis.

and, assuming $\pi(0)$ is a constant, its solution is given by

$$\begin{aligned} \pi(t) &= \pi(0) + \frac{1}{\beta}\left(1 - e^{-\alpha\beta t}\right)(Y(0) - J) \\ &+ \int_0^t \int_0^\tau -\alpha^2 \beta e^{-\alpha\beta(t-s)} dW d\tau \\ &+ \int_0^t \alpha dW \end{aligned} \quad (13)$$

Then the mean value of $\pi(t)$ is

$$\begin{aligned} E[\pi(t)] &= \pi(0) + \frac{1}{\beta}\left(1 - e^{-\alpha\beta t}\right)(Y(0) - J) \\ &+ \underbrace{E\left[\int_0^t \int_0^\tau -\alpha^2 \beta e^{-\alpha\beta(t-s)} dW d\tau\right]}_{=0 \text{ by } (5) \text{ and } (6)} \\ &+ \underbrace{E\left[\int_0^t \alpha dW\right]}_{=0 \text{ by } (5)} \\ &= \pi(0) + \frac{1}{\beta}\left(1 - e^{-\alpha\beta t}\right)(Y(0) - J), \end{aligned} \quad (14)$$

and clearly,

$$E[\pi(\infty)] = \pi(0) + \frac{1}{\beta}(Y(0) - J). \quad (15)$$

As done in the case of Y, it is observed that the tâtonnement process is asymptotically stable in the mean of π (relative to the region H_o) since $\lim_{t\to\infty} E[|\pi(t) - J|] = 0$ for $|\pi(0) - J| < H_0$.

IV. Numerical Results

We perform a numerical experiment in order to verify the stability of the tâtonnement process (1) and (2), and illustrate that the random interference generated by (11) and the expected price given by (14), are asymptotically close to the stable solutions of the tâtonnement process on every time t. First, we generate a cognitive network of 10 PUs and 10 SBSs within a 2000m×2000m square. Then we generate the initial statistical parameters of $Y_l(0)$ for each $l \in \mathcal{L}$, that is, $E[Y_l(0)]$ and $V[Y_l(0)]$ utilizing the random interference model developed in [2], and generate $Y_l(0)$ following $Y_l(0) \sim \mathcal{N}(E[Y_l(0)], V[Y_l(0)])$. We let each l have random J_l^{tol} generated uniformly between -40dB and -20dB. In addition, we assume that there is no change in the average number of transmitting SUs during the tâtonnement process. We initialize $\pi_l(0)$ to 200, and set the average number of transmitting SUs in each SBS as 20. Then, we perform the tâtonnement process 10^5 times, and each tâtonnement is proceeded from $t = 0$ until $t = 1500$ with $\alpha = \beta = 0.1$. On each t, we measure the transition of the average solutions of the tâtonnement process (plotted with ‘Y(t)’ and ‘price(t)’ on the graph in Fig.1) and the transition of the average interference and the expected price (plotted with ‘Ran_Y(t)’ and ‘E[price(t)]’ on the graph, respectively) in two sampled PUs. As shown in the graph, it is observed that the solutions of the tâtonnement process are quite close to the random interference and the expected price in most of the time instances. Moreover, both the random interference and the interference solution of the tâtonnement process, that is, ‘Y(t)’ and ‘Ran_Y(t)’, approach to the interference limit ‘J_tol’ quite closely after around $t = 400$.

As a result, we conclude that the tâtonnement process is easily expected to be asymptotically stable in the mean, and the random interference process and the expected price model given by (11) and (14), respectively, behave approximately well toward the tâtonnement process.

V. Conclusion

In this letter, we model a cognitive radio network as an interference market, and investigate the market equilibrium asymptotically stable in the mean. To this end, we express the tâtonnement process as a pair of SDE, and prove the stability using some properties of stochastic calculus. The numerical results illustrate the stability of the tâtonnement process and show that the stable solutions (i.e., price and interference) are quite close to the market equilibrium.

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