Doctoral theses at NTNU, 2015:265

Nils Melvær Nornes

Degeneration and related partial orders in representation theory.

Nils Melvær Nornes

Doctoral Thesis

ISBN 978-82-326-1186-7 (printed version)
ISBN 978-82-326-1187-4 (electronic version)
ISSN 1503-8181

Doctoral theses at NTNU, 2015:265

NTNU
Norwegian University of
Science and Technology
Faculty of Information Technology,
Mathematics and Electrical Engineering
Department of Mathematical Sciences

NTNU
Norwegian University of
Science and Technology

NTNU

NTNU
Norwegian University of
Science and Technology

Nils Melvær Nornes

Degeneration and related partial orders in representation theory.

Thesis for the degree of Philosophiae Doctor

Trondheim, October 2015

Norwegian University of Science and Technology

NTNU
Norwegian University of
Science and Technology

NTNU
Norwegian University of Science and Technology

Thesis for the degree of Philosophiae Doctor

ISBN 978-82-326-1186-7 (printed version)
ISBN 978-82-326-1187-4 (electronic version)
ISSN 1503-8181

Doctoral theses at NTNU, 2015:265

Printed by Skipnes Kommunikasjon as

CONTENTS

ACKNOWLEDGEMENTS

Wait, I'm actually done now? Huh. Time to thank people then.

Thanks to my advisor, Professor Sverre O. Smal, for advising. And to my other coauthors, Tore Forbregd and Professor Steffen Oppermann, for um, coauthing. It has been a pleasure working with you all.

I am also grateful to the rest of the algebra group at NTNU for many fine discussions over coffee and other beverages, to the department staff for making everything run smoothly, and to the Norwegian tax payers for picking up the tab.

And finally, many thanks to my parents for all sorts of support over the years.

Nils Melvær Nornes
Nornes, September 2015

INTRODUCTION

The general idea behind representation theory is, as the name suggests, to represent complicated objects by something we know better, namely vector spaces and linear transformations. It started out with linear representations of groups, which are homomorphisms from a group to a group of automorphisms on a vector space. Similar constructions were made for Lie groups and Lie algebras, and eventually this was generalized to representations of associative algebras.

Let k be a field and Λ a finitely generated k-algebra. A *finite-dimensional representation* of Λ is an algebra homomorphism from Λ to a full matrix ring over k. Let d be a natural number and $\mathcal{M}_d(k)$ the ring of $d \times d$-matrices over k. We denote by $\operatorname{mod}_d \Lambda$ the set of all d-dimensional representations of Λ, i.e. all algebra homomorphisms from Λ to $\mathcal{M}_d(k)$. The representations in $\operatorname{mod}_d \Lambda$ correspond bijectively to the Λ-module structures on the vector space k^d. The general linear group $\mathrm{GL}_d(k)$ acts on $\operatorname{mod}_d \Lambda$ by conjugation, and the orbits of this action corresponds to isomorphism classes of modules.

When k is algebraically closed, the set $\operatorname{mod}_d \Lambda$ also has the structure of an affine variety. Then the closures of the $\mathrm{GL}_d(k)$-orbits are partially ordered by inclusion, and this gives a partial order called *degeneration* on the set of d-dimensional Λ-modules. There are several other partial orders related to this, and we will study some of them.

The thesis consists of, in addition to this introduction, three papers: *Partial Orders on Representations of Algebras* [7] (cowritten with Tore A. Forbregd and Sverre O. Smalø), *Degenerations of Submodules and Composition Series* [9] (cowritten with Steffen Oppermann) and *Module Degenerations and Finite Field Extensions* [10].

1. Background

Let ρ be a representation in $\operatorname{mod}_d \Lambda$. It defines a module structure on the vector space k^d in the following way. For a $\lambda \in \Lambda$ and $x \in k^d$ let $\lambda \cdot x = \rho(\lambda) \cdot x$ (where the multiplication on the right hand side is just matrix multiplication). Conversely, every module structure on k^d gives us a representation. Given a Λ-module M with underlying vector space k^d, every $\lambda \in \Lambda$ defines a linear transformation $f_\lambda : k^d \to k^d$ by $f_\lambda(x) = \lambda \cdot x$. The function $\rho_M : \Lambda \to \mathcal{M}_d(k)$ given by $\rho_M(\lambda) = f_\lambda$ is an algebra homomorphism, and thus a representation. This gives us a bijection between the set of Λ-module structures on k^d and $\operatorname{mod}_d \Lambda$.

Since Λ is finitely generated, a representation is completely determined by its value on a finite set of elements. Let $\{\lambda_1, \dots, \lambda_n\}$ be a set of generators for Λ, and let $\rho \in \operatorname{mod}_d \Lambda$. If we know the matrices $\rho(\lambda_i)$ for $0 < i \leq n$, then we can reconstruct all of ρ. This means that we can identify $\operatorname{mod}_d \Lambda$ with a subset of $\mathcal{M}_d(k)^n$. This subset is closed in the Zariski topology. To show this we need to construct some polynomials.

An element $(A_1, \dots, A_n) \in \mathcal{M}_d(k)^n$ is in $\operatorname{mod}_d \Lambda$ if and only if for every noncommutative polynomial f such that $f(\lambda_1, \dots, \lambda_n) = 0$ we have $f(A_1, \dots, A_n) = 0$. Let $k[\{X_{abc}\}_{0<a\leq n, 0<b\leq d, 0<c\leq d}]$ be the coordinate ring of $\mathcal{M}_d(k)^n$. For every f with $f(\lambda_1, \dots, \lambda_n) = 0$, we have a matrix

$$f\left(\begin{pmatrix} X_{111} & \cdots & X_{11d} \\ \vdots & \ddots & \vdots \\ X_{1d1} & \cdots & X_{1dd} \end{pmatrix}, \dots, \begin{pmatrix} X_{n11} & \cdots & X_{n1d} \\ \vdots & \ddots & \vdots \\ X_{nd1} & \cdots & X_{ndd} \end{pmatrix}\right).$$

Each entry in this matrix is a polynomial that is 0 on $\operatorname{mod}_d \Lambda$. Let S be the set of all these polynomials for all f such that $f(\lambda_1, \dots, \lambda_n) = 0$. Then $\operatorname{mod}_d \Lambda$ is the zero set of S, and thus an affine variety.

The group variety $\mathrm{GL}_d(k)$ acts on $\operatorname{mod}_d \Lambda$ by conjugation, that is, for $g \in \mathrm{GL}_d(k)$ and $(A_1, \dots, A_n) \in \operatorname{mod}_d \Lambda$ we have $g \star (A_1, \dots, A_n) = (gA_1g^{-1}, \dots, gA_ng^{-1})$. This induces an isomorphism between the module represented by ρ and the module represented by $g \star \rho$, and thus we get a one-to-one correspondence between $\mathrm{GL}_d(k)$-orbits in $\operatorname{mod}_d \Lambda$ and isomorphism classes of d-dimensional Λ-modules. We are now ready for the definition of degeneration.

Definition. If the orbit corresponding to a module N is contained in the closure of the orbit corresponding to the module M, we say that M *degenerates* to N, and denote this by $M \leq_{\text{deg}} N$.

The simplest examples of module varieties occur when $\Lambda = k[X]$, the polynomial ring in one variable. Here we have $\operatorname{mod}_d(k[X]) = \mathcal{M}_d(k)$, and the orbits are just similarity classes of matrices. That makes it easy to decide if two modules are isomorphic, we only have to compare the Jordan forms of their representations. If they have the same eigenvalues and block sizes, they are isomorphic.

It is also easy to decide if one $k[X]$-module degenerates to another. M. Gerstenhaber showed in [8] that for $A, B \in \operatorname{mod}_d k[X]$ we have $A \leq_{\text{deg}} B$ if and only if $\operatorname{rank}(A - \lambda)^i \geq \operatorname{rank}(B - \lambda)^i$ for all $i \in \mathbb{N}$ and all eigenvalues λ of A.

Any minimal degeneration of a $k[X]$-module A can be constructed by reducing the size of a Jordan block by one and increasing the size of a smaller block with the same eigenvalue (or creating a new block of size one).

In 1969, M. Artin [4] observed that for an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have $B \leq_{\mathrm{deg}} A \oplus C$. An immediate consequence of this result is that an orbit is closed if and only if the corresponding module is semisimple.

For some algebras this tells the whole story, every minimal degeneration is of this form. In particular this is true for $k[X]$.

The existence of such sequences gives rise to a coarser order called $\leq_{\mathrm{ext}}$, which was first considered by S. Abeasis and A. Del Fra in [1].

Definition. Let M and N be Λ-modules. $M \leq_{\mathrm{ext}} N$ if for some $n \in \mathbb{N}$ there exist n short exact sequences

$$0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$$

such that $M \simeq B_1$, $N \simeq A_n \oplus C_n$ and $B_i \simeq A_{i-1} \oplus C_{i-1}$ for $2 \leq i \leq n$.

Abeasis and Del Fra also introduced a third order $\leq_{\mathrm{r}}$ based on the ranks of certain matrices.

They showed in [1], [2] and [3] that $\leq_{\mathrm{ext}}$, $\leq_{\mathrm{deg}}$ and $\leq_{\mathrm{r}}$ are the same for all path algebras over A_n-quivers, and over some D_n-quivers. Later K. Bongartz showed in [5] that $\leq_{\mathrm{deg}}$ and $\leq_{\mathrm{ext}}$ are the same for all representation-directed algebras. This includes all path algebras over Dynkin quivers. He also showed this for the Kronecker algebra. As mentioned above, $\leq_{\mathrm{ext}}$ and $\leq_{\mathrm{deg}}$ are also the same for $k[X]$.

In [11], C. Riedtmann proved the following.

Proposition 1. *Let*

$$0 \longrightarrow X \longrightarrow X \oplus M \longrightarrow N \longrightarrow 0$$

be an exact sequence in $\mathrm{mod}\,\Lambda$. *Then* $M \leq_{\mathrm{deg}} N$.

Using this she gave the first example of a proper degeneration $M \leq_{\mathrm{deg}} N$ where N is indecomposable. When $M \leq_{\mathrm{ext}} N$, N is clearly decomposable, so this shows that degeneration is strictly finer than the ext-order. In the same paper she presented an example due to J. Carlson which shows that one cannot cancel common summands in a degeneration. This led her to introduce two new partial orders.

Definition. M *virtually degenerates* to N, denoted $M \leq_{\mathrm{vdeg}} N$, if there exists $Y \in \mathrm{mod}\,\Lambda$ such that $M \oplus Y \leq_{\mathrm{deg}} N \oplus Y$.

Definition. $M \leq_{\text{Hom}} N$ if for all $X \in \operatorname{mod}\Lambda$ we have $[X, M] \leq [X, N]$.

She then showed that $\leq_{\text{vdeg}}$ and $\leq_{\text{Hom}}$ are the same for representation-finite algebras.

In 2000 G. Zwara proved the converse of Proposition 1 in [15].

Theorem 2. *Let M and N be Λ-modules. The following are equivalent.*

(1) $M \leq_{\text{deg}} N$.

(2) *There exists a short exact sequence of Λ-modules*

$$0 \longrightarrow X \longrightarrow X \oplus M \longrightarrow N \longrightarrow 0\ .$$

(3) *There exists a short exact sequence of Λ-modules*

$$0 \longrightarrow N \longrightarrow Y \oplus M \longrightarrow Y \longrightarrow 0.$$

This gives a completely module theoretic description of degenerations. We can use this as an alternative definition of degeneration, and since it does not involve geometry, we can relax the conditions on k. It does not have to be algebraically closed any more, it does not even have to be a field. All we need is a commutative artin ring.

Without geometry it is no longer obvious that degeneration is a partial order, but this was proved by G. Zwara in [13].

Obviously we still have that $M \leq_{\text{ext}} N$ implies $M \leq_{\text{deg}} N$, and $M \leq_{\text{deg}} N$ implies $M \leq_{\text{vdeg}} N$. From the new definition it is also easy to see that $M \leq_{\text{vdeg}} N$ implies $M \leq_{\text{Hom}} N$: If $M \leq_{\text{vdeg}} N$ we have a Riedtmann sequence

$$0 \longrightarrow X \longrightarrow X \oplus M \oplus Y \longrightarrow N \oplus Y \longrightarrow 0\ .$$

For any $Z \in \operatorname{mod}\Lambda$ we apply $\operatorname{Hom}_\Lambda(Z, -)$ to the Riedtmann sequence and get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_\Lambda(Z, X) \longrightarrow \operatorname{Hom}_\Lambda(Z, X) \oplus \operatorname{Hom}_\Lambda(Z, M) \oplus \operatorname{Hom}_\Lambda(Z, Y)$$

$$\longrightarrow \operatorname{Hom}_\Lambda(Z, N) \oplus \operatorname{Hom}_\Lambda(Z, Y)\ ,$$

and summing up the lengths we see that $[Z, M] \leq [Z, N]$.

