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Algebraic invariants of links and 3-manifolds

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Master of Science

Submission date: December 2015

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ABSTRACT

The goal of this thesis is to describe certain algebraic invariants of links, and try to modify them to obtain invariants of 3-manifolds. Racks and quandles are algebraic structures that were invented to give invariants of knots and links. They generalise the classical colouring invariants, and a rack or quandle can be associated to any link, known as its fundamental rack or quandle. In this thesis we explain how to modify the construction of the fundamental rack to obtain an invariant of 3-manifolds, making use of the fact that every 3-manifold can be obtained by integral Dehn surgery on a link in the 3-sphere. Finally, we show how to distinguish the 3-sphere from the Poincaré homology sphere using this invariant.

SAMMENDRAG

Målet med denne oppgaven er å beskrive visse algebraiske invarianter på lenker, og prøve å modifisere dem for å få invarianter på 3-mangfoldigheter. *Racks* og *Quandles* er algebraiske strukturer som ble laget for å gi invarianter på knuter og lenker. De generaliserer den klassiske fargeleggingsinvarianten og en *rack* eller *quandle* kan assosieres med enhver lenke, kjent som *fundamentalracken* eller *-quandlen* til lenka. I denne oppgaven forklarer vi hvordan *fundamentalracks* kan modifiseres slik at de blir invarianter på 3-mangfoldigheter, ved å bruke at enhver 3-mangfoldighet kan dannes ved Dehn-kirurgi på 3-sfæren. Til slutt viser vi hvordan vi kan benytte denne invarianten til å skille 3-sfæren fra Poincaré-sfæren.

ACKNOWLEDGEMENT

This thesis marks the end of my time as a student at the LUR-program and the Department of Mathematical Sciences at NTNU.

Firstly, I would like to thank Markus Szymik, my supervisor, for excellent supervision. He has helped me with everything from the technical details of proofs to how a thesis should be written up in a nice and neat way. I would also like to thank Richard Williamson for his engagement in the thesis, suggesting an interesting topic and continuous guidance. A big thank you goes to my friends in Matteland for brightening up the days, and especially to Håvard and Gunnar, who in addition to nice lunch breaks and Hanabi, have helped me with technical details in my thesis.

I would also like to thank all the nice people I met in Vektorprogrammet, my time as a student would not have been the same without you! Lastly, I would like to thank my family for the support during my time at NTNU, and Gard for supporting me through frustrating days and proof reading.

Reidun Persdatter Ødegaard

01.12.2015

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INTRODUCTION

The theory of 3-manifolds is rich and complicated, and 3-manifolds can be hard to distinguish. In this thesis, we construct a simple invariant of 3-manifolds which allows us to distinguish the 3-sphere from the Poincaré homology sphere, and which can be used to distinguish many other 3-manifolds.

Our approach relies upon several foundational theorems concerning 3-manifolds. The first of these, often attributed to Lickorish and Wallace [5] [9], states that every closed, connected, orientable 3-manifold can be obtained by Dehn surgery on links with integral framing in the 3-sphere, and it is therefore possible to study the links presenting 3-manifolds instead of the 3-manifolds themselves. The second foundational theorem, due to Kirby [4] states that, given a pair of 3-manifolds M and N , and links L_M and L_N with respect to which we obtain M and N via integral Dehn surgery in the 3-sphere, we have that M and N are homeomorphic if and only if L_M can be obtained from L_N by a finite sequence of a pair of moves, now known as Kirby moves.

One of the Kirby moves is global, and in order to study the link diagrams presenting the links, local moves are more convenient. Fenn and Rourke [1] demonstrated that the Kirby moves can be replaced by local moves known as blow ups and blow downs, with an arbitrary number of strands. Building upon this, Martelli [6] was recently able to prove that a finite collection of local moves suffice. This last result is the point of departure for the construction of the invariant we shall describe.

The algebraic structures *racks* were invented to obtain an invariant of knots. They

generalise the classical n -colourability invariant. Joyce [3] was the first to explore racks in the setting of knot theory, introducing the construction of the fundamental quandle of a link. Quandles are racks which satisfy an additional axiom, which ensures that the fundamental quandle of a link is invariant under the Reidemeister moves. Fenn and Rourke [2] laid the foundations for the theory of racks in the setting of knot theory, associating to a framed link a rack known as its fundamental rack, which is invariant under the framed Reidemeister moves. In constructing our invariant of 3-manifolds, we modify the definition of the fundamental rack of a framed link in such a way as to ensure that it is invariant under a finite collection of local Kirby moves to which Martelli's theorem applies.

In this thesis we begin by recalling the theory of knots, and that of racks as an invariant on knots. We first describe knot diagrams, their framing and the colouring invariant in Chapter 1. Thereafter we describe the algebraic structure of racks and quandles in Chapter 2, before we discuss the construction of the fundamental rack of a framed link in Chapter 3. We use the fundamental racks for the unknot and trefoil to show that they are not isotopic.

Next, we begin working towards a modification of the construction of the fundamental rack to obtain an invariant of 3-manifolds. In Chapter 4, we recall the notion of Dehn surgery on a link in S^3 , before describing the Kirby moves. Following this, we define the local moves known as blow ups and blow downs, and recall the theorem of Fenn and Rourke concerning these. We then state Martelli's theorem and several corollaries, and prove them.

In Chapter 5, we construct our invariant of 3-manifolds, associating to a framed link a rack which we refer to as its Fenn-Rourke-local fundamental rack. We prove that the Fenn-Rourke-local fundamental rack of a framed link is invariant under the Martelli moves of one of the corollaries of Chapter 4. Finally, we use our invariant to distinguish the 3-sphere from the Poincaré homology sphere.

CHAPTER 1

LINKS

In this chapter we will describe link diagrams and framed link diagrams, and recall the classical n -colourability link invariant.

1.1 Link diagrams

Links are geometric objects in three dimensional space, often the 3-sphere S^3 or the real 3-dimensional space \mathbb{R}^3 . To present links we use link diagrams. A link diagram is the projection p of a link down on \mathbb{R}^2 , such that for every point x in \mathbb{R}^2 with the property that $p^{-1}(x)$ is not unique, there is a crossing in the link diagram. We know from Reidemeister's theorem [7] that a pair of links are isotopic if and only if we can obtain the diagram of one from the diagram of the other by a finite sequence of the moves R1, R2 and R3 depicted Figure 1.1. The Reidemeister moves are local, which means that one can one “draw a fence” around the arcs and crossings of a link diagram affected by the move in such a way that no other arcs or crossings are inside the fence. The knot diagram inside the fence can be replaced by another knot diagram, but the endpoints of the knots have to be the same before and after the move.

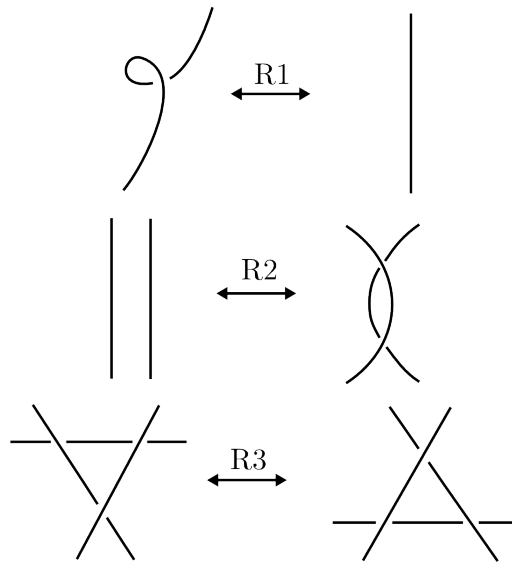


Figure 1.1: The Reidemeister moves.

1.2 Colourability

Colourability of a link is defined in the following way. We prove in Proposition 1.2.3 that whether or not a link is n -colourable is invariant under the Reidemeister moves.

Definition 1.2.1. A link L is n -colourable if to every arc one can assign an integer such that

1. at each crossing

$$\begin{array}{c} a \\ | \\ \hline c \\ | \\ b \end{array}$$

we have that $a + b \equiv 2c \pmod{n}$,

2. not every arc is assigned the same number, i.e. no link is 1-colourable.

Example 1.2.2. The trefoil is 3-colourable. If we assign the integers 0, 1 and 2 to the three arcs in the trefoil as in the following figure, the crossings satisfy the condition

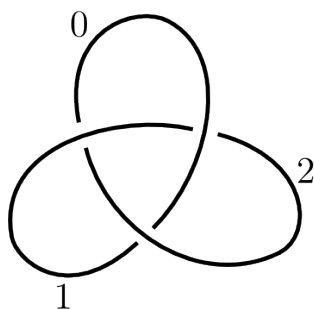
for 3-colourability, since the equations

$$0 + 2 = 2 \cdot 1 \pmod{3}$$

$$1 + 0 = 2 \cdot 2 \pmod{3}$$

$$2 + 1 = 2 \cdot 0 \pmod{3}$$

are all satisfied.



Proposition 1.2.3. *Let n be an integer. Whether or not a link is n -colourable is invariant under the Reidemeister moves.*

Proof. For each of the Reidemeister moves, we assign a letter to the arcs involved in the move, and check that, after applying the move, we can still find a valid labelling.

For the first Reidemeister move, R1, we assign the letter x to one of the arcs, as shown in Figure 1.2. The other arc has to be assigned $2x - x = x$ according to the definition of n -colourability. Thus The R1 does not affect our ability to find a valid labelling.