K. Bongartz has shown in [6] that for tame hereditary algebras over algebraically closed fields $\leq_{\text{deg}}$ and $\leq_{\text{Hom}}$ are the same. G. Zwara showed the same for representation-finite algebras over algebraically closed fields in [14]. Zwara's result was later generalized to any representation-finite artin algebra by S. O. Smalø in [12].

The examples from Riedtmann and Carlson show that for arbitrary algebras $\leq_{\text{ext}}$ is strictly coarser than $\leq_{\text{deg}}$, which is again coarser than $\leq_{\text{vdeg}}$. It is still not known if $\leq_{\text{Hom}}$ is different from $\leq_{\text{vdeg}}$.

2. Overview of the thesis

In [12], S. O. Smalø introduced the following family of quasiorders. Let M be a Λ-module and n a natural number. An $n \times n$-matrix A with entries in Λ induces a k-linear map from M^n to itself. Denote the image of this map by AM^n.

Definition. $M \leq_{\mathrm{n}} N$ if $\ell(M^n/AM^n) \leq \ell(N^n/AN^n)$ for all $n \times n$-matrices A with entries in Λ.

Equivalently, $M \leq_{\mathrm{n}} N$ if each matrix has greater or equal rank as a M^n-endomorphism than as a N^n-endomorphism. This generalizes the rank order of Abeasis and Del Fra.

Clearly the relation $\leq_{\mathrm{n}}$ is reflexive and transitive, but for small n it is not always antisymmetric.

In [7], cowritten with my fellow student Tore A. Forbregd and our adviser Professor Sverre O. Smalø, we show that $\leq_{\mathrm{d}^3}$ always is a partial order on $\mathrm{mod}_d\,\Lambda$. It seems like for large enough n, $\leq_{\mathrm{n}}$ is equivalent to $\leq_{\mathrm{Hom}}$. In the paper we claimed this as a fact, but we did not give a proof. When the reviewer requested a proof, we realized that the proof we had in mind was incomplete. We decided to remove the statement, but unfortunately we wrote it twice and deleted it once, so it still appears in the published version.

While we have not found a proof, we have not found any counterexample either. In fact, in all examples we have looked at, $\leq_{\mathrm{n}}$ is either not a partial order or equivalent to $\leq_{\mathrm{Hom}}$. We still have no examples where $\leq_{\mathrm{n}}$ is a partial order but is different from $\leq_{\mathrm{Hom}}$. However, we have such an example for the closely related quasiorders $\leq_{\mathrm{Hom-n}}$ obtained by loosening the conditions of the Hom-order.

Definition. Let M and N be Λ-modules and n a natural number. $M \leq_{\mathrm{Hom-n}} N$ if $[X, M] \leq [X, N]$ for all Λ-modules X with $\ell X \leq n$.

Example 1. Let Q be the Kronecker quiver,

$$Q:\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2\ ,$$

and let $\Lambda = kQ$ be the path algebra. The representations

$$P_1:\ k \underset{\binom{0}{1}}{\overset{\binom{1}{0}}{\rightrightarrows}} k^2\ , \qquad P_2:\ 0 \rightrightarrows k$$

are the indecomposable projective modules, and

$$I_1:\ k \rightrightarrows 0\ , \qquad k^2 \underset{(0\ 1)}{\overset{(1\ 0)}{\rightrightarrows}} k$$

are the indecomposable injective modules. Let $M = P_2 \oplus I_2$ and $N = P_1 \oplus I_1$. We have $M \leq_{\text{Hom}-4} N$, but $[M, P_2] > [N, P_2]$, so M and N are not comparable in the Hom-order. We also have that $\ell(\operatorname{End}_\Lambda M) = \ell(\operatorname{End}_\Lambda N) = 3$, which is something that cannot happen with a proper degeneration.

In [9], cowritten with Professor Steffen Oppermann, we show that a degeneration $M \leq_{\text{deg}} N$ induces degenerations from submodules of M to submodules of N. Given a submodule $M' \subseteq M$ and a Riedtmann sequence, we construct a submodule $N' \subseteq N$ such that $M' \leq_{\text{deg}} N'$. This construction gives rise to a function from the set of submodules of M to the set of submodules of N, but this function does not seem to have any nice properties. We give examples that show, among other things, that it is neither injective nor surjective.

Since submodules degenerate to submodules we also have that composition series in some sense degenerate to composition series. We give a geometric interpretation of this degeneration order using the subset of mod_Λ consisting of those homomorphisms whose images are contained in the ring of upper triangular matrices. Such a representation can be viewed as a representation of a composition series. With the right group action we get a correspondence between orbits and isomorphism classes of composition series, and orbit closures give rise to degenerations.

In [10] we study the degeneration order for some algebras over fields that are not algebraically closed. In particular we look at modules over $K \otimes_k \Lambda$, where K is a finite extension of the base field. These modules can also be viewed as Λ-modules, and we try to show how isomorphism classes and degenerations differ depending on which algebra we work over. The Λ-isomorphism class of a module may contain several different $K \otimes_k \Lambda$-isomorphism classes, and in the case where K is a normal extension we give a complete description of these. We show several examples where modules degenerate as Λ-modules but not as $K \otimes_k \Lambda$-modules. We also find some examples where $M \oplus M \leq_{\text{deg}} N \oplus N$ but M does not degenerate to N.

References

[1] Abeasis, S. and Del Fra, A. Degenerations for the representations of an equioriented quiver of type A_m. Boll. Un. Mat. Ital. Suppl. (1980), no. 2, 157–171

[2] Abeasis, S. and Del Fra, A. Degenerations for the representations of an equioriented quiver of type D_m. Adv in Math 52 (1980), no. 2, 157–171

[3] Abeasis, S. and Del Fra, A. Degenerations for the representations of a quiver of type A_m. J. Algebra 93 (1985), no. 2, 376-412

[4] Artin, M. On Azumaya Algebras and Finite Dimensional Representations of Rings. J. Algebra 11 (1969), 532–563
[5] Bongartz, K. On Degenerations and Extensions of Finite Dimensional Modules. Adv. in Math. 121 (1996), 245–287.
[6] Bongartz, K. Degenerations for representations of tame quivers. Ann. scient. Ec. Norm. Sup., 4. serie, t. 28, (1995), 647–668.
[7] Forbregd, T. A., Nornes, N. M., Smalø, S. O. Partial Orders on Representations of Algebras. J. Algebra 323 (2010), no. 7, 2058–2062.
[8] Gerstenhaber, M. On nilalgebras and linear varieties of nilpotent matrices. Ann. of Math. 70 (1959), 167–205.
[9] Nornes, N. M., Oppermann, S. Degenerations of Submodules and Composition Series. Submitted to Algebr. Represent. Theory.
[10] Nornes, N. M. Module Degenerations and Finite Field Extensions.
[11] Riedtmann, C. Degenerations for representations of quivers with relations. Ann. Sci. Ecole Nomale Sup. 4 (1986), 275–301.
[12] Smalø, S. O. Degenerations of representations of associative algebras. Milan Journal of Math. 76, 1 (2008), 135–164.
[13] Zwara, G. A degeneration-like order for modules. Arch. Math. 71 (1998), 437-444
[14] Zwara, G. Degenerations for modules over representation-finite algebras. Proc. Amer. Math. Soc. 127, 5 (1999), 1313–1322.
[15] Zwara, G. Degenerations of finite-dimensional modules are given by extensions. Composito Math. 121 (2000), 205–218.

I

PARTIAL ORDERS ON REPRESENTATIONS OF ALGEBRAS

TORE A. FORBREGD, NILS M. NORNES, AND SVERRE O. SMALØ

Published in Journal of Algebra 323 (2010), no. 7, 2058-2062.

PARTIAL ORDERS ON REPRESENTATIONS OF ALGEBRAS

TORE A. FORBREGD, NILS M. NORNES, AND SVERRE O. SMALØ

Abstract. Let k be a commutative artin ring and let Λ be an artin k-algebra. For each natural number d let $\operatorname{rep}_d \Lambda$ be the set of isomorphism classes of Λ-modules with k-length equal to d. For each natural number n, an $n \times n$-matrix with entries in Λ, can be considered as a k-endomorphism of M^n, where M^n denotes the direct sum of n copies of the Λ-module M. The quasiorder $\leq_n$ on $\operatorname{rep}_d \Lambda$ is defined by $M \leq_n N$ if for every $n \times n$-matrix φ, with entries in Λ, we have that $\ell_k(M^n/\varphi M^n) \leq \ell_k(N^n/\varphi N^n)$.

We show that the quasiorder $\leq_n$ is a partial order on $\operatorname{rep}_d \Lambda$ for $n \geq d^3$.

1. Introduction

For an artin k-algebra Λ, where k is a commutative artin ring, and a natural number d, let $\operatorname{rep}_d \Lambda$ be the set of isomorphism classes of Λ-modules X such that the k-length of X, $\ell_k(X)$, is d. One can define several partial orders on $\operatorname{rep}_d \Lambda$, such as the degeneration order $\leq_{\text{deg}}$, the virtual degeneration order $\leq_{\text{vdeg}}$ and the Hom-order $\leq_{\text{hom}}$. The two first of these orders come from geometry when k is an algebraically closed field. However, due to a result of C. Riedtmann combined with a result of G. Zwara (see [6] and [8]) these orders have a purely module theoretical interpretation. Namely, $M \leq_{\text{deg}} N$ is equivalent to the existence of a short exact sequence of Λ-modules of the form

$$0 \longrightarrow X \longrightarrow X \oplus M \longrightarrow N \longrightarrow 0 .$$

and thus this can be taken as the definition for the relation $\leq_{\text{deg}}$ on $\operatorname{rep}_d \Lambda$ for all d. Furthermore, Zwara also showed that this is equivalent to the existence of a short exact sequence of the form

$$0 \longrightarrow N \longrightarrow M \oplus X \longrightarrow X \longrightarrow 0 .$$

The relation $\leq_{\text{vdeg}}$ is defined by $M \leq_{\text{vdeg}} N$ if there exists a Λ-module Y such that $Y \oplus M \leq_{\text{deg}} Y \oplus N$. Finally, the Hom-relation is defined by $M \leq_{\text{hom}} N$ if $\ell_k(\operatorname{Hom}_\Lambda(X, M)) \leq \ell_k(\operatorname{Hom}_\Lambda(X, N))$ for all Λ-modules X. The fact that $\leq_{\text{hom}}$ is a partial order is due to a result of M. Auslander. In [1] he showed that $M \simeq N$ if and only if

$\ell(\mathrm{Hom}_\Lambda(X, M)) = \ell(\mathrm{Hom}_\Lambda(X, N))$ for all finitely generated Λ-modules X. From Proposition 5 in the main section of this paper it follows that one does not need to consider all X to show that $M \simeq N$, it is enough to look at a certain set of submodules of M and N. This is proved using a construction due to O. Iyama [4].

It is known that $M \leq_{\mathrm{deg}} N$ implies $M \leq_{\mathrm{vdeg}} N$ which implies $M \leq_{\mathrm{hom}} N$. However, due to an example of J. Carlson we do not have that $\leq_{\mathrm{vdeg}}$ is equivalent to $\leq_{\mathrm{deg}}$ in general. It is still open whether $\leq_{\mathrm{vdeg}}$ is equivalent to $\leq_{\mathrm{hom}}$. For some classes of algebras it is known that these three orders coincide, e.g. algebras of finite representation type and tame hereditary algebras. For a quick overview the reader is referred to [7].

Here we will look at the Hom-order and the quasiorders $\leq_n$. If for every $n \times n$-matrix, φ, with entries in Λ, we have that $\ell_k(M^n/\varphi M^n) \leq \ell_k(N^n/\varphi N^n)$, then we write $M \leq_n N$. If k is a field there is a strong link between these quasiorders and the Hom-order, in the sense that there exists a natural number n_d such that $\leq_{n_d}$ is a partial order on $\mathrm{rep}_d\,\Lambda$ and this partial order coincides with $\leq_{\mathrm{hom}}$.[1] In the case where Λ is of finite representation type, it is known that there is a universal n such that $\leq_n$ is a partial order on $\mathrm{rep}_d\,\Lambda$ for any d (see [5]). The main result, Theorem 6, states that $\leq_{d^3}$ is a partial order.