For the R2 move, we label the arcs on the left side of the move x and y , as in Figure 1.3. On the right hand side, the arc y is unchanged. The arc x goes beneath the arc labelled y , and the crossing gives that the part between the two crossings is labelled $2y - x$. This arc goes underneath y once again, and this crossing yields that the arc to the top right of Figure 1.3 is labelled $2y - (2y - x) = x$. Hence the R2 move does not affect our ability to find a valid labelling.

The labelling of the arcs before and after the R3 move is shown in Figure 1.4. We see that that R3 move does not affect our ability to find a valid labelling.

Analogous proofs can be given for all other combinations of over and under crossings for the Reidemester moves. \square

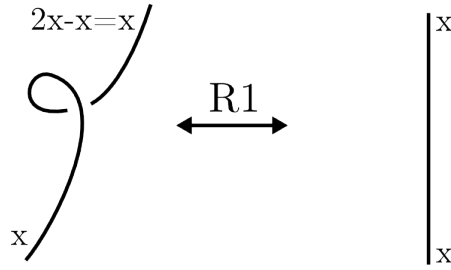


Figure 1.2: The labelling of the endpoints are unchanged under the R1 move.

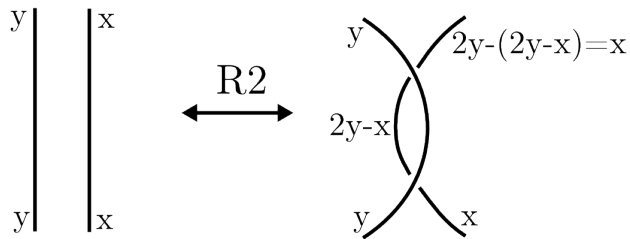


Figure 1.3: The labelling of the endpoints are unchanged under the R2 move.

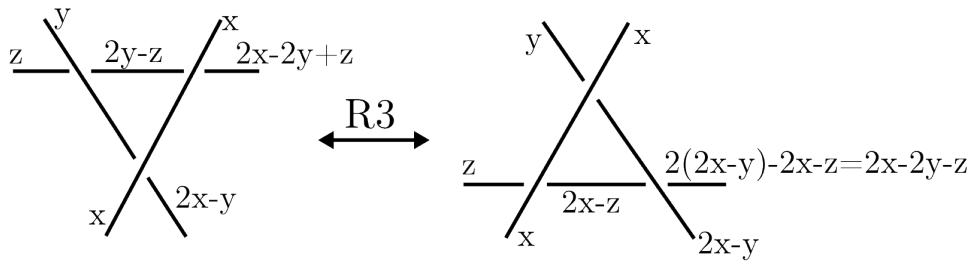
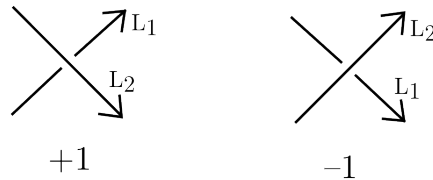


Figure 1.4: The R3 move does not affect the labelling of the endpoints.

1.3 Linking number and framing

The linking number of a link L can be defined in various ways. We will use the following from Saveliev [8].

Definition 1.3.1. Let L_1 and L_2 be disjoint oriented knots in S^3 , and consider the projection of $L_1 \cup L_2$, i.e. its link diagram. Each time L_1 crosses under L_2 , assign the crossing the integer $+1$ or -1 , according to which of the configuration of orientations in the following figure holds at the crossing. The *linking number*, $\text{lk}(L_1, L_2)$, of L_1 and L_2 is the sum over all crossings of L_1 under L_2 .



The linking number is a tool in determining the canonical meridian-longitude pair for a knot k . When $\text{lk}(k, l) = 0$, the choice of longitude, l , gives the canonical meridian-longitude pair.

Example 1.3.2. The longitude l which runs alongside the trefoil k as in left hand side of Figure 1.5 has the property of $\text{lk}(k, l) = 3$. Thus this choice of longitude does not determine the canonical meridian-longitude pair. If the longitude is twisted three times around the trefoil, as in the right hand picture of Figure 1.5, then $\text{lk}(k, l) = 0$, and this choice of longitude gives the canonical meridian-longitude pair.

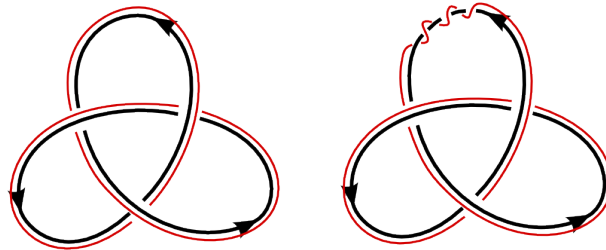


Figure 1.5: The choice of longitude as in the right hand diagram gives the canonical meridian-longitude pair. The longitude is drawn in red.

We say that the longitude *frames* the knot, and the linking number of the knot and the longitude defines the framing. If $\text{lk}(k, l) = n$, we say that the knot has framing n .

CHAPTER 2

ALGEBRAIC DESCRIPTION OF RACKS AND QUANDLES

In this chapter we introduce the algebraic structure of a rack. We will use this in the next chapter to show that racks can be used to distinguish framed links.

2.1 The algebra of racks and quandles

The following definition of a rack is adapted from Fenn and Rourke [2]. The notation is due to Joyce [3], whose work concerns *quandles*, which are racks which satisfy an additional axiom, and which we will also describe in this section.

Definition 2.1.1. A *rack* is a set X together with a binary operation $\triangleright : X \times X \longrightarrow X$, satisfying the following axioms.

1. For every $y, z \in X$, there is a unique element $x \in X$ such that $x \triangleright y = z$.
2. For all $x, y, z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

As a consequence of the first axiom, the map $x \mapsto x \triangleright y$ is a bijection for any given $y \in X$. The inverse map \triangleright^{-1} has the property that $(x \triangleright y) \triangleright^{-1} y = x$. Here \triangleright^{-1} is a right inverse. To prove that it is also a left inverse, i.e. that $(x \triangleright^{-1} y) \triangleright y = x$, we use the first axiom and set $x = x' \triangleright y$. We then have that

$$(x \triangleright^{-1} y) \triangleright y = ((x' \triangleright y) \triangleright^{-1} y) \triangleright y.$$

Since \triangleright^{-1} is a right inverse, $(x' \triangleright y) \triangleright^{-1} y = x'$, and we obtain that

$$((x' \triangleright y) \triangleright^{-1} y) \triangleright y = x' \triangleright y = x.$$

The binary operations $x \triangleright y$ and $x \triangleright^{-1} y$ can be read “ y acts on x ”.

Example 2.1.2. The group $\mathbb{Z}/3\mathbb{Z}$ can be equipped with the structure of a rack in the following way. Let \triangleright and \triangleright^{-1} be given by $x \triangleright y = 2y - x$ and $x \triangleright^{-1} y = 2y - x$. We check that the axioms for a rack are satisfied.

The left side of the first axiom, $(x \triangleright y) \triangleright^{-1} y$, gives

$$2y - (2y - x) = 2y - 2y + x = x$$

in $\mathbb{Z}/3\mathbb{Z}$. Thus the first axiom, $(x \triangleright y) \triangleright^{-1} y = x$, is satisfied in $\mathbb{Z}/3\mathbb{Z}$.

The left side of the second axiom, $(x \triangleright y) \triangleright z$, gives

$$(x \triangleright y) \triangleright z = 2z - (2y - x) = 2z - 2y + x$$

in $\mathbb{Z}/3\mathbb{Z}$. The right side of this axiom, $(x \triangleright z) \triangleright (y \triangleright z)$, gives

$$(x \triangleright z) \triangleright (y \triangleright z) = 2(2z - y) - (2z - x) = 4z - 2y - 2z + x = 2z - 2y + x$$

in $\mathbb{Z}/3\mathbb{Z}$. Thus the third axiom, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$, is satisfied in $\mathbb{Z}/3\mathbb{Z}$. This rack is referred to as the *core rack* in Example 3 of Fenn and Rourke [2], and we will denote it by $Core(\mathbb{Z}/3\mathbb{Z})$.

Definition 2.1.3. A *quandle* is a set X together with a binary operation $\triangleright : X \times X \longrightarrow X$, satisfying the following axioms.

1. For all $x \in X$, $x \triangleright x = x$.
2. For every $y, z \in X$, there is a unique element $x \in X$ such that $x \triangleright y = z$.

3. For all $x, y, z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

Example 2.1.4. The core rack $Core(\mathbb{Z}/3\mathbb{Z})$ from Example 2.1.2 is in fact a quandle, since the choice of \triangleright as in Example 2.1.2, in addition to the second and third axiom, satisfies the first axiom,

$$x \triangleright x = 2x - x = x,$$

and hence equips $\mathbb{Z}/3\mathbb{Z}$ with the structure of a quandle.

Every abelian group can be equipped with the structure of a quandle by the same choice of binary operation \triangleright as in Example 2.1.2.

2.2 Morphisms of racks

Fundamental racks, which will be described in Chapter 3, are large and complicated structures, and in order to make them easier to investigate we typically look for morphisms from them to racks of simpler structure.

Definition 2.2.1. Let (X, \triangleright) and (Y, \triangleright) be racks. A *morphism* from (X, \triangleright) to (Y, \triangleright) is a map φ from X to Y such that $\varphi(x \triangleright y) = \varphi(x) \triangleright \varphi(y)$ for all x and y which belong to X .