2. Preliminaries

Let k be a commutative artin ring and let Λ be an artin k-algebra. For a ring R we have that if M is a right R-module, then it is in a natural way a left R^{op}-module. Therefore, throughout this article all modules will be unital left modules. Denote by mod Λ the category of finitely generated Λ-modules and for M in mod Λ let $\mathrm{add}\, M$ be the additive closure of M in mod Λ. If not specified otherwise, the length, $\ell(M)$, of a Λ-module M will mean its length as a k-module. We will write ${}_R(-,-)$ instead of $\mathrm{Hom}_R(-,-)$. For a Λ-module M let $\mathrm{rad}\, M$ be the Jacobson radical of M,that is the submodule of M given by the intersection of all maximal submodules of M. For artin algebras it is known that $\mathrm{rad}\, M = (\mathrm{rad}\,\Lambda) \cdot M$. Moreover, the socle of M, $\mathrm{soc}\, M$, is the sum of all simple submodules of M, i.e. the largest semisimple submodule of M.

Let $\mathcal{K}_0(\mathrm{mod}\ \Lambda) = F(\mathrm{mod}\ \Lambda)/R(\mathrm{mod}\ \Lambda)$ denote the Grothendieck group of mod Λ, where $F(\mathrm{mod}\ \Lambda)$ is the free abelian group on the isomorphism classes of Λ-modules and $R(\mathrm{mod}\ \Lambda)$ is the subgroup generated by all short exact sequences of Λ-modules. Let $[M]$ denote the

[1]We don't have a proof of this. See introduction.

element in $\mathcal{K}_0(\text{mod}\ \Lambda)$ corresponding to the module M in mod Λ. Furthermore, for a ring R, denote by $\mathscr{P}(R)$ the full subcategory of $\text{mod}\, R$ consisting of projective R-modules.

For a module M, M^n denotes the direct sum of n copies of M. Let $\mathscr{M}_n(\Lambda)$ be the set of $n \times n$-matrices with entries in Λ. A matrix $\varphi \in \mathscr{M}_n(\Lambda)$ induces a k-endomorphism on M^n by matrix multiplication from the left, and we denote the image of this homomorphism by φM^n. It also induces a Λ-endomorphism on Λ^n by matrix multiplication from the right, and we denote the image of this by $\Lambda^n\varphi$.

Definition. The Hom-relation $\leq_{\text{hom}}$ on $\text{rep}_d\,\Lambda$ is defined by $M \leq_{\text{hom}} N$ if $\ell(\text{Hom}_\Lambda(X, M)) \leq \ell(\text{Hom}_\Lambda(X, N))$ for all X in mod Λ.

It is obvious that the relation $\leq_{\text{hom}}$ is reflexive and transitive. In [1] Auslander showed that $M \simeq N$ if and only if $\ell({}_\Lambda(X, M)) = \ell({}_\Lambda(X, N))$ for all X in mod Λ, and thus that $\leq_{\text{hom}}$ is antisymmetric and hence a partial order. The result also holds in the more general setting of a commutative ring R and an R-linear abelian category where all morphism sets have finite lengths as R-modules. This generalization was proved by Bongartz in [3].

One can equivalently define a Hom-order $\leq'_{\text{hom}}$ by looking at ${}_\Lambda(M, X)$ and ${}_\Lambda(N, X)$, however this gives the same partial order as $\leq_{\text{hom}}$. For the convenience of the reader we will recall a proof of this fact.

Proposition 1. *Let M and N be modules in $\text{rep}_d\,\Lambda$. Then $M \leq_{\text{hom}} N$ if and only if $M \leq'_{\text{hom}} N$.*

Proof. We first consider the case where $\ell({}_\Lambda(P, M) < \ell({}_\Lambda(P, N))$ for an indecomposable projective Λ-module P. Since $\ell(M) = \ell(N)$ and $\ell(M) = \ell({}_\Lambda(\Lambda, M))$ there exists another indecomposable projective Λ-module P' with $\ell({}_\Lambda(P', M)) > \ell({}_\Lambda(P', N))$, so M and N are incomparable. Moreover, for an indecomposable projective Λ-module P, $\ell_{\text{End}_\Lambda(P)^{\text{op}}}({}_\Lambda(P, M))$ is equal to the number of times the simple module $P/\operatorname{rad} P$ occurs as a composition factor of M. Likewise, for an indecomposable injective Λ-module I, $\ell_{\text{End}_\Lambda(I)}({}_\Lambda(M, I))$ is equal to the number of times the simple module $\operatorname{soc} I$ occurs as a composition factor of M. Since P and I are indecomposable, $\text{End}_\Lambda(I)$ and $\text{End}_\Lambda(P)$ are local rings, thus we see that

$$\ell_k({}_\Lambda(P, M)) = \ell_{\text{End}_\Lambda(P)^{\text{op}}}({}_\Lambda(P, M)) \cdot \ell_k(\text{End}_\Lambda(P)/\operatorname{rad}(\text{End}_\Lambda(P)))$$
$$\ell_k({}_\Lambda(M, I)) = \ell_{\text{End}_\Lambda(I)}({}_\Lambda(M, I)) \cdot \ell_k(\text{End}_\Lambda(I)/\operatorname{rad}(\text{End}_\Lambda(I))).$$

Let I be the indecomposable injective Λ-module corresponding to P, i.e. I is the injective envelope of $P/\operatorname{rad} P$. By the above we have that $\ell({}_\Lambda(P, M) < \ell({}_\Lambda(P, N))$ implies $\ell({}_\Lambda(M, I)) < \ell({}_\Lambda(N, I))$. Hence we get that $\ell({}_\Lambda(P,M)) = \ell({}_\Lambda(P,N))$ for all projective Λ-modules P if and only if $\ell({}_\Lambda(M,I)) = \ell({}_\Lambda(N,I))$ for all injective Λ-modules I.

Now if $\ell({}_\Lambda(P,M)) = \ell({}_\Lambda(P,N))$ for all projective Λ-modules P, it follows from Corollary IV.4.3 in [2] that $\ell({}_\Lambda(M, D\operatorname{Tr} X)) \geq \ell({}_\Lambda(N, D\operatorname{Tr} X))$ if $\ell({}_\Lambda(X,M)) \geq \ell({}_\Lambda(X,N))$, where $D\operatorname{Tr}\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ is the Auslander-Reiten translate on mod Λ. Hence, we get that $\ell({}_\Lambda(X,M)) \geq \ell({}_\Lambda(X,N))$ for all X in mod Λ, if and only if $\ell({}_\Lambda(M,X)) \geq \ell({}_\Lambda(N,X))$ for all X in mod Λ. □

Definition. Let $M, N \in \operatorname{rep}_d \Lambda$. We say that $M \leq_n N$ if for every $\varphi \in \mathscr{M}_n(\Lambda)$ we have that $\ell(M^n/\varphi M^n) \leq \ell(N^n/\varphi N^n)$.

In general this gives a quasi-ordering on $\operatorname{rep}_d \Lambda$, however it is not always antisymmetric. It is known that $\leq_{d^5}$ is a partial order on $\operatorname{rep}_d \Lambda$ (see [5]).

We now give some basic facts about these quasi-orderings.

Proposition 2. *Let M and N be modules in $\operatorname{rep}_d \Lambda$, and let m and n be natural numbers. Then*

(1) *$M \leq_n N$ implies $M \leq_m N$ whenever $m \leq n$. In particular, if $\leq_m$ is a partial order, then so is $\leq_n$.*
(2) *$M \leq_{\hom} N$ implies $M \leq_n N$*

Proof. Part 1 follows from the fact that if $m \leq n$, every $m \times m$-matrix can be expanded to a $n \times n$-matrix simply by filling in enough zeros.

To show part 2, we consider the following exact sequence

$$\Lambda^n \xrightarrow{-\cdot\varphi} \Lambda^n \longrightarrow \Lambda^n/\Lambda^n\varphi \longrightarrow 0$$

with $\varphi \in \mathscr{M}_n(\Lambda)$. By applying ${}_\Lambda(-,M)$ to the sequence above we get the following exact commutative diagram.

$$\begin{array}{ccccccccc} 0 \longrightarrow {}_\Lambda(\Lambda^n/\Lambda^n\varphi, M) \longrightarrow & {}_\Lambda(\Lambda^n, M) & \xrightarrow{{}_\Lambda(-\cdot\varphi, M)} & {}_\Lambda(\Lambda^n, M) & & & \\ & \wr\!\! \| & & \wr\!\! \| & & & \\ & M^n & \xrightarrow{\varphi\cdot -} & M^n & \longrightarrow & M^n/\varphi M^n & \longrightarrow 0 \end{array}$$

Since the alternating sum of the lengths of the modules in an exact sequence equals zero, this yields that $\ell(M^n/\varphi M^n) = \ell({}_\Lambda(\Lambda^n/\Lambda^n\varphi, M))$, and hence $M \leq_{\hom} N$ implies that $M \leq_n N$ for all n. □

3. The Main Result

We begin by stating and proving the following lemma.

Lemma 3. *Let $M, N \in \operatorname{rep}_d \Lambda$ and $X \in \operatorname{rep}_s \Lambda$. If $M \leq_{d^2s} N$ and $N \leq_{d^2s} M$, then $l({}_\Lambda(X,M)) = l({}_\Lambda(X,N))$.*

Proof. Note first that without loss of generality we may assume that $\operatorname{ann} M = \operatorname{ann} N$, since otherwise there is a $\lambda \in \Lambda$ such that $\ell(M/\lambda M) \neq \ell(N/\lambda N)$. Let $\Gamma = \Lambda/\operatorname{ann} M$. We then have that $\Gamma \subseteq \operatorname{End}_k(M)$ and hence $\ell(\Gamma) \leq d^2$, and ${}_\Gamma(X/\operatorname{ann} M \cdot X, M) \simeq {}_\Lambda(X, M)$ and ${}_\Gamma(X/\operatorname{ann} M \cdot X, N) \simeq {}_\Lambda(X, N)$. Then we may find a free resolution of X over Γ which is of the form

$$\Gamma^{d^2 s} \longrightarrow \Gamma^s \longrightarrow X/\operatorname{ann} M \cdot X \longrightarrow 0.$$

By adding $d^2 s - s$ copies of the algebra Γ to the second and third terms, we get the exact sequence $\Gamma^{d^2 s} \xrightarrow{\varphi} \Gamma^{d^2 s} \longrightarrow X/\operatorname{ann} M \cdot X \oplus \Gamma^{d^2 s - s} \longrightarrow 0$, where φ can be described by a matrix in $\mathscr{M}_{d^2 s}(\Lambda)$. By applying ${}_\Gamma(-, M)$ and ${}_\Gamma(-, N)$ and counting lengths in the resulting sequences, we get that

$$\ell({}_\Gamma(X/\operatorname{ann} M \cdot X, M) + \ell(\varphi M^{d^2 s}) = s \cdot \ell(M)$$
$$\ell({}_\Gamma(X/\operatorname{ann} M \cdot X, N) + \ell(\varphi N^{d^2 s}) = s \cdot \ell(N).$$

Since M and N are both i $\operatorname{rep}_d \Lambda$, we have $\ell(M) = \ell(N) = d$, and $M \leq_{d^2 s} N$ and $N \leq_{d^2 s} M$ implies that $\ell(\varphi M^{d^2 s}) = \ell(\varphi N^{d^2 s})$. Hence,

$$\ell({}_\Lambda(X, M)) = \ell({}_\Gamma(X/\operatorname{ann} M \cdot X, M)) = \ell({}_\Gamma(X/\operatorname{ann} M \cdot X, N)) = \ell({}_\Lambda(X, N)).$$

□

Note that if one considers all matrices, rather than just square matrices, adding $\Gamma^{d^2 s - s}$ is not necessary. In other words, it is sufficient to look at $d^2 s \times s$-matrices.

In [4] O. Iyama showed that for each L in $\operatorname{mod} \Lambda$ we can find submodules $L_i \subset L$ for $i = 1, \ldots, r$ such that $\operatorname{gl.dim.} \operatorname{End}_\Lambda(\bigoplus_{i=0}^r L_i) < \infty$ where $L_0 = L$. In [4] these submodules are given by $L_{i+1} = \operatorname{rad}(\operatorname{End}_\Lambda(L_i)) \cdot L_i$, i.e. L_{i+1} is the submodule generated by the images of all maps in the Jacobson radical of $\operatorname{End}_\Lambda(L_i)$.

Remark 4. This construction behaves nicely with respect to direct sums, that is if $L = M \oplus N$ then $\operatorname{rad}(\operatorname{End}_\Lambda(L)) \cdot L \simeq M_1 \oplus N_1$ with $M_1 \subset M$ and $N_1 \subset N$.

Proposition 5. *Let M and N be in $\operatorname{rep}_d \Lambda$ and let $L_0 = M \oplus N$ and $L_{i+1} = \operatorname{rad}(\operatorname{End}_\Lambda(L_i)) \cdot L_i$ for $i = 1, \ldots, r$ with $L_{r+1} = 0$. Then $M \simeq N$ if and only if $\ell({}_\Lambda(X, M)) = \ell({}_\Lambda(X, N))$ for all X in $\operatorname{add} \bigoplus_{i=0}^r L_i$.*

Proof. Clearly $M \simeq N$ implies $\ell({}_\Lambda(X, M)) = \ell({}_\Lambda(X, N))$ for all X. So let $C = \bigoplus_{i=0}^r L_i$ and $\Gamma = \operatorname{End}_\Lambda(C)^{\mathrm{op}}$, and suppose that $\ell({}_\Lambda(X, M)) = \ell({}_\Lambda(X, N))$ for all X in $\operatorname{add} C$. It is sufficient to consider the indecomposable objects in $\operatorname{add} C$. Let $C \simeq \bigoplus_{j=1}^t C_j$ be a decomposition of C into indecomposable Λ-modules. We then have an equivalence of subcategories ${}_\Lambda(C,-)\colon \operatorname{add} C \twoheadrightarrow \mathscr{P}(\Gamma)$. For each C_j we have a k-isomorphism ${}_\Gamma({}_\Lambda(C,C_j),{}_\Lambda(C,M)) \simeq {}_\Lambda(C_j,M)$. Since C_j is in $\operatorname{add} C$, by assumption we get that $\ell({}_\Gamma({}_\Lambda(C,C_j),{}_\Lambda(C,M))) = \ell({}_\Gamma({}_\Lambda(C,C_j),{}_\Lambda(C,N)))$. As C_j is indecomposable, $\operatorname{End}_\Lambda(C_j)$ is a local ring. Therefore we have that

$$\ell({}_\Gamma({}_\Lambda(C,C_j),{}_\Lambda(C,M))) = \ell({}_\Gamma({}_\Lambda(C,C_j),{}_\Lambda(C,N)))$$

also when we consider lenghts over $\operatorname{End}_\Lambda(C_j)^{\mathrm{op}}$. This common number, denoted m_j, is the multiplicity of the simple Γ-module $S_j = {}_\Lambda(C,C_j)/\operatorname{rad}({}_\Lambda(C,C_j))$ as a composition factor of ${}_\Lambda(C,M)$, and as a composition factor of ${}_\Lambda(C,N)$, for $j = 1,\dots,t$. We then have that $[{}_\Lambda(C,M)] = \sum_{j=1}^t m_j[S_j] = [{}_\Lambda(C,N)]$ as elements in the Grothendieck group $\mathcal{K}_0(\operatorname{mod}\Gamma)$ of Γ. By [4], Γ has finite global dimension, and thus we have that the indecomposable projective Γ-modules constitute a basis for $\mathcal{K}_0(\operatorname{mod}\Gamma)$. This implies that every projective Γ-module is determined by its composition factors, and therefore ${}_\Lambda(C,M) \simeq {}_\Lambda(C,N)$. Through the equivalence ${}_\Lambda(C,-)\colon \operatorname{add} C \twoheadrightarrow \mathscr{P}(\Gamma)$ we get that $M \simeq N$. □

By combining Lemma 3 and Proposition 5 we get the following result.

Theorem 6. *The relation $\leq_{d^3}$ is a partial order on $\operatorname{rep}_d \Lambda$.*

Proof. It is enough to prove that the relation $\leq_{d^3}$ is antisymmetric. Suppose that $M \leq_{d^3} N$ and $N \leq_{d^3} M$. Let $L_0 = M \oplus N$ and let $C = \bigoplus_{i=0}^r L_i$ where $L_{i+1} = \operatorname{rad}(\operatorname{End}_\Lambda(L_i))\cdot L_i$ for $i = 1,\dots,r$ with $L_{r+1} = 0$. Let $C = \bigoplus_{j=1}^t C_j$ be a decomposition of C into indecomposable Λ-modules. By Remark 4 we have that $C_j \subset M$ or $C_j \subset N$ for all $1 \leq j \leq t$, and therefore $s_j = \ell(C_j) \leq d$. Since $d^3 \geq s_j d^2$, Lemma 3 yields that $\ell({}_\Lambda(C_j,M)) = \ell({}_\Lambda(C_j,N))$ for each indecomposable summand C_j of C, and hence $\ell({}_\Lambda(X,M)) = \ell({}_\Lambda(X,N))$ for all X in $\operatorname{add} C$. Thus the conditions of Proposition 5 are satisfied and we have that $M \simeq N$ and the relation $\leq_{d^3}$ is a partial order on $\operatorname{rep}_d \Lambda$. □

4. Closing Comment

For an M in $\operatorname{rep}_d \Lambda$, the length of the first syzygy of M as a module over $\Lambda/\operatorname{ann} M$ is bounded by $d^3 - d$. Therefore, $\leq_n$ will be a partial order on $\operatorname{rep}_d \Lambda$ for $n \geq \max\{d^3 - d, d\}$.

References

[1] Auslander, M. Representation theory of finite dimensional algebras. Contemp. Math. 13 (1982), 27–39.

[2] Auslander, M., Reiten, I. and Smalø, S. O. Representation Theory of Artin Algebras. Cambridge University Press, 1994.

[3] Bongartz, K. A generalization of a theorem of M. Auslander. Bull. London Math. Soc. 21, 3 (1989), 255–256.

[4] Iyama, O. Finiteness of representation dimension. Proceedings of the American Mathematical Society 131, 4 (2002), 1011–1014.

[5] Nornes, N. Partial orders related to the hom-order and degenerations. São Paulo Journal of Math. Sci. (To appear).

[6] Riedtmann, C. Degenerations for representations of quivers with relations. Ann. Sci. Ecole Nomale Sup. 4 (1986), 275–301.

[7] Smalø, S. O. Degenerations of representations of associative algebras. Milan Journal of Math. 76, 1 (2008), 135–164.

[8] Zwara, G. Degenerations of finite-dimensional modules are given by extensions. Composito Math. 121 (2000), 205–218.

II

DEGENERATIONS OF SUBMODULES AND COMPOSITION SERIES

NILS M. NORNES AND STEFFEN OPPERMANN

Submitted to Algebras and Representation Theory.

DEGENERATIONS OF SUBMODULES AND COMPOSITION SERIES

NILS M. NORNES AND STEFFEN OPPERMANN

Abstract. Let M and N be modules over an artin algebra such that M degenerates to N. We show that any submodule of M degenerates to a submodule of N. This suggests that a composition series of M will in some sense degenerate to a composition series of N.

We then study a subvariety of the module variety, consisting of those representations where all matrices are upper triangular. We show that these representations can be seen as representations of composition series, and that the orbit closures describe the above mentioned degeneration of composition series.

1. Introduction

Let k be an algebraically closed field, and let Λ be a finite dimensional associative k-algebra with unity. We denote by $\operatorname{mod}\Lambda$ the category of finite dimensional unital left modules over Λ. For natural numbers m and n, let $\mathcal{M}_{m\times n}(k)$ denote the set of $m \times n$-matrices with entries in k, let $\mathcal{M}_n(k)$ denote the k-algebra of $n \times n$-matrices and $\mathcal{U}_n(k) \subseteq \mathcal{M}_n(k)$ the subalgebra of upper triangular matrices. $\mathrm{GL}_n(k) \subseteq \mathcal{M}_n(k)$ denotes the general linear group, and $U_d(k) \subseteq \mathrm{GL}_d(k)$ denotes the subgroup of upper triangular matrices.

Fix a natural number d. We want to study the set of left Λ-module structures on the vector space k^d. We have a one-to-one correspondence between this set and the set of k-algebra homomorphisms from Λ to $\mathcal{M}_d(k)$. If f is such a homomorphism, we obtain a module structure by setting $\lambda \cdot \mathbf{v} := f(\lambda)\mathbf{v}$ for $\lambda \in \Lambda$ and $\mathbf{v} \in k^d$. Conversely, if we have a module structure, we get a k-algebra homomorphism g by setting $g(\lambda) := \begin{pmatrix}\lambda \cdot \mathbf{u}_1 & \dots & \lambda \cdot \mathbf{u}_d\end{pmatrix}$, where $\mathbf{u}_i$ is the ith unit column vector. Such a homomorphism is called a *d-dimensional representation of* Λ, and we denote the set of all d-dimensional representations of Λ by $\operatorname{mod}_d \Lambda$.

Let $\{\lambda_1, \dots, \lambda_n\}$ be a generating set of Λ. Then a representation $\rho \in \operatorname{mod}_d \Lambda$ is completely determined by its values on λ_i, so we can view $\operatorname{mod}_d \Lambda$ as a subset of $\mathcal{M}_d(k)^n$. This subset is Zariski closed, so

$\operatorname{mod}_d \Lambda$ has the structure of an affine variety. The group variety $\mathrm{GL}_d(k)$ acts on $\operatorname{mod}_d \Lambda$ by conjugation, and its orbits correspond bijectively to the isomorphism classes of modules. We can now give the definition of degeneration of modules.

Definition. Let M and N be Λ-modules with representations μ and ν in $\operatorname{mod}_d \Lambda$. M *degenerates to* N if ν lies in the closure of the $\mathrm{GL}_d(k)$-orbit of μ. This is denoted by $M \leq_{\deg} N$.

Degeneration is a partial order on the set of isomorphism classes of d-dimensional modules. The *codimension* of a degeneration $M \leq_{\deg} N$, denoted $\operatorname{codim}(M, N)$, is the codimension of the orbit corresponding to N in the closure of the orbit corresponding to M. The dimension of an orbit $\mathrm{GL}_d(k) * \mu$ can be computed by the formula $\dim \mathrm{GL}_d(k) * \mu = d^2 - [M, M]$, where $[M, M]$ denotes the k-dimension of $\operatorname{Hom}_\Lambda(M, M)$. From that we get $\operatorname{codim}(M, N) = [N, N] - [M, M]$.

In [8] G. Zwara, building on earlier work of C. Riedtmann in [4], gave a nice module-theoretic description of this partial order:

Theorem 1. *Let M and N be Λ-modules. Then the following are equivalent:*

(1) $M \leq_{\deg} N$
(2) *There exists a short exact sequence* $0 \to N \to M \oplus Z \to Z \to 0$ *in* $\operatorname{mod} \Lambda$ *for some* $Z \in \operatorname{mod} \Lambda$.
(3) *There exists a short exact sequence* $0 \to X \to M \oplus X \to N \to 0$ *in* $\operatorname{mod} \Lambda$ *for some* $X \in \operatorname{mod} \Lambda$.

The short exact sequences in Theorem 1 are called *Riedtmann-sequences*. In this paper we will use Riedtmann-sequences of the form $0 \to X \to M \oplus X \to N \to 0$, but all our results work equally well for sequences of the other form.

Now one can extend the notion of degeneration to algebras over arbitrary fields, and even over commutative artin rings, by using the existence of Riedtmann-sequences as the definition. G. Zwara showed in [7] that degeneration is a partial order also in this case. Here we define the codimension of $M \leq_{\deg} N$ to be $[N, N] - [M, M]$ (where $[X, X]$ denotes length of $\operatorname{Hom}_\Lambda(X, X)$ as a k-module.)

One problem with the degeneration order is that in general one cannot cancel common summands, that is $X \oplus M \leq_{\deg} X \oplus N$ does not imply $M \leq_{\deg} N$. This led to the introduction of a new partial order called *virtual degeneration* in [4].

Definition. Let M and N be Λ-modules. M *virtually degenerates to* N if there exists a module $X \in \operatorname{mod} \Lambda$ such that $X \oplus M \leq_{\deg} X \oplus N$. This is denoted by $M \leq_{\mathrm{vdeg}} N$.

The following proposition gives an alternative way of describing virtual degenerations. For a proof of the proposition see [6], section 2.

Proposition 2. *Let M and N be Λ-modules. Then $M \leq_{\mathrm{vdeg}} N$ if and only if there is some finitely presented functor $\delta : \operatorname{mod}\Lambda \to \operatorname{mod} k$ such that $\ell(\delta(X)) = [X,N] - [X,M]$ for all $X \in \operatorname{mod}\Lambda$.*

If δ is such a functor, we say that the degeneration is *given by δ*.

In section 2 we will prove the following:

Theorem 3. *Let M and N be Λ-modules and let $M' \subseteq M$ be a submodule.*

(1) *If $M \leq_{\mathrm{deg}} N$, then there exists a submodule $N' \subseteq N$ such that $M' \leq_{\mathrm{deg}} N'$.*
(2) *If $M \leq_{\mathrm{vdeg}} N$, then there exists a submodule $N' \subseteq N$ such that $M' \leq_{\mathrm{vdeg}} N'$.*

In section 3 we look at representations whose images are contained in $\mathcal{U}_d(k)$, which we call *triangular representations*. We show that these can be viewed as representations of composition series, and then we prove the following analogue of Theorem 1.