A rack can be constructed from generators and relations between the generators. Let $\langle G \mid J \rangle$ denote a rack, where G is the set of generators and J is the set of relations on the generators. The rack $\langle G \mid \emptyset \rangle$ is the free rack generated by G . Let R be another rack, then there is a set of morphisms $\tilde{\varphi}$ from $\langle G \mid \emptyset \rangle$ to R , $Mor(\langle G \mid \emptyset \rangle, R)$. This set is in natural bijection to the set of maps φ from G to R , $Mor(G, R)$. The bijection is given by restricting $\tilde{\varphi}$ to the set of generators. The two maps $f, g : J \rightarrow \langle G \mid \emptyset \rangle$ send both sides of the relations in J to $\langle G \mid \emptyset \rangle$. For instance if j is an element of J , then $f(j) = x \triangleright y$ and $g(j) = z$. Then the set of morphisms φ from $\langle G \mid J \rangle$ to R is

$$Mor(\langle G \mid J \rangle, R) = \{\varphi : G \rightarrow R \mid J \xrightarrow{f} \langle G \mid \emptyset \rangle \xrightarrow{\tilde{\varphi}} R, \tilde{\varphi} \circ g = \varphi \circ g\},$$

where $\tilde{\varphi} \circ f = \varphi \circ f$.

CHAPTER 3

FUNDAMENTAL RACKS

In this chapter we will describe the construction of a fundamental rack, and show that fundamental racks are an invariant of oriented framed links. As mentioned in the introduction, quandles define invariants of links. However, it do not keep track of framings since the R1 move (corresponding to the first axiom in Definition 2.1.3) changes framings. We therefore only allow the moves R2 and R3 in order to keep track of framings, and show that racks are invariant under these moves.

3.1 The fundamental rack for a framed link

To any link diagram we can associate the algebraic structure of a rack which is generated by the arcs of the link diagram where the binary operation is the relation between the arcs in a crossing. In order to equip a link diagram with the structure of a rack, we need a relationship between racks and links. This relationship was established by Joyce [3] in 1982. We assign a letter to each arc in a link diagram, and each crossing of the link diagram is a relation between the arcs, as seen in Figure 3.1.

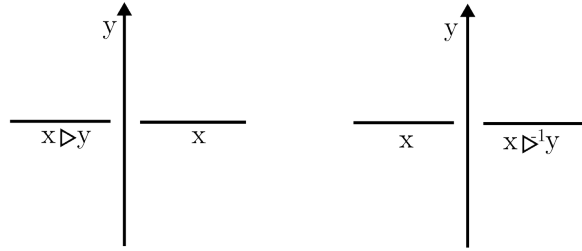


Figure 3.1: Relationship between the link diagrams and the binary operation \triangleright and its inverse.

Definition 3.1.1. The *fundamental rack* of a link diagram is the quotient of the free rack on the arcs of this diagram in which we impose the relations indicated in Figure 3.1 for each crossing.

Example 3.1.2. Since the link diagram of the *unknot* contains only one arc, and the fundamental rack for the unknot is generated by only one generator, x . Since there are no crossings in the link diagram for the unknot, there are no relations between the arcs. The fundamental rack for the link diagram of the unknot is presented by $R_U = \langle x \mid \emptyset \rangle$. In other words the fundamental rack for the unknot is the free rack generated by x .

Example 3.1.3. The following link diagram for the *trefoil* has three arcs. We denote the three generators of the fundamental rack for the trefoil corresponding to these arcs by x , y and z . For the orientation chosen in Figure 3.2, the relations from the crossings are $x \triangleright y = z$, $y \triangleright z = x$ and $z \triangleright x = y$. The fundamental rack R_T for this diagram of the trefoil is thus

$$R_T = \langle x, y, z \mid x \triangleright y = z, y \triangleright z = x, z \triangleright x = y \rangle.$$

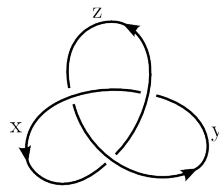


Figure 3.2:

Proposition 3.1.4. *If two diagrams L and L' for oriented framed links can be obtained from one another by a finite sequence of Reidemeister moves of type R2 and R3, then the fundamental racks for L and L' are isomorphic.*

Proof. By definition of the fundamental rack of an oriented framed link, the R2 and R3 Reidemeister moves have the effect on its generators and relations depicted in Figure 3.3. In order for the fundamental rack to be invariant under these moves, we require that

$$x = (x \triangleright y) \triangleright^{-1} y,$$

and

$$(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z).$$

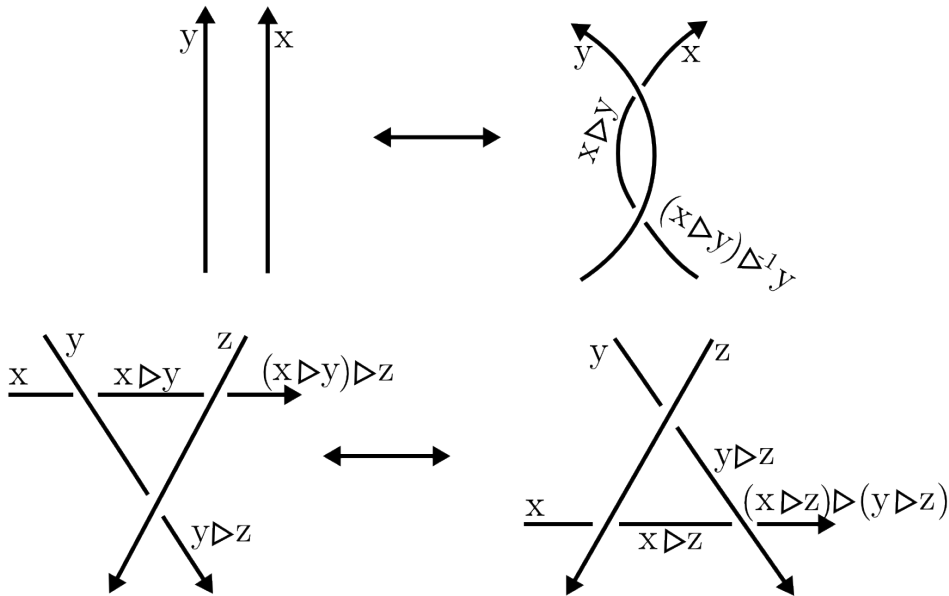


Figure 3.3: The arcs and their respective labels before and after an R2 and R3 move.

This is precisely ensured by the axioms for a rack. Hence, the fundamental racks for two link diagrams presenting the same oriented framed links, are isomorphic. \square

Proposition 3.1.5. *For every n -colourable link L there is a morphism f of racks from the fundamental rack R_L for a diagram of L to the core rack $\text{Core}(\mathbb{Z}/n\mathbb{Z})$.*

Proof. Generalising Example 2.1.2, we have that defining \triangleright by $x \triangleright y = 2y - x$ equips $\mathbb{Z}/n\mathbb{Z}$ with the structure of a rack. Let f be the map from the set of generators of R_L to $\mathbb{Z}/n\mathbb{Z}$ which assigns to an arc of L the integer with which it is labelled in a given n -colouring of L . The first condition in the definition of n -colourability of L exactly ensures that this map defines a morphism of racks from R_L to $\text{Core}(\mathbb{Z}/n\mathbb{Z})$. \square

3.2 The unknot and the trefoil are not isotopic

In this section we will give an example of how racks can be used to distinguish framed links. In particular, we will show that the unknot and the trefoil are not isotopic for any choice of framing.

Recall that in Example 3.1.2 and 3.1.3 we showed that the fundamental racks for the link diagrams of the trefoil and unknot are

$$R_U = \langle x \mid \emptyset \rangle$$

and

$$R_T = \langle x, y, z \mid x \triangleright y = z, y \triangleright z = x, z \triangleright x = y \rangle.$$

Lemma 3.2.1. *There is a surjective morphism of racks from the fundamental rack for the trefoil R_T to the core rack $\text{Core}(\mathbb{Z}/3\mathbb{Z})$.*

Proof. From Example 9 by Fenn and Rourke [2], we know that every free rack has the property that any function $S \rightarrow R$, where S is a set and R is any rack, extends uniquely to a morphism of racks $\langle S \mid \emptyset \rangle \rightarrow R$. Hence every map $\varphi : \{x, y, z\} \rightarrow \text{Core}(\mathbb{Z}/3\mathbb{Z})$ extends uniquely to a morphism $\tilde{\varphi} : \langle \{x, y, z\} \mid \emptyset \rangle \rightarrow \text{Core}(\mathbb{Z}/3\mathbb{Z})$ of racks because $\langle \{x, y, z\} \mid \emptyset \rangle$ is free. If φ is bijective, then $\tilde{\varphi}$ is surjective.

In $\text{Core}(\mathbb{Z}/3\mathbb{Z})$ the set of relations on R_T is satisfied since the three relations

$$x \triangleright y = z, y \triangleright z = x \text{ and } z \triangleright x = y$$

all translate to $x + y + z = 0$, which is satisfied in $\text{Core}(\mathbb{Z}/3\mathbb{Z})$ if x , y and z are all different or equal.