Theorem 4. *Let μ and ν be triangular Λ-representations, and let respectively $M_1 \overset{i_1}{\hookrightarrow} \dots \overset{i_{d-1}}{\hookrightarrow} M_d$ and $N_1 \overset{j_1}{\hookrightarrow} \dots \overset{j_{d-1}}{\hookrightarrow} N_d$ be the corresponding composition series. Then $\nu \in \overline{U_d(k) * \mu}$ if and only if there exists a commutative diagram*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
X_1 \arrow[r, "h_1"] \arrow[d] & X_2 \arrow[r, "h_2"] \arrow[d] & \cdots \arrow[r, "h_{d-1}"] & X_d \arrow[d] \\
X_1 \oplus M_1 \arrow[r, "\begin{pmatrix} h_1 & 0 \\ 0 & i_1 \end{pmatrix}"] \arrow[d] & X_2 \oplus M_2 \arrow[r, "\begin{pmatrix} h_2 & 0 \\ 0 & i_2 \end{pmatrix}"] \arrow[d] & \cdots \arrow[r, "\begin{pmatrix} h_{d-1} & 0 \\ 0 & i_{d-1} \end{pmatrix}"] & X_d \oplus M_d \arrow[d] \\
N_1 \arrow[r, "j_1"] \arrow[d] & N_2 \arrow[r, "j_2"] \arrow[d] & \cdots \arrow[r, "j_{d-1}"] & N_d \arrow[d] \\
0 & 0 & & 0
\end{tikzcd}$$

with exact columns.

To study degenerations of modules, one can look at the variety of quiver representations, $\operatorname{rep}_{\mathbf{d}}(Q,\rho)$, instead of $\operatorname{mod}_d \Lambda$. Let Q be a quiver with vertices $Q_0 = \{1,\dots,n\}$ and arrows Q_1, and let $\mathbf{d} =$

$(d_1, \ldots, d_n) \in \mathbb{N}^n$. Then $\operatorname{rep}_{\mathbf{d}} Q = \coprod_{\alpha \in Q_1} \mathcal{M}_{d_{e(\alpha)} \times d_{s(\alpha)}}(k)$, where $s(\alpha)$ and $e(\alpha)$ are respectively the start and end points of the arrow α, consists of all representations with dimension vector $\mathbf{d}$. The group variety $G_{\mathbf{d}} = \mathrm{GL}_{d_1}(k) \times \ldots \times \mathrm{GL}_{d_n}(k)$ acts on $\operatorname{rep}_{\mathbf{d}} Q$, and the orbits correspond to isomorphism classes. Given a set of relations ρ on Q, $\operatorname{rep}_{\mathbf{d}}(Q, \rho)$ is the subvariety of $\operatorname{rep}_{\mathbf{d}} Q$ consisting of all representations that satisfy the relations in ρ. K. Bongartz showed in [2] that the degeneration order we get from $\operatorname{rep}_{\mathbf{d}}(Q, \rho)$ is the same as the one we get from $\operatorname{mod}_d kQ/\langle \rho \rangle$. He also showed a deeper geometric connection between these varieties, but we will not go into that in this paper. Usually $\operatorname{rep}_{\mathbf{d}}(Q, \rho)$ is much smaller than $\operatorname{mod}_d kQ/\langle \rho \rangle$, which makes it easier to perform computations.

In section 4 we introduce a similar smaller variety that can be used to study degenerations of composition series.

For general background on representation theory of algebras we refer the reader to [1]. For an introduction to the topic of module degenerations, see [5].

2. Degenerations of submodules

In this section, let k be a commutative artin ring and let Λ be an artin k-algebra. All modules considered in this paper have finite length.

We first prove part 1 of Theorem 3.

Proposition 5. *Let M and N be Λ-modules and let $M' \subseteq M$ be a submodule. If $M \leq_{\deg} N$, then there exists a submodule $N' \subseteq N$ such that $M' \leq_{\deg} N'$.*

Proof. Assume that $M \leq_{\deg} N$ and let $M' \subseteq M$ be a submodule. Then there exists an exact sequence

$$\eta : \ 0 \longrightarrow X \xrightarrow{\binom{f}{g}} X \oplus M \longrightarrow N \longrightarrow 0 \ .$$

Let $X' = \{x \in X \mid gf^n(x) \in M' \quad \forall n \geq 0\}$, let $i_X : X' \to X$ and $i_M : M' \to M$ be the submodule inclusions. From the definition of X', we see that $f(X') \subseteq X'$ and $g(X') \subseteq M'$. Thus, by restricting $\binom{f}{g}$ to X', we get a homomorphism $\binom{f}{g}\big|_{X'}^{X' \oplus M'} : X' \to X' \oplus M'$. Let

$N' = \operatorname{coker} \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)\Big|_{X'}^{X'\oplus M'}$. We then have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X' \arrow[r, "\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)\big|_{X'}^{X'\oplus M'}"] \arrow[d, "i_X"] & X' \oplus M' \arrow[r] \arrow[d, "\left(\begin{smallmatrix} i_X & 0 \\ 0 & i_M \end{smallmatrix}\right)"] & N' \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & X \arrow[r, "\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)"] \arrow[d] & X \oplus M \arrow[r] \arrow[d] & N \arrow[r] & 0 \\
 & X/X' \arrow[r, "\left(\begin{smallmatrix} \overline{f} \\ \overline{g} \end{smallmatrix}\right)"] \arrow[d] & X/X' \oplus M/M' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. Since the top row is exact we have $M' \leq_{\deg} N'$, so it remains to show that α is a monomorphism. We have

$$\begin{aligned} \ker \overline{f} &= \{(x + X') \in X/X' \mid f(x) \in X'\} \\ &= \{(x + X') \in X/X' \mid gf^n(x) \in M' \forall n \geq 1\}. \end{aligned}$$

If $(x + X')$ is a non-zero element in $\ker \overline{f}$ then $x \notin X' = \{x \in X \mid gf^n(x) \in M' \forall n \geq 0\}$, so we must have $g(x) \notin M'$ and hence $(x + X') \notin \ker \overline{g}$. This means that $\ker \left(\begin{smallmatrix} \overline{f} \\ \overline{g} \end{smallmatrix}\right) = \ker \overline{f} \cap \ker \overline{g} = (0)$. Then by the Snake Lemma we get that $\ker \alpha = (0)$. $\square$

To prove the same result for virtual degenerations, we will need the following simple lemma.

Lemma 6. *Let X and Y be Λ-modules, and let $M \subseteq X \oplus Y$ be a submodule. Then there exist submodules $X' \subseteq X$ and $Y' \subseteq Y$ such that $M \leq_{\deg} X' \oplus Y'$.*

Proof. Let $i : M \to X \oplus Y$ be the inclusion and $p : X \oplus Y \to X$ the projection on the first summand. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & X \oplus Y \arrow[r, "p"] & X \arrow[r] & 0 \\
0 \arrow[r] & \ker pi \arrow[r] \arrow[u, hook] & M \arrow[r] \arrow[u, hook, "i"] & \operatorname{im} pi \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

with exact rows. From the bottom row we make an exact sequence

$$0 \to \ker pi \to \ker pi \oplus M \to \ker pi \oplus \operatorname{im} pi \to 0,$$

which shows that $M \leq_{\deg} \operatorname{im} pi \oplus \ker pi$. $\square$

We can now complete the proof of Theorem 3.

Theorem 3. *Let M and N be Λ-modules and let $M' \subseteq M$ be a submodule.*

(1) *If $M \leq_{\mathrm{deg}} N$, then there exists a submodule $N' \subseteq N$ such that $M' \leq_{\mathrm{deg}} N'$.*
(2) *If $M \leq_{\mathrm{vdeg}} N$, then there exists a submodule $N' \subseteq N$ such that $M' \leq_{\mathrm{vdeg}} N'$.*

Proof. Part 1 was proved in Proposition 5, so it remains to prove part 2.

Assume that $M \leq_{\mathrm{vdeg}} N$. Then there exists some $Y \in \operatorname{mod} \Lambda$ so that $M \oplus Y \leq_{\mathrm{deg}} N \oplus Y$. We have a submodule $M' \subseteq M$, and we want to find submodules $N' \subseteq N$ and $Y' \subseteq Y$ such that $M' \oplus Y' \leq_{\mathrm{deg}} N' \oplus Y'$. To do so we construct two descending chains of submodules $Y = Y_1 \supseteq Y_2 \supseteq \dots$ and $N = N_1 \supseteq N_2 \supseteq \dots$, where $M' \oplus Y_i \leq_{\mathrm{deg}} N_{i+1} \oplus Y_{i+1}$ for all i.

We have that $M' \oplus Y \subseteq M \oplus Y$, so by Proposition 5, there exists a submodule $Z_1 \subseteq N \oplus Y$ such that $M' \oplus Y \leq_{\mathrm{deg}} Z_1$. Then by Lemma 6, there exist submodules $N_2 \subseteq N$ and $Y_2 \subseteq Y$ such that $Z_1 \leq_{\mathrm{deg}} N_2 \oplus Y_2$, so we have $M' \oplus Y_1 \leq_{\mathrm{deg}} N_2 \oplus Y_2$.

For $i > 1$, assume that we have $M' \oplus Y_{i-1} \leq_{\mathrm{deg}} N_i \oplus Y_i$ and $Y_i \subseteq Y_{i-1}$. Then $M' \oplus Y_i \subseteq M' \oplus Y_{i-1}$, and we can again apply Proposition 5 and Lemma 6 to find $N_{i+1} \subseteq N_i$ and $Y_{i+1} \subseteq Y_i$ such that $M' \oplus Y_i \leq_{\mathrm{deg}} N_{i+1} \oplus Y_{i+1}$.

Since Y is artin there is some j such that $Y_j = Y_{j-1}$, so we have $M' \oplus Y_j \leq_{\mathrm{deg}} N_j \oplus Y_j$ and thus $M' \leq_{\mathrm{vdeg}} N_j$. □

For a module M, let $\operatorname{Sub} M$ denote the set of submodules of M. The construction in the proof of Proposition 5 induces a function $\phi_\eta : \operatorname{Sub} M \to \operatorname{Sub} N$. Note that if θ is a different Riedtmann-sequence for the same degeneration, the functions ϕ_η and ϕ_θ may be different. There are several questions that are natural to ask here, for example

- Is ϕ_η surjective?
- Is it injective?
- Is the codimension of $M' \leq_{\mathrm{deg}} N'$ bounded by the codimension of $M \leq_{\mathrm{deg}} N$?
- If $M \leq_{\mathrm{deg}} N$ is given by a finitely presented functor δ, is $M' \leq_{\mathrm{deg}} N'$ given by a subfunctor of δ?

As the following examples show, the answer to each of these questions is in general no.

Example 7. *Let k be a field, Q the Kronecker quiver,*

$$Q:\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2\ ,$$

and consider the path algebra kQ and the kQ-modules given by the quiver representations

$$I_2 =\ k^2 \underset{(0\ 1)}{\overset{(1\ 0)}{\rightrightarrows}} k\ , \qquad S_1 =\ k \underset{(0)}{\overset{(0)}{\rightrightarrows}} 0\ ,$$

$$S_2 =\ 0 \underset{(0)}{\overset{(0)}{\rightrightarrows}} k\ , \qquad R =\ k \underset{(0)}{\overset{(1)}{\rightrightarrows}} k$$

$$D\,\mathrm{Tr}\,S_1 =\ k^3 \underset{\left(\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right)}{\rightrightarrows}} k^2\ .$$

We have a degeneration $I_2 \leq_{\deg} R \oplus S_1$ given by a Riedtmann-sequence

$$\eta:\ 0 \longrightarrow R \longrightarrow R \oplus I_2 \longrightarrow R \oplus S_1 \longrightarrow 0\ .$$

Any $(1,1)$-dimensional regular module R' is isomorphic to a submodule of I_2, but when $R' \not\simeq R$ the only submodule of $R \oplus S_1$ it can degenerate to is the socle. Thus we see that ϕ_η is not injective. On the other hand, there is a k-family of submodules of $R \oplus S_1$ that are isomorphic to R. But there is only one submodule of I_2 that can degenerate to any of these, so ϕ_η is not surjective either.

Note also that we have $[D\,\mathrm{Tr}\,S_1, R \oplus S_1] - [D\,\mathrm{Tr}\,S_1, I_2] = 1 \leq [D\,\mathrm{Tr}\,S_1, S_1 \oplus S_2] - [D\,\mathrm{Tr}\,S_1, R'] = 3$, so if $R' \leq_{\deg} S_1 \oplus S_2$ is given by a functor δ, then δ can not be a subfunctor of any functor giving the degeneration $I_2 \leq_{\deg} R \oplus S_1$.

In the above example the codimension of the degeneration decreases when we go to the submodules, that is, for modules $M \leq_{\deg} N$ and submodules $M' \leq_{\deg} N'$ we have $\operatorname{codim}(M', N') \leq \operatorname{codim}(M, N)$. As the next example shows, this does not hold in general.