This implies that for every bijective φ , the morphism $\tilde{\varphi}$ factors through R_T .

$$\begin{array}{ccc}
\langle \{x, y, z\} \mid \emptyset \rangle & \xrightarrow{\tilde{\varphi}} & \text{Core}(\mathbb{Z}/3\mathbb{Z}) \\
\downarrow & \nearrow & \\
R_T & \xrightarrow{\tilde{\varphi}} &
\end{array}$$

Hence $\tilde{\varphi}$ is surjective. □

Lemma 3.2.2. *There is no surjective morphism of racks from the fundamental rack for the unknot R_U into the core rack $\text{Core}(\mathbb{Z}/3\mathbb{Z})$.*

Proof. Let $\tilde{\varphi}$ be a morphism of racks $R_U = \langle \{x\} \mid \emptyset \rangle \rightarrow \text{Core}(\mathbb{Z}/3\mathbb{Z})$ and φ be the corresponding map $\{x\} \rightarrow \text{Core}(\mathbb{Z}/3\mathbb{Z})$. The fundamental rack for the unknot is generated by $\{x\}$ with no relations, i.e. the fundamental rack for the unknot is the free rack generated by $\{x\}$. From Example 9 by Fenn and Rourke [2], φ extends uniquely to $\tilde{\varphi}$ in the following diagram.

$$\begin{array}{ccc}
\langle \{x\} \mid \emptyset \rangle & \xrightarrow{\tilde{\varphi}} & \text{Core}(\mathbb{Z}/3\mathbb{Z}) \\
\uparrow & \nearrow & \\
\{x\} & \xrightarrow{\varphi} &
\end{array}$$

Since R_U is generated by one generator, the image of $\tilde{\varphi}$ is generated by one generator. In the rack $\text{Core}(\mathbb{Z}/3\mathbb{Z})$ a subrack generated by one element a is $\{a\}$, and the map $\tilde{\varphi}$ is constant, and therefore not surjective. □

Corollary 3.2.3. *The fundamental racks for the trefoil and unknot are not isomorphic.*

Proof. If i were an isomorphism from R_U to R_T and $\tilde{\varphi}$ were the surjective morphism from R_T to $\text{Core}(\mathbb{Z}/3\mathbb{Z})$ of Lemma 3.2.1, then $\tilde{\varphi} \circ i : R_U \rightarrow \text{Core}(\mathbb{Z}/3\mathbb{Z})$ would be surjective. This is a contradiction since there is no surjective morphism $R_U \rightarrow \text{Core}(\mathbb{Z}/3\mathbb{Z})$ by Lemma 3.2.2. □

Corollary 3.2.4. *The framed trefoil and unknot are not isotopic.*

Proof. If U were isotopic to T , then Reidemeister's theorem [7] would imply that U and T can be obtained from one another by a finite sequence of Reidemeister moves.

Proposition 3.1.4 then gives that $R_U \cong R_T$, in contradiction to Corollary 3.2.3. Hence U and T are not isotopic. \square

CHAPTER 4

COMBINATORIAL DESCRIPTION OF 3-MANIFOLDS

In order to study 3-manifolds, several tools have been developed to simplify their presentations, and the goal of this chapter is to describe a simple diagrammatic presentation of 3-manifolds. Lickorish and Wallace's theorem [5][9] states that every closed orientable 3-manifold can be obtained by surgery on a link with integral framing in S^3 . This theorem allows us to study the links presenting the 3-manifolds, instead of the 3-manifolds themselves. Kirby [4] stated a theorem where he described two different moves, one local and one global, on a framed link which do not change the 3-manifold. The theorem states that if and only if one can obtain one framed link from another by a finite sequence of Kirby moves, then the two links represent the same 3-manifold. The Kirby moves are described in Section 4.2. Fenn and Rourke [1] further developed Kirby's theorem to only involve local moves, but they were infinitely many. Martelli [6] was able to define five local moves, which are sufficient to cover all of the infinite number of blow ups and blow downs. In this chapter we will briefly describe the surgery on links from which we can obtain any 3-manifold. Thereafter we will give a short description of Kirby's theorem as well as and Fenn and Rourke's theorem. Martelli's theorem will be stated and proved in the last section of this chapter.

4.1 Surgery on links

Let k be a knot in a closed orientable 3-manifold M . The knot can be thickened to obtain its tubular neighbourhood $N(k)$. The boundary of $N(k)$ is homeomorphic to the 2-torus, T^2 , and the closure of $N(k)$ is the solid torus $D^2 \times S^1$. The boundary of $M \setminus D^2 \times S^1$ is also the 2-torus, T^2 . The tubular neighbourhood of the link, $N(k)$, can be cut out of the 3-manifold M , and then be glued back in again to obtain a new 3-manifold. We say that the new 3-manifold is obtained by surgery on L . The homomorphism along which the tubular neighbourhood is glued determines the 3-manifold obtained by this operation. The tubular neighbourhood is glued in along the longitude of T^2 , and therefore we say that the framing of the knot determines the 3-manifold obtained by surgery. We refer the reader to Chapter 2 of Saveliev [8] for a detailed description of Dehn surgery.

Theorem 4.1.1. *Every 3-manifold can be obtained by an surgery on links with integral framing in the 3-sphere [5][9].*

4.2 Kirby moves

Since any closed orientable 3-manifold can be obtained from surgery on framed links in S^3 , it is of interest to determine whether two links give the same 3-manifold. In order to do this Kirby [4] defined two operations that do not change the 3-manifold. These two operations are called *the Kirby moves*.

Definition 4.2.1 (Kirby moves). The following two moves do not change the 3-manifold.

- \mathcal{O}_1 Add or delete a circle with framing ± 1 which belongs to a 3-ball that does not intersect with any of the components of the framed link \mathcal{L} .
- \mathcal{O}_2 Let L_1 and L_2 be two link components of the link $\mathcal{L} = L_1 \cup L_2$ with framing n_1 and n_2 respectively. Add L_1 to the framing of L_2 by gluing in a band between L_1 and the framing of L_2 . The new link is $\mathcal{L} = L_{\#} \cup L_2$ with framing $n_1 + n_2 + 2\text{lk}(L_1, L_2)$, where $L_{\#}$ is L_1 together with the framing of L_2 and the band glued in between L_1 and the framing of L_2 .

The first Kirby move is called a blow up when adding a circle with framing ± 1 and

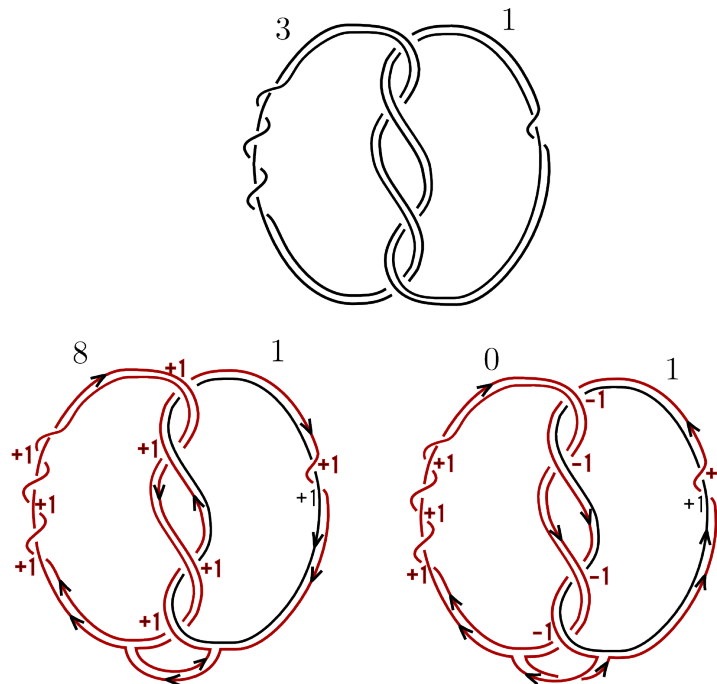


Figure 4.1: Kirby move \mathcal{O}_2 for a link \mathcal{L} with two components L_1 and L_2 , where L_1 has framing 3 and L_2 has framing 1. By choosing the orientation on L_1 clockwise, the two possible orientations for L_2 are shown for the two possible ways to glue in the band. The new component $L_{\#}$ is coloured in red and the framing is calculated with the red numbers.

blow down when deleting a circle with framing ± 1 . The second Kirby move is a handle slide, sliding L_1 around L_2 . It is necessary to know the orientation on L_1 and L_2 to compute the linking number of L_1 and L_2 . Choose orientation on L_1 , the orientation on L_2 then depends on how the band between L_1 and the framing of L_2 is glued in. The link component L_2 is oriented in such a way that together with L_1 it defines an orientation on $L_{\#}$. See figure 4.1 for an example of a handle slide. The handle slide move is an example of a move which is not local.

Theorem 4.2.2. *Two 3-manifolds are homeomorphic if and only if the links presenting the 3-manifolds by surgery are connected by a finite sequence of Kirby moves [4].*

4.3 Fenn and Rourke's theorem

The Kirby moves are global moves which affect the entire manifold. In order to find local moves, Fenn and Rourke [1] showed that blow ups and blow downs with an arbitrary number of strands are sufficient to deform one link to another link, both presenting the same 3-manifold.

Theorem 4.3.1. *Two 3-manifolds are homoeomorphic if and only if the links presenting the 3-manifolds by surgery are connected by a finite sequence of blow ups and blow downs with an arbitrary number of strands [1].*

A blow down with three strands is presented in Figure 4.2, while a picture of a full clockwise twist is presented in Figure 4.3.

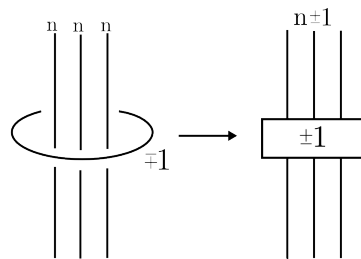


Figure 4.2: A topological blow down for three strands. The framing of the strands change by ± 1 when a circle with framing ∓ 1 is deleted. The box with the sign ± 1 means that the strands have been twisted one full counterclockwise twist or clockwise twist, respectively.

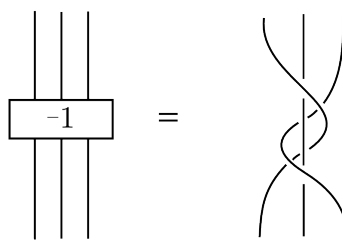


Figure 4.3: One full clockwise twist with three strands.

4.4 Martelli's theorem

While Kirby's moves are not local, the local moves suggested by Fenn and Rourke are infinitely many. Martelli [6] defined five local moves which, together with their inverses, are sufficient to cover all blow downs and blow ups with an arbitrary number of strands. These five Martelli moves are presented in Figure 4.4.

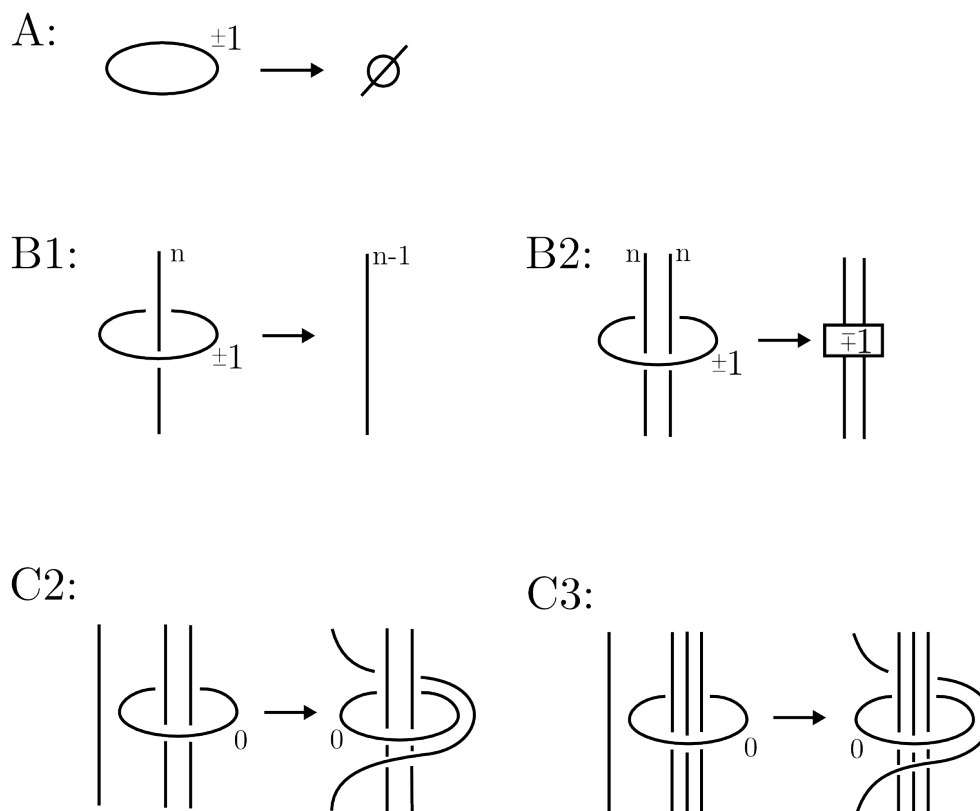


Figure 4.4: The five local moves defined by Martelli.

Theorem 4.4.1. *Let \mathcal{L} and \mathcal{L}' be two framed links in S^3 such that the 3-manifolds obtained by surgery on \mathcal{L} and \mathcal{L}' are homeomorphic. Then the link \mathcal{L}' can be obtained from the link \mathcal{L} through a finite sequence of the moves from Figure 4.4 and their inverses.*

Proof. It is possible to create a zero framed Hopf link with these moves, starting

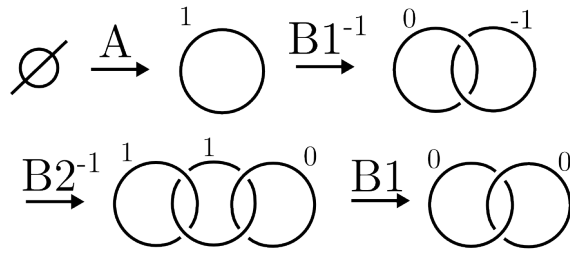


Figure 4.5: Creating a zero framed Hopf link.

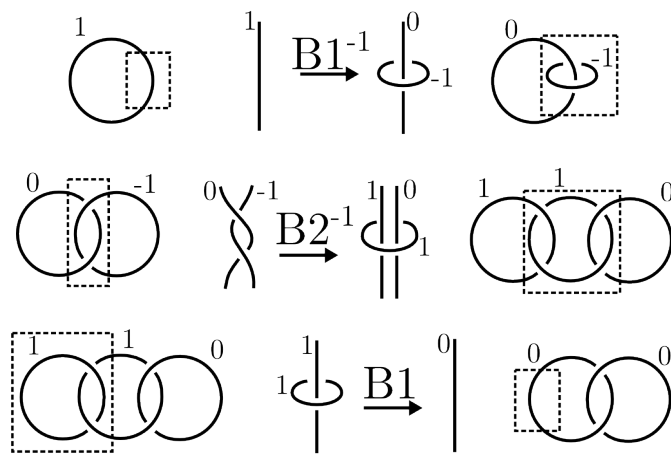


Figure 4.6: The local details of the blow ups $B1^{-1}$ and $B2^{-1}$ and the blow down $B1$.

with a circle with framing +1. Then add a circle with framing -1 to the circle with framing +1, obtaining the new framing 0. Then twist the two circles apart and add a circle of framing +1 connecting them. The two circles change framing by +1. Delete the circle with framing +1 from the circle of framing +1 that was added in the previous blow up. The two remaining circles have framing 0 and is connected in a Hopf link. How to create a zero framed Hopf link is shown in Figure 4.5. The details of each step is shown in Figure 4.6.

We prove Theorem 4.4.1 for a blow down with three strands. Blow downs for an arbitrary number of strands are done equivalently, while blow ups for an arbitrary number of strands are the inverses of the blow downs.

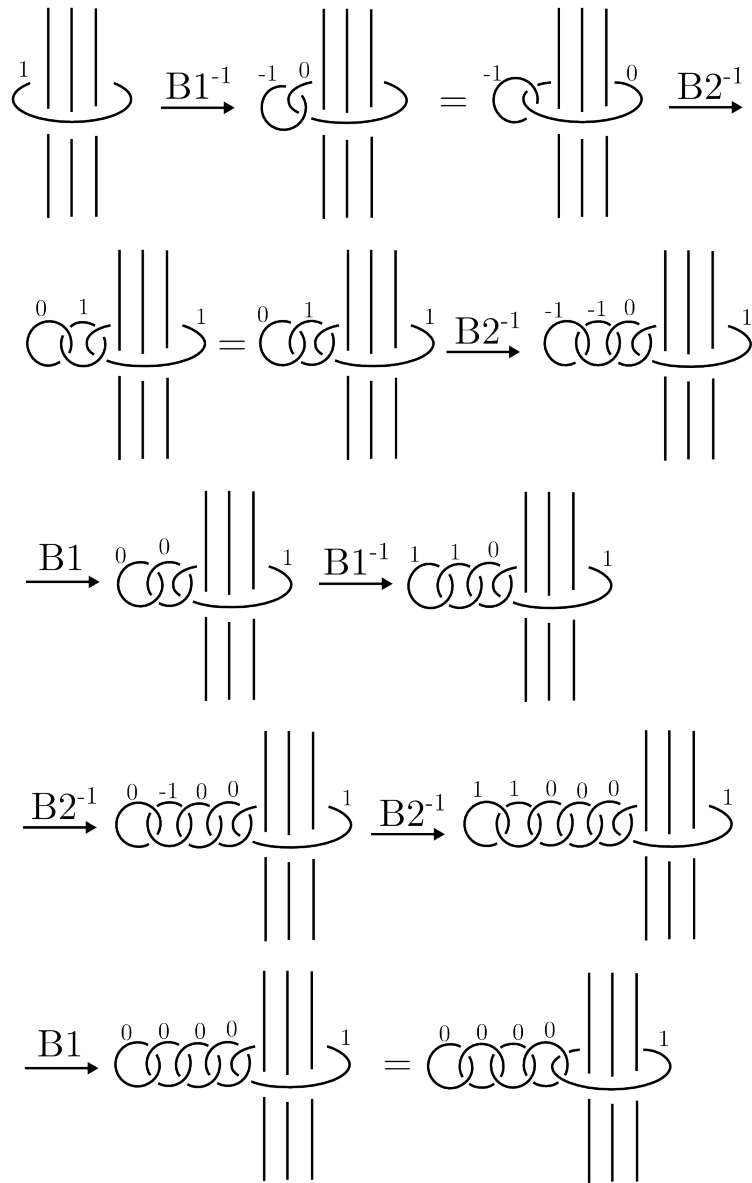


Figure 4.7: Adding a tail of four zero framed circles. The details are the same as for creating a zero framed Hopf link.

We start by assigning the circle around the three strands with the framing +1. We then add a tail of two zero framed Hopf links, see Figure 4.7.

By two C2 moves, we slide the three strands into different circles. Details are shown in Figure 4.8. The moves are described in Figure 4.9.

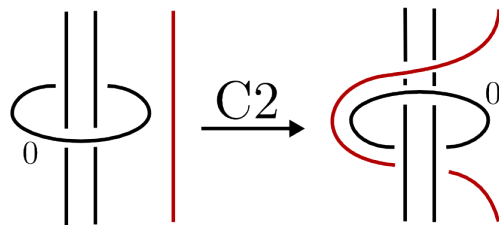


Figure 4.8: C2-move for one strand.