Example 8. *Let k be a field and $\Lambda = k[X]/(X^2)$, let S be the simple Λ-module and let $p : \Lambda \twoheadrightarrow S$ and $i : S \hookrightarrow \Lambda$ be the natural projection and inclusion. From the Riedtmann-sequence*

$$\eta:\ 0 \longrightarrow S \xrightarrow{\left(\begin{smallmatrix}0\\i\\0\end{smallmatrix}\right)} S \oplus \Lambda^2 \xrightarrow{\left(\begin{smallmatrix}0&0&1\\0&p&0\\1&0&0\end{smallmatrix}\right)} \Lambda \oplus S^2 \longrightarrow 0$$

we see that $\Lambda^2 \leq_{\deg} \Lambda \oplus S^2$*. Let* $M \subseteq \Lambda^2$ *be the image of* $\Lambda \oplus S \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)} \Lambda^2$ *. Then* $\phi_\eta(M) \simeq S^3$*, and* $\operatorname{codim}(\Lambda^2, \Lambda \oplus S^2) = 2$*, while* $\operatorname{codim}(M, S^3) = 4$*. However, for the Riedtmann-sequence*

$$\theta : \; 0 \longrightarrow S \xrightarrow{\left(\begin{smallmatrix} 0 \\ 0 \\ i \end{smallmatrix}\right)} S \oplus \Lambda^2 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 & p \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)} \Lambda \oplus S^2 \longrightarrow 0$$

we get $\phi_\theta(M) \simeq \Lambda \oplus S$*, and then* $\operatorname{codim}(M, \phi_\theta(M)) = 0$.

Applying Theorem 3 repeatedly we get a connection between the composition series of a module and the composition series of its degenerations.

Corollary 9. *Let* M *and* N *be* Λ*-modules such that* $M \leq_{\deg} N$ *(*$M \leq_{\mathrm{vdeg}} N$*), and let* $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d = M$ *be a composition series of* M*. Then there is a composition series* $(0) = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_d = N$ *of* N *such that for* $1 \leq i \leq d$ *we have* $M_i \leq_{\deg} N_i$ *(*$M_i \leq_{\mathrm{vdeg}} N_i$*). In particular,* $M_i/M_{i-1} \simeq N_i/N_{i-1}$*.*

So given a composition series $(0) \subseteq M_1 \subseteq \cdots \subseteq M_d$ of M and a Riedtmann-sequence of a degeneration $M \leq_{\deg} N$, we get a composition series $(0) \subseteq N_1 \subseteq \cdots \subseteq N_d$ of N that seems to be some kind of degeneration of $(0) \subseteq M_1 \subseteq \cdots \subseteq M_d$. If we are working over an algebraically closed field, it seems like there should be a variety of composition series where $(0) \subseteq N_1 \subseteq \cdots \subseteq N_d$ is in the orbit closure of $(0) \subseteq M_1 \subseteq \cdots \subseteq M_d$. In the next section we will describe such a variety.

3. Triangular representations

In this section let k be an algebraically closed field, and let Λ be a basic finite-dimensional k-algebra. We are going to look at the following subvariety of $\operatorname{mod}_d \Lambda$.

Definition. We call a representation $\rho \in \operatorname{mod}_d \Lambda$ *triangular* if $\operatorname{im} \rho \subseteq \mathcal{U}_d(k)$. We denote the set of all triangular representations in $\operatorname{mod}_d \Lambda$ by $T_d(\Lambda)$.

Given any subset of $\operatorname{mod}_d \Lambda$, an obvious question to ask is which d-dimensional Λ-modules have representations in the subset. As we shall see, all d-dimensional Λ-modules have representations in $T_d(\Lambda)$.

Clearly $T_d(\Lambda)$ is a closed subset of $\operatorname{mod}_d \Lambda$, so it is an affine variety. The group variety $U_d(k)$ acts on it by conjugation. In $\operatorname{mod}_d \Lambda$, orbits correspond to isoclasses of modules, and orbit closures can be described using Riedtmann-sequences. We are going to give a similar description of orbits and orbit closures in $T_d(\Lambda)$.

We will first show how a triangular representation can be viewed as a representation of a module and one of its composition series. Then we show that $U_d(k)$-orbits correspond 1-1 to isoclasses of composition series. Finally, we prove Theorem 4, which gives an algebraic description of the orbit closures, and shows that degeneration in $T_d(\Lambda)$ is the same as the degeneration of composition series suggested by Corollary 9.

Given a triangular representation $\mu = \mu_d$ we obtain a composition series in the following way: Let M_d be k^d with the module structure obtained from μ in the usual way. For each i let M_i be the submodule generated by the unit vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_i\}$. Then we get a representation μ_i of M_i simply by deleting the rightmost column and the bottom row of each of the matrices in μ_{i+1}.

Given a composition series $(0) \subseteq M_1 \subseteq \ldots \subseteq M_d$ we must choose a basis of M_d in order to construct a representation. Choosing the basis $\{\mathbf{x}_1, \dots, \mathbf{x}_d\}$ such that $\mathbf{x}_i \in M_i$ for all i, we get a representation that is triangular.

Since triangular representations represent composition series, and all modules have composition series, it follows that all modules have triangular representations.

We say that two composition series $(0) \subseteq M_1 \subseteq \cdots \subseteq M_d$ and $(0) \subseteq N_1 \subseteq \cdots \subseteq N_d$ are isomorphic if $M_i \simeq N_i$ for all i and these isomorphisms commute with the submodule inclusions. In $\operatorname{mod}_d \Lambda$ the isomorphism classes correspond to $\mathrm{GL}_d(k)$-orbits, and we want a similar correspondence for $T_d(\Lambda)$. We will now show that the orbits of $U_d(k)$ in $T_d(\Lambda)$ correspond to isomorphism classes of composition series.

If μ and ν are triangular representations of $(0) \subseteq M_1 \subseteq \ldots \subseteq M_d$ and $(0) \subseteq N_1 \subseteq \ldots \subseteq N_d$, and $\nu = g * \mu$ for some $g \in U_d(k)$, then since $g \in \mathrm{GL}_d(k)$ we have an isomorphism between M_d and N_d. Let $g_d = g$ and for $1 \leq i < d$ let g_i be the matrix obtained from g_{i+1} by deleting the bottom row and rightmost column. Then for each i, g_i gives us an isomorphism between M_i and N_i, and the isomorphisms commute with the inclusions, so the two composition series are isomorphic.

Conversely, let μ and ν be triangular representations where we have an isomorphism f between the corresponding composition series

$$\begin{tikzcd} M_1 \arrow[r, hook, "m_1"] \arrow[d, "f_1"] & M_2 \arrow[r, hook, "m_2"] \arrow[d, "f_2"] & \cdots \arrow[r, hook, "m_{d-1}"] & M_d \arrow[d, "f_d"] \\ N_1 \arrow[r, hook, "n_1"] & N_2 \arrow[r, hook, "n_2"] & \cdots \arrow[r, hook, "n_{d-1}"] & N_d \end{tikzcd} .$$

The matrices of m_i and n_i with respect to the standard bases of k^i and k^{i+1} are

$$A(m_i) = A(n_i) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \end{pmatrix}.$$

It is then easy to check that the matrix of f_i, $A(f_i)$, will be upper triangular for each i, and $\nu = A(f_d) * \mu$.

A composition series of a d-dimensional module can also be viewed as a "representation" of the quiver

$$\mathbb{A}_d : \ 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow d \ ,$$

but with Λ-modules and homomorphisms instead of vector spaces and linear maps. That is, we have a category $\Lambda\text{-rep}\,\mathbb{A}_d$, where the objects are series of d Λ-modules and $d-1$ Λ-homomorphisms

$$M_1 \xrightarrow{m_1} M_2 \xrightarrow{m_2} \cdots \xrightarrow{m_{d-1}} M_d$$

and morphisms are commutative diagrams

$$\begin{tikzcd} M_1 \arrow[r, "m_1"] \arrow[d, "f_1"] & M_2 \arrow[r, "m_2"] \arrow[d, "f_2"] & \cdots \arrow[r, "m_{d-1}"] & M_d \arrow[d, "f_d"] \\ N_1 \arrow[r, "n_1"] & N_2 \arrow[r, "n_2"] & \cdots \arrow[r, "n_{d-1}"] & N_d \end{tikzcd} ,$$

and the composition series are objects in this category. Similarly to the case of ordinary representations of $\mathbb{A}_d$, we have an equivalence between $\Lambda\text{-rep}\,\mathbb{A}_d$ and $\operatorname{mod}\mathcal{U}_d(\Lambda)$.

We can now consider degenerations in $T_d(\Lambda)$. Clearly $\nu \in \overline{U_d(k) * \mu}$ implies $\nu \in \overline{\mathrm{GL}_d(k) * \mu}$, but the converse does not hold.

Example 10. *Let $\Lambda = k[X]/(X^3)$ and consider $\operatorname{mod}_3 \Lambda$. Any representation is completely determined by its value on X, so we identify $\operatorname{mod}_d \Lambda$ with the set of nilpotent 3×3-matrices. Let*

$$\mu = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \nu = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In $\operatorname{mod}_3 \Lambda$, μ and ν are in the same orbit, but in $T_3(\Lambda)$ we have

$$\mu \in \overline{U_3(k) * \nu} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle|\, a, b \in k \right\},$$

but

$$\nu \notin \overline{U_3(k) * \mu} = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a \in k. \right\}$$

So as a triangular representation, μ is a proper degeneration of ν, even though as ordinary representations they are isomorphic.

Let S be the simple Λ-module and Y the 2-dimensional indecomposable Λ-module, and let i denote the inclusion $S \hookrightarrow Y$. Both μ and ν represent $S \oplus Y$, and the corresponding composition series are

$$\mu: \; 0 \hookrightarrow S \xhookrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} S \oplus S \xhookrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)} S \oplus Y$$

$$\nu: \; 0 \hookrightarrow S \xhookrightarrow{i} Y \xhookrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} S \oplus Y \, .$$

In Example 10, we have a degeneration at each level of the composition series. That is a necessary condition for having a degeneration in $T_d(\Lambda)$, but as the next example shows, it is not sufficient.

Example 11. *Keep the notation from Example 10, and let*

$$\nu' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This corresponds to the composition series

$$\nu': \; S \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} S \oplus S \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)} S \oplus Y \, .$$

Between μ and ν' we have isomorphisms at each level of the composition series, but the isomorphisms do not commute with the inclusions. Thus they are not isomorphic as composition series, and μ and ν are in different G'-orbits. As a triangular representation, μ is a proper degeneration of ν'. Despite ν'_i being a degeneration of μ_i for each i, ν' is not a degeneration of μ in $T_d(\Lambda)$.

In order to get a degeneration in $T_d(\Lambda)$ we somehow need the module degenerations to "commute" with the inclusions. More precisely, there must be Riedtmann-sequences for the module degenerations that form a commutative diagram with the composition series.

Theorem 4. *Let μ and ν be triangular Λ-representations, and let respectively $M_1 \xhookrightarrow{i_1} \dots \xhookrightarrow{i_{d-1}} M_d$ and $N_1 \xhookrightarrow{j_1} \dots \xhookrightarrow{j_{d-1}} N_d$ be the corresponding composition series. Then $\nu \in \overline{U_d(k) * \mu}$ if and only if there* *exists a commutative diagram*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
X_1 \arrow[r, "h_1"] \arrow[d] & X_2 \arrow[r, "h_2"] \arrow[d] & \cdots \arrow[r, "h_{d-1}"] & X_d \arrow[d] \\
X_1 \oplus M_1 \arrow[r, "\left(\begin{smallmatrix} h_1 & 0 \\ 0 & i_1 \end{smallmatrix}\right)"] \arrow[d] & X_2 \oplus M_2 \arrow[r, "\left(\begin{smallmatrix} h_2 & 0 \\ 0 & i_2 \end{smallmatrix}\right)"] \arrow[d] & \cdots \arrow[r, "\left(\begin{smallmatrix} h_{d-1} & 0 \\ 0 & i_{d-1} \end{smallmatrix}\right)"] & X_d \oplus M_d \arrow[d] \\
N_1 \arrow[r, "j_1"] \arrow[d] & N_2 \arrow[r, "j_2"] \arrow[d] & \cdots \arrow[r, "j_{d-1}"] & N_d \arrow[d] \\
0 & 0 & & 0
\end{tikzcd}$$

with exact columns.