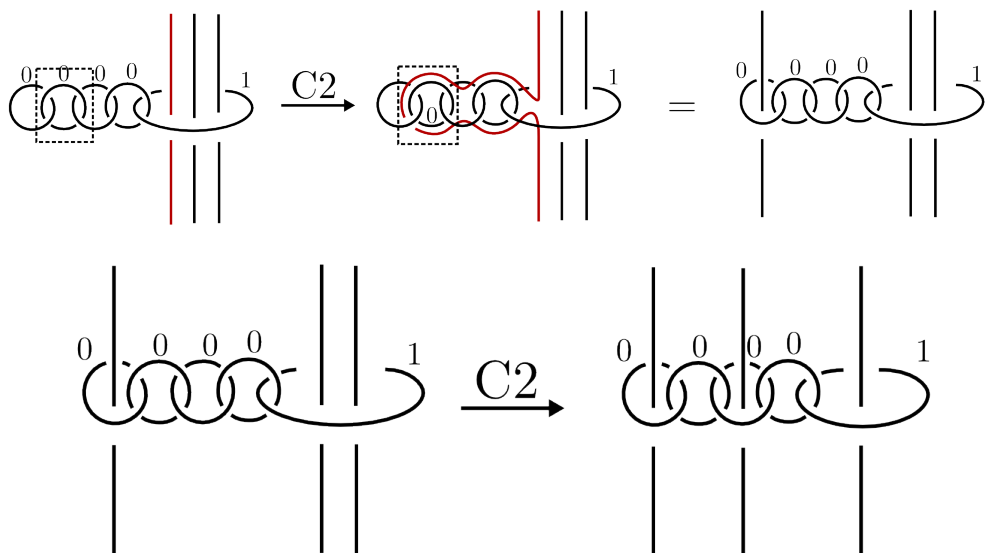


Figure 4.9: Sliding the strands to different circles.

By applying a B2 move, we can remove the circle with framing +1. The strand and the circle are twisted together and their framing changes by -1 . Details are shown in Figure 4.10. Subsequently a C2 move slides the strand to the right around the circle to

the left. Figure 4.11 shows the move, and Figure 4.8 is a detailed image. The circle to the right has framing -1 , and through a **B1** move, we delete it from the circle it is attached to, which changes the framing by $+1$, see Figure 4.12.

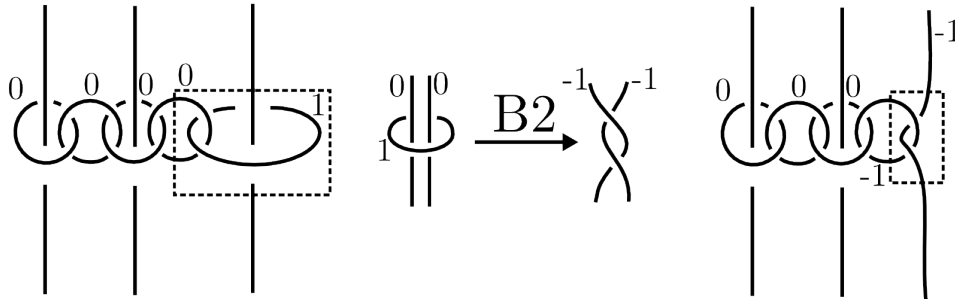


Figure 4.10: Removing the circle of framing $+1$ via a **B2** move.

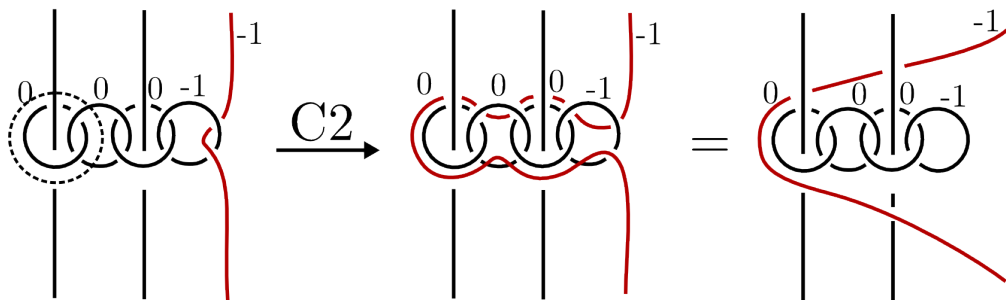


Figure 4.11: Sliding the red strand around the circle to the left via a **C2** move.

By repeating the sequence of moves, **B2**, **C2** and **B1**, the link we are left with is the one presented in the second row second from the left in Figure 4.13. By performing another **B1** move, the blow down with three strands is complete. \square

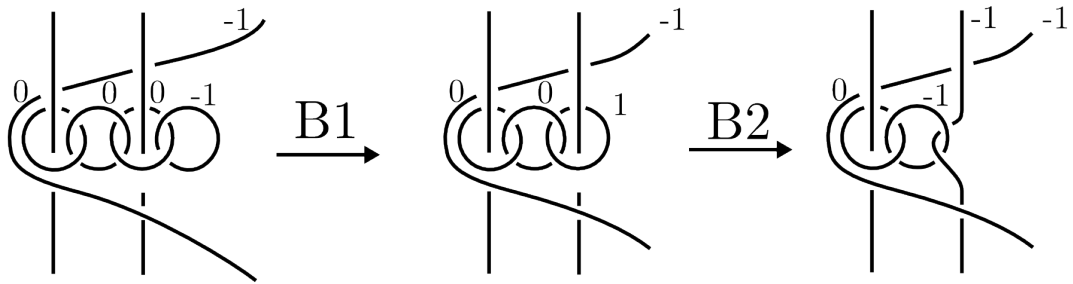


Figure 4.12: A B1 move deletes the circle to the right. Then a B2 move deletes the circle with framing +1, and twists the remaining circle and strand together.

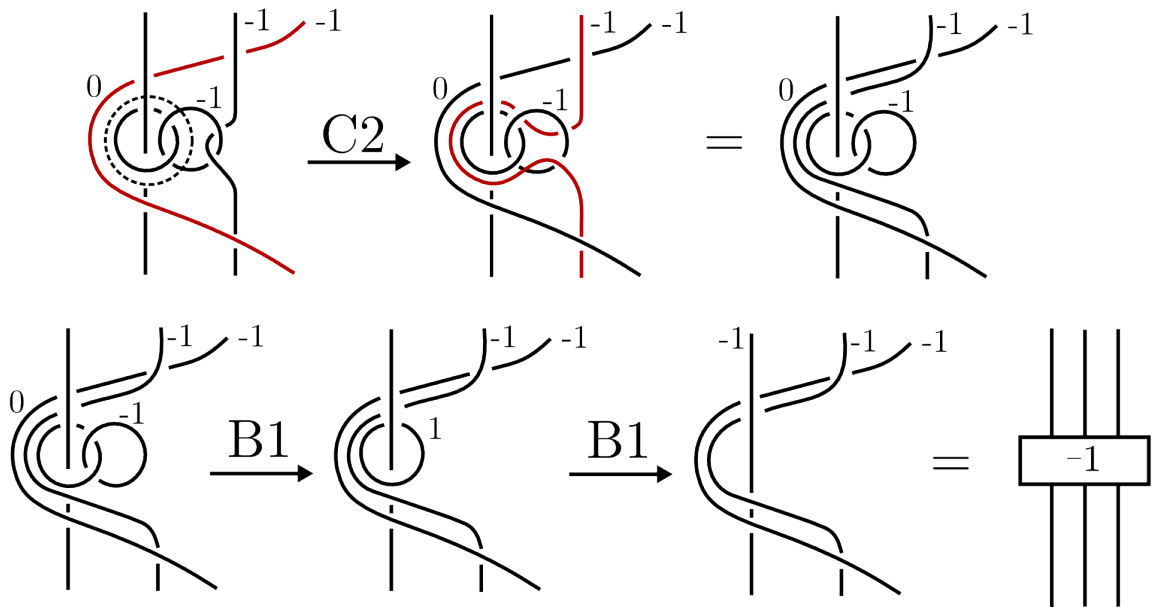


Figure 4.13:

Corollary 4.4.2. *Let \mathcal{L} and \mathcal{L}' be two framed links in S^3 such that the 3-manifolds obtained from surgery on \mathcal{L} and \mathcal{L}' are homeomorphic. Then the link \mathcal{L}' can be obtained from the link \mathcal{L} through a finite sequence of Fenn and Rowke's blow ups and blow downs with up to five strands. [6].*

Proof. We prove that a handle slide with three strands, a C3 Martelli move can be obtained by a finite sequence of blow ups and blow downs with up to five strands. An analogous proof follows to prove that the C2 Martelli move can be obtained by blow ups and blow downs with up to five strands.

We start with the left hand side of the C3 Martelli move. We can apply a move of the type in Corollary 4.4.3 to obtain the diagram on right hand side of Figure 4.14.

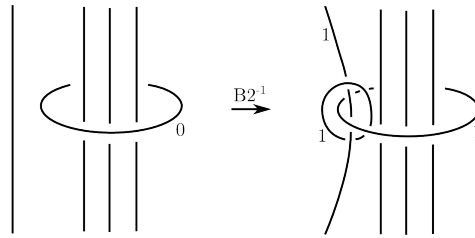


Figure 4.14:

By applying a blow down with three strands we obtain the diagram in Figure 4.15. This diagram is described in detail in Figure 4.16.

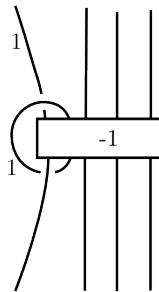


Figure 4.15:

The right hand side of Figure 4.16 is equal to the left hand side of Figure 4.17. By applying a blow up with four strands, we obtain the diagram in the center of Figure 4.17, to which we can apply a blow down with one strand to complete the proof.

□

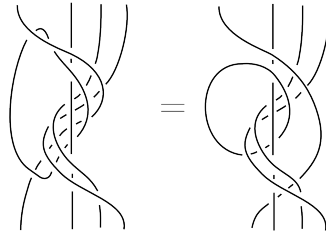


Figure 4.16:

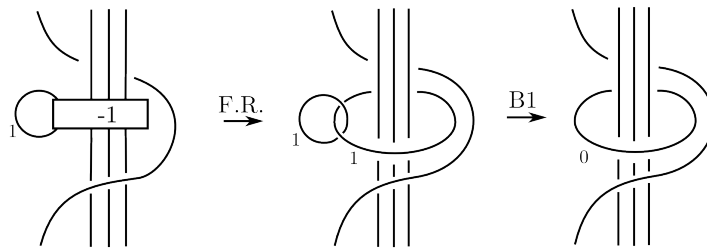


Figure 4.17:

Lemma 4.4.3. *There is a finite sequence of Reidemeister moves R2 and R3 and Martelli moves between the two link diagrams in Figure 4.18.*

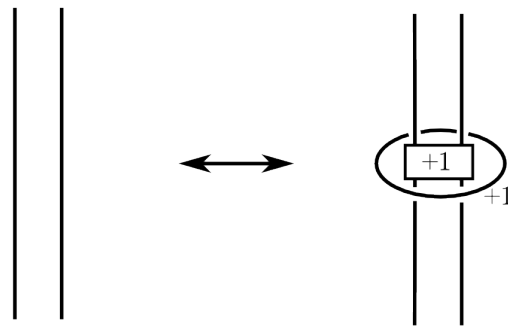


Figure 4.18:

Proof. Take two strands to begin with and perform two Reidemeister moves of type R2 to obtain the diagram in the middle of Figure 4.19. The twist at the top of the

diagram is a clockwise twist, i.e. a -1 twist. The lower twist is a counterclockwise twist. By applying the inverse B2 Martelli move on both those twists we obtain the diagram to the right of Figure 4.19.

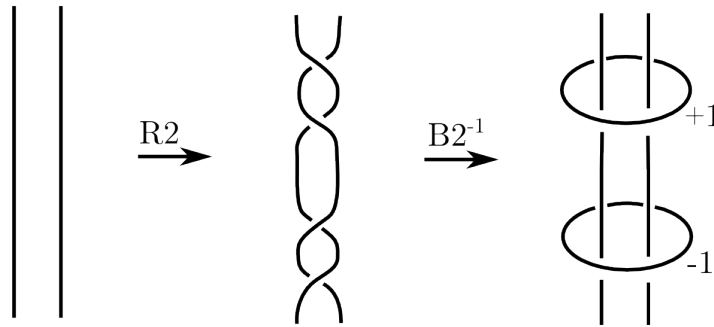


Figure 4.19:

By applying four Reidemeister moves of type R3, we obtain the diagram to the left of Figure 4.20, to which we can apply a B2 Martelli move to obtain the diagram to the right of Figure 4.20.

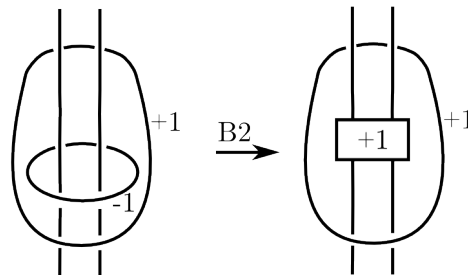


Figure 4.20:

Analogous proofs follow for an arbitrary number of strands. \square

Remark 4.4.4. The move from left to right in Figure 4.18 is hereafter referred to as *special blow ups*.

Corollary 4.4.5. Let \mathcal{L} and \mathcal{L}' be two framed links in S^3 such that the 3-manifolds obtained from surgery on \mathcal{L} and \mathcal{L}' are homeomorphic. Then the link \mathcal{L}' can be obtained

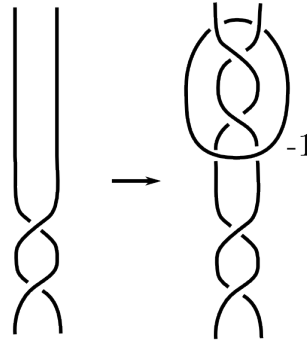


Figure 4.21:

from the link \mathcal{L} through a finite sequence of Fenn and Rourke's blow downs with up to five strands and special blow ups with up to five strands.

Proof. From Corollary 4.4.2 we have that the 3-manifolds obtained by surgery on the links \mathcal{L} and \mathcal{L}' , are homeomorphic if \mathcal{L} and \mathcal{L}' can be obtained from one another by a finite sequence of blow ups and blow downs with up to five strands. It remains for us to show that the blow ups with up to five strands can be obtained from the special blow ups.

Take two strands with a counterclockwise twist, and apply the special blow up above the twist to obtain the right hand side of Figure 4.21. By applying two Reidemeister moves of type R3, we obtain the left hand side of Figure 4.22, to which we can apply two R2 moves in order to twist the opposite strands apart and obtain the right hand side of Figure 4.22.

Analogous proofs can be given for special blow ups with three, four and five strands. \square

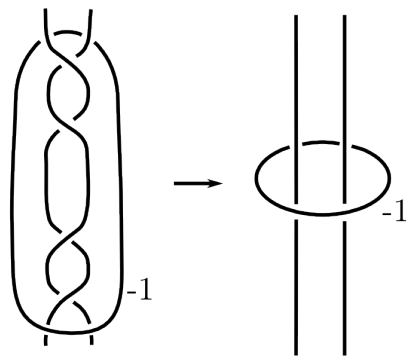


Figure 4.22:

CHAPTER 5

AN INVARIANT OF 3-MANIFOLDS

In this chapter we will define an invariant of 3-manifolds, which we refer to as the *Fenn-Rourke-local rack*. Let M and N be 3-manifolds, and let L_M and L_N be links which determine M and N by integral Dehn surgery in the 3-sphere. By Corollary 4.4.5, M and N are homeomorphic if and only if L_M can be obtained from L_N by a finite sequence of blow down with up to five strands and special blow downs with up to five strands. In order to demonstrate that the Fenn-Rourke-local rack is an invariant, it therefore suffices to demonstrate that it is invariant under these moves and under the R2 and R3 Reidemeister moves.

When defining the Fenn-Rourke-local fundamental rack of a framed link, we ignore its framing. Thus it can distinguish 3-manifolds obtained by integral Dehn surgery on different links, but not those obtained by integral Dehn surgery on the same link (up to isotopy). It is straightforward to adapt the construction of the Fenn-Rourke-local fundamental rack as we define it to take into account framings, and thus to obtain an invariant of 3-manifolds per se.

5.1 The Fenn-Rourke-local fundamental rack of a link diagram

We will associate to any link diagram a rack which we will refer to as its Fenn-Rourke-local fundamental rack.

Definition 5.1.1. The *Fenn-Rourke-local fundamental rack* of a link L is the rack obtained by the following procedure. We begin by taking the free rack on the arcs of L , except those arcs which are disjoint circles. We then take a quotient of this free rack which imposes the following relations.

1. Given any crossing of L as in Figure 3.1 in which all arcs belong to the same component of L , impose the relation from the crossing.
2. Let y and z be arcs of L for which L looks locally as on the left hand side of Figure 5.1; for which y and z belong to the same component of L ; for which the two pieces of the broken circle belong to the same component of L , and this component is not the same component as that to which y and z belong; and for which, if we remove all components of L except the component to which y and z belong, Figure 5.1 can be manipulated using R2 and R3 moves to a single arc joining y and z . We then require that $y = z$, and that $y \triangleright y = y$.
3. Let y, z, y' , and z' be arcs of L for which L looks locally as on the right hand side of Figure 5.1; for which the two pieces of the broken circle belong to the same component of L , and this component is not the same component as that to which any of the arcs y, z, y' , and z' belong; and for which, if we remove all components of L except those to which y, z, y' , and z' belong, we can manipulate the right hand side of Figure 5.1 using R2 and R3 moves to obtain a pair of disjoint arcs joining y to z and joining y' to z' . We then require that $y = z = y' = z'$, and that $y \triangleright y = y$.
4. If L looks locally as any of the diagrams Figure 5.2, and we have the same conditions as in 3 for up to five strands, we then require that $y = z = y' = z' = u = v = u' = v' = u'' = v''$ and that $y \triangleright y = y$.

The breaks in Figure 5.1 and 5.2 indicates that we place no requirements on how L looks locally between the broken arcs. Other crossings may for instance occur.