For Example 10, we have this diagram (where p is the projection $Y \twoheadrightarrow Y/S \simeq S$):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\chi : & S \arrow[r, "1"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] & S \arrow[r, "i"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ i \end{smallmatrix}\right)"] & Y \arrow[d, "\left(\begin{smallmatrix} 0 \\ -p \\ 1 \end{smallmatrix}\right)"] \\
\chi \oplus \nu : & S \oplus S \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)"] \arrow[d, "(1\ 0)"] & S \oplus Y \arrow[r, "\left(\begin{smallmatrix} i & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & p \\ 1 & 0 \end{smallmatrix}\right)"] & Y \oplus (S \oplus Y) \arrow[d, "\left(\begin{smallmatrix} 0 & 1 & p \\ 1 & 0 & 0 \end{smallmatrix}\right)"] \\
\mu : & S \arrow[r, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[d] & S \oplus S \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)"] \arrow[d] & S \oplus Y \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

And for Example 11, we have this diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\chi' : & 0 \arrow[r] \arrow[d] & S \arrow[r, "1"] \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \\ -1 \end{smallmatrix}\right)"] & S \arrow[d, "\left(\begin{smallmatrix} 0 \\ 1 \\ -i \end{smallmatrix}\right)"] \\
\chi' \oplus \nu' : & S \arrow[r, "\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)"] \arrow[d, "1"] & S \oplus (S \oplus S) \arrow[r, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)"] & S \oplus (S \oplus Y) \arrow[d, "\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & i & 1 \end{smallmatrix}\right)"] \\
\mu : & S \arrow[r, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[d] & S \oplus S \arrow[r, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)"] \arrow[d] & S \oplus Y \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}
$$

We now come to the proof of Theorem 4.

Proof. We first assume that we have a commutative diagram

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] \\
X_1 \arrow[r, "h_1"] \arrow[d, "\left(\begin{smallmatrix} f_1 \\ g_1 \end{smallmatrix}\right)"] & X_2 \arrow[r, "h_2"] \arrow[d, "\left(\begin{smallmatrix} f_2 \\ g_2 \end{smallmatrix}\right)"] & \cdots \arrow[r, "h_{d-1}"] & X_d \arrow[d, "\left(\begin{smallmatrix} f_d \\ g_d \end{smallmatrix}\right)"] \\
X_1 \oplus M_1 \arrow[r, "\left(\begin{smallmatrix} h_1 & 0 \\ 0 & i_1 \end{smallmatrix}\right)"'] \arrow[d] & X_2 \oplus M_2 \arrow[r, "\left(\begin{smallmatrix} h_2 & 0 \\ 0 & i_2 \end{smallmatrix}\right)"'] \arrow[d] & \cdots \arrow[r, "\left(\begin{smallmatrix} h_{d-1} & 0 \\ 0 & i_{d-1} \end{smallmatrix}\right)"'] & X_d \oplus M_d \arrow[d] \\
N_1 \arrow[r, "j_1"] \arrow[d] & N_2 \arrow[r, "j_2"] \arrow[d] & \cdots \arrow[r, "j_{d-1}"] & N_d \arrow[d] \\
0 & 0 & & 0
\end{tikzcd}
$$

with exact columns, and show that this implies that $\nu \in \overline{U_d(k) * \mu}$.

The maps i_n and j_n are monomorphisms for all n, and we start by showing that h_n can also be assumed to be monic.

Let r be the highest number such that h_r is not monic. Let $\pi : X_r \to \operatorname{im} h_r$ be the natural projection and $\iota : \operatorname{im} h_r \to X_{r+1}$ the natural injection. We make a new commutative diagram by replacing the rth column with the image of the chain complex map $(h_r, \left(\begin{smallmatrix} h_r & 0 \\ 0 & i_r \end{smallmatrix}\right), j_r)$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & X_{r-1} \arrow[r, "\pi h_r"] \arrow[d, "\binom{f_{r-1}}{g_{r-1}}"] & \operatorname{im} h_r \arrow[r, "\iota"] \arrow[d, "\alpha"] & X_{r+1} \arrow[r] \arrow[d, "\binom{f_{r+1}}{g_{r+1}}"] & \cdots \\
\cdots \arrow[r] & X_{r-1} \oplus M_{r-1} \arrow[r, "\begin{pmatrix} \pi h_{r-1} & 0 \\ 0 & i_{r-1} \end{pmatrix}"'] \arrow[d] & \operatorname{im} h_r \oplus M_r \arrow[r, "\begin{pmatrix} \iota & 0 \\ 0 & i_r \end{pmatrix}"'] \arrow[d, "\beta"] & X_{r+1} \oplus M_{r+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & N_{r-1} \arrow[r, "j_{r-1}"] \arrow[d] & N_r \arrow[r, "j_r"] \arrow[d] & N_{r+1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The new column is a subcomplex of a short exact sequence, so α is a monomorphism, and it is also a quotient of a short exact sequence, so β is an epimorphism. Since $\dim_k(\operatorname{im} h_r \oplus M_r) = \dim_k \operatorname{im} h_r + \dim_k N_r$ it is exact. By induction, we can construct a diagram of the desired form where all the horizontal maps are monic.

We now use a modification of Riedtmann's proof that a Riedtmann-sequence implies degeneration. We want to find a family of representations $\{\nu^t\}_{t \in S} \subseteq T_d(\Lambda)$, where S is an open subset of k, $\nu^t \in U_d(k) * \mu$ for all $t \neq 0$, and $\nu^0 \in U_d(k) * \nu$. We choose a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_d\}$ for a complement of $\operatorname{im} \binom{f_d}{g_d}$ in $X_d \oplus M_d$, in such a way that $\mathbf{b}_i \in X_i \oplus M_i$ for all i. Let V be the span of B. Then we explicitly construct the modules N_d^t that will correspond to the representations ν^t. For each $t \in k$ we have a homomorphism

$$\phi_t : \ X_d \xrightarrow{\binom{f_d + t \cdot 1_{X_d}}{g_d}} X_d \oplus M_d \ .$$

Let S be the set of all $t \in k$ such that ϕ_t is a monomorphism and $\operatorname{im} \phi_t$ is a complement of V. As a vector space, N_d^t is V. To multiply with an element in Λ, we multiply in $X_d \oplus M_d$ and project the product onto N_d^t along the image of ϕ_t. For $t \neq 0$ ϕ_t is a split monomorphism, so we get an isomorphism between N_d^t and M_d. Restrictions of this yields an isomorphism between composition series, and thus we get that $\nu^t \in U_d(k) * \mu$. The map sending t to ν^t is continuous, so ν^0 must be in $\overline{U_d(k) * \mu}$.

To show the other implication, we embed $T_d(\Lambda)$ in $\operatorname{mod}_a(\mathcal{U}_d(\Lambda))$, where $a = \frac{d(d+1)}{2}$. Let $\{\lambda_1 = 1_\Lambda, \lambda_2, \ldots, \lambda_n\}$ be a generating set of Λ, and let $E_{i,j}$ denote the matrix where the jth entry of the ith row is 1, and all other entries are 0. Then $\mathcal{U}_d(\Lambda)$ is generated by the matrices

$$L_j = \begin{pmatrix} \lambda_j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_j \end{pmatrix}$$

for $1 \leq j \leq n$, $E_{i,i}$ for $1 \leq i \leq d$ and $E_{i,i+1}$ for $1 \leq i \leq d-1$. Let $\psi : T_d(\Lambda) \to \operatorname{mod}_a \mathcal{U}_d(\Lambda)$ be the morphism given by the following block matrices. Here I_n denotes the $n \times n$ identity matrix and 0_n the $n \times n$ zero matrix.

$$\psi(\mu)(L_j) = \begin{pmatrix} \mu_1(\lambda_j) & 0 & & 0 \\ 0 & \mu_2(\lambda_j) & & 0 \\ & & \ddots & \\ 0 & 0 & & \mu_d(\lambda_j) \end{pmatrix}$$

$$\psi(\mu)(E_{i,i}) = \begin{pmatrix} 0_1 & & 0 & 0 & 0 \\ & \ddots & & & \\ 0 & & 0_{i-1} & 0 & 0 \\ 0 & & 0 & I_i & 0 \\ 0 & & 0 & 0 & 0 \end{pmatrix}$$

$$\psi(\mu)(E_{i,i+1}) = \begin{pmatrix} 0_1 & & 0 & 0 & 0 \\ & \ddots & & & \\ 0 & & 0_i & I_i & 0 \\ 0 & & 0 & 0 & 0 \end{pmatrix}$$

Clearly ψ is a morphism of varieties, and $U_d(k)$-orbits in $T_d(\Lambda)$ are mapped into $\mathrm{GL}_a(k)$-orbits in $\operatorname{mod}_a \mathcal{U}_d(\Lambda)$. Thus $\nu \in \overline{U_d(k) * \mu}$ implies $\psi(\nu) \in \overline{\mathrm{GL}_a(k) * \psi(\mu)}$, and by Theorem 1 we then have an exact sequence of $\mathcal{U}_d(\Lambda)$-modules $0 \to \hat{X} \to \hat{X} \oplus \hat{M} \to \hat{N} \to 0$. Since $\operatorname{mod} \mathcal{U}_d(\Lambda) \simeq \Lambda\text{-rep}\, \mathbb{A}_d$, this gives us an exact sequence in $\Lambda\text{-rep}\, \mathbb{A}_d$, which is the commutative diagram we are looking for.

□

4. Smaller varieties of triangular representations

When studying degeneration of modules, one can replace $\operatorname{mod}_d \Lambda$ with a variety of quiver representations, which is usually much smaller. We want to find a similar variety smaller than $T_d(\Lambda)$.

As in the previous section, let k be an algebraically closed field, and let Λ be a basic finite-dimensional k-algebra. Then there is a quiver Q and a set of admissible relations ρ such that $\Lambda \simeq kQ/\langle\rho\rangle$. Let $\mathbf{d} = (d_1, \ldots, d_n)$ be a dimension vector over Q, and let $d = d_1 + \ldots + d_n$.

In the path algebra of a quiver we have some distinguished idempotents, namely the trivial paths $\{e_1, \ldots, e_n\}$. Choosing suitable idempotent matrices $A_i \in \mathcal{M}_d(k)$, we can identify $\operatorname{rep}_{\mathbf{d}}(Q, \rho)$ with the subvariety of $\operatorname{mod}_d \Lambda$ consisting of all representations μ such that $\mu(e_i) = A_i$. We want to construct a similar subvariety of $T_d(\Lambda)$.

Recall that a set of idempotents $\{e_1, \ldots, e_n\}$ in Λ is called *orthogonal* if $e_i e_j = 0$ when $i \neq j$, and a non-zero idempotent is called *primitive* if it cannot be written as a sum of two non-zero orthogonal idempotents. An orthogonal set of primitive idempotents is called *complete* if it is not a proper subset of a larger orthogonal set of primitive idempotents. If the orthogonal set $\{e_1, \ldots, e_n\}$ is complete, then for any simple Λ-module S we have $S \simeq e_i\Lambda / \operatorname{rad} e_i\Lambda$ for some i. The set of trivial paths in a path algebra is an example of a complete orthogonal set of primitive idempotents.

Let $E = \{e_1 \ldots, e_n\} \subseteq \Lambda$ be an orthogonal set of primitive idempotents. We want to fix some idempotent matrices $A_i \in \mathcal{U}_d(\Lambda)$ and look at the subvariety of $T_d(\Lambda)$ consisting of representations μ such that $\mu(e_i) = A_i$. When we make this restriction in $\operatorname{mod}_d \Lambda$, we go from having representations of all d-dimensional modules to having just those with a particular set of composition factors. When we do the same in $T_d(\Lambda)$, the sequence in which the factors occur in the composition series also matters.

Proposition 12. *Let M and N be d-dimensional Λ-modules. The following are equivalent:*

(1) *There exist composition series $(0) = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d = M$ and $(0) = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_d = N$ such that $M_i/M_{i-1} \simeq N_i/N_{i-1}$ for $1 \leq i \leq d$.*
(2) *For any orthogonal set E of idempotents in Λ, there exist triangular representations $\mu, \nu \in \operatorname{mod}_d \Lambda$ of M and N respectively, such that $\mu(e) = \nu(e)$ for all $e \in E$.*
(3) *There exists a complete orthogonal set E of primitive idempotents in Λ and triangular representations $\mu, \nu \in \operatorname{mod}_d \Lambda$ of M and N respectively, such that $\mu(e) = \nu(e)$ for all $e \in E$.*

Proof. We first show that 1 implies 2. Let E be an orthogonal set of idempotents. Since any idempotent can be written as a sum of primitive idempotents, and any orthogonal set can be expanded to a complete orthogonal set, we may assume that E is a complete orthogonal set of primitive idempotents.

When $d = 1$, $1 \Rightarrow 2$ is obvious. Assume it holds for $d = l - 1$ and let M and N be l-dimensional modules satisfying 1. Then M_{l-1} and N_{l-1} have triangular representations μ and ν where $\mu(e) = \nu(e)$

for all $e \in E$. We now want to construct suitable bases for M and N. Let $(m_1, \dots, m_{l-1})$ and $(n_1, \dots, n_{l-1})$ be bases for M_{l-1} and N_{l-1} corresponding to μ and ν. Choose elements $m \in M \setminus M_{l-1}$ and $n \in N \setminus N_{l-1}$. Since M/M_{l-1} is simple there is exactly one element $e \in E$ such that $eM/M_{l-1} \neq 0$, and since $M/M_{l-1} \simeq N/N_{l-1}$ we also have $eN/N_{l-1} \neq 0$. We set $m_l = em$ and $n_l = en$. Then $(m_1, \dots, m_l)$ and $(n_1, \dots, n_l)$ are bases for M and N, and we let μ' and ν' be the corresponding representations.