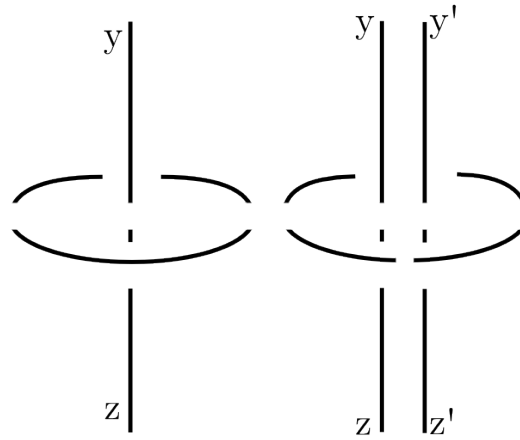


Figure 5.1:

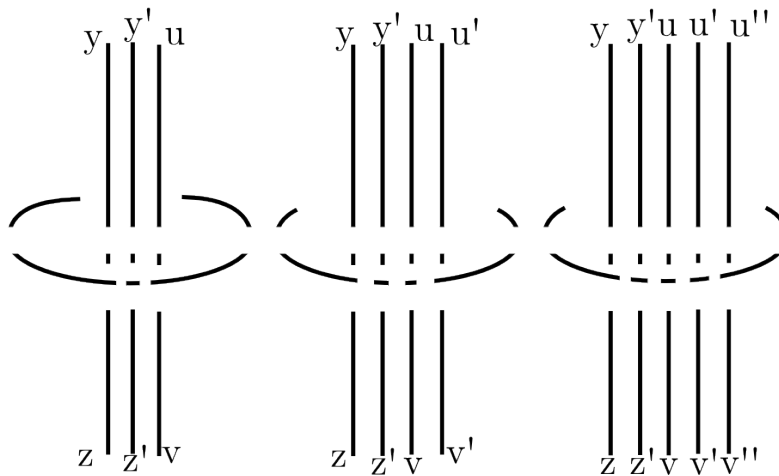


Figure 5.2:

Theorem 5.1.2. *If a pair of link diagrams L and L' can be obtained from one another by a finite sequence of the moves from Corollary 4.4.5, then the Fenn-Rourke-local fundamental racks of L and L' are isomorphic.*

Proof. We first observe that the whether or not the conditions of 2 in Definition 5.1.1 are satisfied for a given pair of arcs is invariant under the moves of Corollary 4.4.5 and,

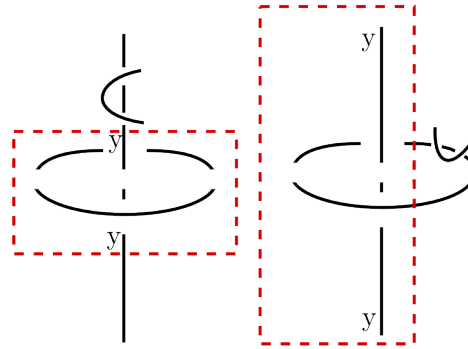


Figure 5.3:

and the same holds for 3 and 4. Figure 5.3 indicates this for 2 in the case of a couple of the possible R2 moves. As a consequence of this, the same proof as in Proposition 3.1.4 demonstrates that the Fenn-Rourke-local fundamental rack is invariant under the R2 and R3 moves.

It then suffices to show that the Fenn-Rourke-local fundamental racks for link diagrams, connected by a finite sequence of blow downs with up to five strands and special blow ups with up to five strands, are isomorphic. Since a disjoint circle component of the link is excluded from the Fenn-Rourke-local fundamental rack, two link diagrams connected by blow ups and blow downs with zero strands have isomorphic Fenn-Rourke-local fundamental racks.

For a blow up with one strand the condition from 2 of Definition 5.1.1 ensures that the labelling of the strands is unaffected by this move, and 1 together with the fact that disjoint circles are ignored ensures that the circle on the left hand side of Figure 5.4 does not give any generators in the Fenn-Rourke-local fundamental rack, and that crossings involving the circle do not impose any relations. Hence the Fenn-Rourke-local fundamental rack is unaffected by the blow up with one strand. By the same argument the it is also unaffected by the blow down with one strand, and the Fenn-Rourke-local fundamental rack is invariant under blow ups and blow downs with one strand.

For a blow down with two strands, condition 3 ensures that that the arcs must be as in the left hand side of Figure 5.5, and in addition that $y \triangleright y = y$. Condition 1 together with the fact that disjoint circles are ignored, ensures that the circle does not give a generator in the Fenn-Rourke-local fundamental rack, and the crossings involving the circle do not impose any relations. Hence the Fenn-Rourke-local fundamental rack is

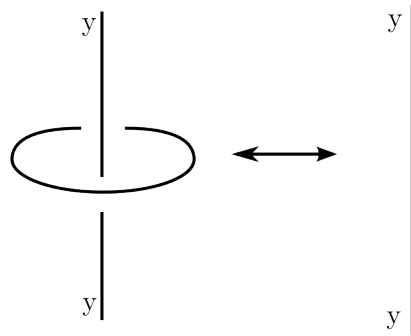


Figure 5.4:

invariant under blow downs with two strands.

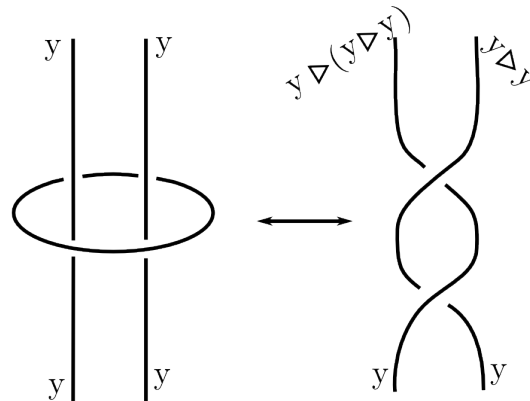


Figure 5.5:

For a special blow up with two strands, 1 of Definition 5.1.1, together with the fact that disjoint circle components are ignored when forming the set of generators of a Fenn-Rourke-local fundamental rack, ensures that the circle is not a generator, and that crossings involving the circle do not contribute any relations, in the right hand side of Figure 5.6. Thus we can replace the left hand side of Figure 5.6 by the right hand side without altering the Fenn-Rourke-local fundamental rack: the unlabelled arcs on the right hand side of Figure 5.6 must be $y \triangleright x$ and $x \triangleright (y \triangleright x)$, and thus can be removed from the generators without altering the rack, since no relations involving these two elements are forced.

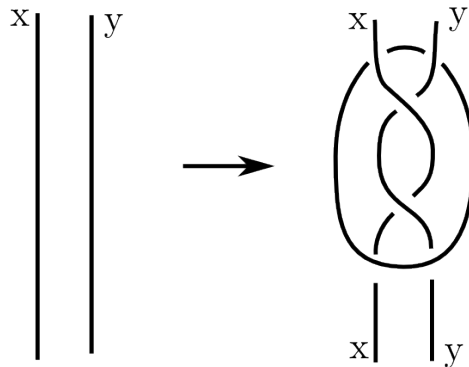


Figure 5.6:

A similar argument can be carried out for blow downs and special blow ups with three, four, and five strands. \square

Example 5.1.3. The unknot consists of a single arc, which is a circle. This circle is excluded from the set of generators of the Fenn-Rourke-local fundamental rack. Hence the Fenn-Rourke-local rack of the unknot, which we will denote by FR_U , is $\langle \emptyset \mid \emptyset \rangle$. This is an empty rack.

Example 5.1.4. The trefoil consists of three arcs, and since none of them are the disjoint circle we get three generators for the Fenn-Rourke-local fundamental rack. All the crossings in the trefoil consist of arcs from the same component, hence we get a relation from every crossing. The Fenn-Rourke-local fundamental rack of the trefoil is presented by

$$FR_T = \langle \{x, y, z\} \mid x \triangleright y = z, y \triangleright z = x, z \triangleright x = y \rangle.$$

The Fenn-Rourke-local fundamental rack of the trefoil is the exact same rack as the fundamental rack for the trefoil.

5.2 The 3-sphere and the Poincaré homology sphere are not homeomorphic

In this section we will use the Fenn-Rourke-local fundamental rack to distinguish between different 3-manifolds.

Lemma 5.2.1. *There is a surjective morphism $\tilde{\varphi}$ of racks from the Fenn-Rourke-local fundamental rack of the trefoil, C_T , into $\text{Core}(\mathbb{Z}/3\mathbb{Z})$.*

Proof. By the argument from the proof for Lemma 3.2.1, we know that the morphism from R_T to $\text{Core}(\mathbb{Z}/3\mathbb{Z})$ is surjective, hence the morphism

$$\tilde{\varphi} : U_T \longrightarrow \text{Core}(\mathbb{Z}/3\mathbb{Z})$$

is also surjective. □

Corollary 5.2.2. *There is no isomorphism between the Fenn-Rourke-local rack of the trefoil and the Fenn-Rourke-local of the unknot.*

Proof. Since the Fenn-Rourke-local of the unknot, FR_U , is the empty set, we know that there is only one morphism from FR_U , that is the morphism with an empty set as an image. The Fenn-Rourke-local rack of the trefoil, FR_T , is not empty since it is generated by three generators. Hence there is no morphism from FR_U into FR_T , and in particular no isomorphism between FR_U and FR_T . □

Corollary 5.2.3. *The 3-sphere and the Poincaré sphere are not homeomorphic.*

Proof. The Poincaré sphere can be obtained by surgery on the trefoil [1] and the 3-sphere can be obtained by surgery on the unknot [4]. By Corollary 4.4.5, links presenting homeomorphic 3-manifolds can be obtained from one another by a finite sequence of blow downs and special blow ups with up to five strands. From Theorem 5.1.2 we know that if two links can be connected by a finite sequence of Reidemeister moves R2 and R3 and blow downs and special blow ups with up to five strands, then the Fenn-Rourke-local racks for the links are isomorphic. Since the Fenn-Rourke-local racks for the trefoil and unknot are not isomorphic, their respective link diagrams cannot be obtained from one another by a finite sequence of R2 and R3 moves and blow ups and blow downs with up to five strands. Hence no 3-manifold obtained by surgery on the unknot can be homeomorphic to a 3-manifold obtained by surgery on the trefoil. □

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