We now have that for any $x \in \Lambda$,

$$\mu'(x) = \left(\begin{array}{ccc|c} & & & s_1^x \\ & \mu(x_i) & & \vdots \\ & & & s_{l-1}^x \\ \hline 0 & \cdots & 0 & s_l^x \end{array}\right)$$

where $s_i^x \in k$. The $l-1$ first entries in row l are all 0 because $M_{l-1} \subseteq M$ is a submodule. Since $\mu(x)$ is upper triangular, $\mu'(x)$ is too. Thus we have that μ' is triangular. Similarly we see that ν' is triangular. Furthermore we have

$$\mu'(e) = \left(\begin{array}{ccc|c} & & & 0 \\ & \mu(e) & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) = \nu'(e),$$

and for any other $e' \in E$ we have

$$\mu'(e') = \left(\begin{array}{ccc|c} & & & 0 \\ & \mu(e') & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right) = \nu'(e').$$

Thus we have $\mu'(e) = \nu'(e)$ for all $e \in E$. By induction we get that $1 \Rightarrow 2$.

Obviously 2 implies 3, so it remains to show that 3 implies 1. Again this is obvious for $d = 1$. Assume that it holds for $d = l - 1$ and let M and N be l-dimensional modules satisfying 3. Let $(m_1, \dots, m_l)$ and $(n_1, \dots, n_l)$ be the bases corresponding to μ and ν. Since μ is triangular, $\{m_1, \dots, m_{l-1}\}$ spans a submodule which we call M_{l-1}. We construct N_{l-1} in the same way. M_{l-1} and N_{l-1} satisfy 3, so by assumption they also satisfy 1. All that is left to check is that $M/M_{l-1} \simeq N/N_{l-1}$. Let $x \in E$ be the idempotent with $xM/M_{l-1} \neq 0$. Then we have

$$xm_l \notin M_{l-1} \Leftrightarrow \mathbf{u}_l^T \mu(x) \mathbf{u}_l = \mathbf{u}_l^T \nu(x) \mathbf{u}_l \neq 0 \Leftrightarrow xn_l \notin N_{l-1} \Leftrightarrow xN/N_{l-1} \neq 0,$$

which shows that $M/M_{l-1} \simeq N/N_{l-1}$. $\square$

Two modules may have the same dimension vector, yet not have compatible composition series as above. Thus, when we restrict to triangular representations with fixed values on E, we get representations of at most one of them.

Example 13. *Let Q be the quiver* $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ *, and let* $\Lambda = kQ/(\alpha\beta, \beta\alpha)$*. Λ is generated by $\{e_1, e_2, \alpha, \beta\}$, where e_i is the trivial path corresponding to the vertex i. Consider the quiver representations*

$$M: \; k \underset{0}{\overset{1}{\rightleftarrows}} k \, , \qquad N: \; k \underset{1}{\overset{0}{\rightleftarrows}} k \, .$$

M and N both have simple socles, but the socles are not isomorphic. Thus they do not satisfy statement 1 in Proposition 12. $\{e_1, e_2\}$ is a complete set of primitive orthogonal idempotents, so if μ and ν are representations of M and N, and we have $\mu(e_1) = \nu(e_1)$ and $\mu(e_2) = \nu(e_2)$, then by Proposition 12 μ and ν cannot both be triangular.

For example, let $\mu, \nu \in \mathrm{mod}_2 \Lambda$ be the functions given by

$$(\mu(e_1), \mu(e_2), \mu(\alpha), \mu(\beta)) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right),$$

$$(\nu(e_1), \nu(e_2), \nu(\alpha), \nu(\beta)) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right).$$

Then μ represents M and ν represents N. We see that $\mu(e_i) = \nu(e_i)$ for $i = 1, 2$ but $\mu(\alpha)$ is not upper triangular, so μ is not a triangular representation. If we instead use a triangular representation of M, say μ' given by

$$(\mu'(e_1), \mu'(e_2), \mu'(\alpha), \mu'(\beta)) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right),$$

we get $\mu'(e_i) \neq \nu(e_i)$ (and in fact the only nonzero idempotent e such that $\mu'(e) = \nu(e)$ is the identity).

So we want an analogue of dimension vectors that also records the sequence of the composition factors.

Definition. The *composition vector* of a composition series $(0) = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_d$ is an element $\mathbf{c} = (c_1, \ldots, c_d) \in E \times \ldots \times E$ such that for all i we have $M_i/M_{i-1} \simeq c_i\Lambda/\operatorname{rad} c_i\Lambda$.

Now given a composition vector $\mathbf{c}$ we can construct a subvariety $T_{\mathbf{c}}(\Lambda) \subseteq T_d(\Lambda)$ in the following way. For $1 \leq i \leq n$ let $A_i^{\mathbf{c}}$ be the diagonal $d \times d$-matrix where the jth element on the diagonal is 1 if $c_j = e_i$ and 0 otherwise. Then let $T_{\mathbf{c}}(\Lambda) = \{\mu \in T_d(\Lambda) \mid \mu(e_i) = A_i^{\mathbf{c}}\}$.

Any representation in $T_{\mathbf{c}}(\Lambda)$ represents a composition series with composition vector $\mathbf{c}$, and from Proposition 12 we see that all composition series with this composition vector are represented in $T_{\mathbf{c}}(\Lambda)$.

We also need a suitable group variety to act on $T_{\mathbf{c}}(\Lambda)$. The normalizer of a closed subset is itself closed, and thus an affine group variety (see e.g. [3], Lemma 2.5.1). Since $T_{\mathbf{c}}(\Lambda)$ is a closed subset both in $T_d(\Lambda)$ and in $\operatorname{mod}_d \Lambda$, we could use its normalizer in either $\mathrm{GL}_d(k)$ or $U_d(k)$. We denote these normalizers $N_{\mathrm{GL}_d(k)}(T_{\mathbf{c}}(\Lambda))$ and $N_{U_d(k)}(T_{\mathbf{c}}(\Lambda))$ respectively.

If we choose $N_{\mathrm{GL}_d(k)}(T_{\mathbf{c}}(\Lambda))$, then the group action no longer preserves composition series. This is shown in the next example.

Example 14. *Let Λ be the Kronecker algebra as in Example 7, and consider the modules R and R_2 given by the following quiver representations.*

$$R: \; k \underset{0}{\overset{1}{\rightrightarrows}} k \, , \qquad R_2: \; k^2 \underset{\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}{\rightrightarrows}} k^2$$

R_2 has triangular representations μ and ν given by

$$\mu(e_1) = \nu(e_1) = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}$$

$$\mu(e_2) = \nu(e_2) = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

$$\mu(\alpha) = \nu(\alpha) = \begin{pmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

$$\mu(\beta) = \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \qquad \nu(\beta) = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

The corresponding composition series are $0 \subseteq S_2 \subseteq S_2^2 \subseteq P_1 \subseteq R_2$ for μ and $0 \subseteq S_2 \subseteq S_2^2 \subseteq R \oplus S_2 \subseteq R_2$ for ν. They both have composition vector $\mathbf{c} = (e_2, e_2, e_1, e_1)$, but they are not isomorphic. $T_{\mathbf{c}}(\Lambda)$ is isomorphic to the variety of quiver representations, $\mathrm{rep}_{(2,2)}(Q) \simeq \mathcal{M}_2(k) \times \mathcal{M}_2(k)$, and we have $N_{\mathrm{GL}_4(k)}(T_{\mathbf{c}}(\Lambda)) \simeq \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$.

Since μ and ν both represent the module R_2, they are in the same $N_{\mathrm{GL}_4(k)}(T_{\mathbf{c}}(\Lambda))$-orbit.

Example 14 shows that $N_{\mathrm{GL}_d(k)}(T_{\mathbf{c}}(\Lambda))$ is a poor choice for the group action. The action of $N_{U_d(k)}(T_{\mathbf{c}}(\Lambda))$ on the other hand, obviously does preserve composition series. In fact, we can restate Theorem 4 with $T_{\mathbf{c}}(\Lambda)$ in the place of $T_d(\Lambda)$.

Theorem 15. *Let $\mathbf{c}$ be a composition vector, let $\mu, \nu \in T_{\mathbf{c}}(\Lambda)$, and let respectively $M_1 \xhookrightarrow{i_1} \dots \xhookrightarrow{i_{d-1}} M_d$ and $N_1 \xhookrightarrow{j_1} \dots \xhookrightarrow{j_{d-1}} N_d$ be the corresponding composition series. Then $\nu \in \overline{N_{U_d(k)}(T_{\mathbf{c}}(\Lambda)) * \mu}$ if and only if there exists a commutative diagram*

$$
\begin{tikzcd}[ampersand replacement=\&]
0 \arrow[d] \& 0 \arrow[d] \& \& 0 \arrow[d] \\
X_1 \arrow[r, "h_1"] \arrow[d] \& X_2 \arrow[r, "h_2"] \arrow[d] \& \cdots \arrow[r, "h_{d-1}"] \& X_d \arrow[d] \\
X_1 \oplus M_1 \arrow[r, "\begin{pmatrix} h_1 & 0 \\ 0 & i_1 \end{pmatrix}"] \arrow[d] \& X_2 \oplus M_2 \arrow[r, "\begin{pmatrix} h_2 & 0 \\ 0 & i_2 \end{pmatrix}"] \arrow[d] \& \cdots \arrow[r, "\begin{pmatrix} h_{d-1} & 0 \\ 0 & i_{d-1} \end{pmatrix}"] \& X_d \oplus M_d \arrow[d] \\
N_1 \arrow[r, "j_1"] \arrow[d] \& N_2 \arrow[r, "j_2"] \arrow[d] \& \cdots \arrow[r, "j_{d-1}"] \& N_d \arrow[d] \\
0 \& 0 \& \& 0
\end{tikzcd}
$$

with exact columns.

The proof is the same as for Theorem 4, we just have to choose the basis for V a little more carefully. Here we need to have $c_i b_i = b_i$ for all i.

References

[1] Auslander, M., Reiten, I. and Smalø, S. O.: Representation Theory of Artin Algebras. Cambridge University Press (1994)

[2] Bongartz, K.: A geometric version of the Morita equivalence. J. Algebra 139, 150–171 (1991)

[3] Geck, M.: An Introduction to Algebraic Geometry and Algebraic Groups. Oxford University Press (2003)

[4] Riedtmann, C.: Degenerations for representations of quivers with relations. Ann. Sci. Ecole Nomale Sup. 4, 275–301 (1986)

[5] Smalø, S. O.: Degenerations of representations of associative algebras. Milan Journal of Math. 76, 1, 135–164 (2008)

[6] Smalø, S.O. and Valenta, A.: Finitely presented functors and degenerations. Comm. Algebra 34, no. 5, 1861–1889, (2006)

[7] Zwara, G.: A degeneration-like order for modules. Arch. Math. 71, 437–444 (1998)

[8] Zwara, G.: Degenerations of finite-dimensional modules are given by extensions. Composito Math. 121, 205–218 (2000)

III

MODULE DEGENERATIONS AND FINITE FIELD EXTENSIONS

NILS M. NORNES

MODULE DEGENERATIONS AND FINITE FIELD EXTENSIONS

NILS M. NORNES

ABSTRACT. Degeneration of modules is usually defined geometrically, but due to results of Zwara and Riedtmann we can also define it in purely homological terms. This homological definition also works over fields that are not algebraically closed. Let k be a field, K a finite extension of k and Λ a k-algebra. Then any $K \otimes_k \Lambda$-module is also a Λ-module. We study how the isomorphism classes, degeneration and hom-order differ depending on whether we work over Λ or $K \otimes_k \Lambda$.

1. INTRODUCTION

Let k be a field, K a normal finite field extension of k and Q a quiver. Since K-vector spaces are also k-vector spaces and all K-linear maps are k-linear, any K-representation of Q is also a k-representation. But, since not all k-linear maps are K-linear, two nonisomorphic K-representations may be isomorphic as k-representations.

Example 1. Let Q be the Kronecker quiver and consider the $\mathbb{C}Q$-modules

$$M: \mathbb{C} \overset{1}{\underset{i}{\rightrightarrows}} \mathbb{C}, \qquad N: \mathbb{C} \overset{1}{\underset{-i}{\rightrightarrows}} \mathbb{C}.$$

M and N are not isomorphic as $\mathbb{C}Q$-modules, but if we view them as $\mathbb{R}Q$-modules, there is an isomorphism given by complex conjugation.

More generally, if Λ is a k-algebra, then two $K \otimes_k \Lambda$-modules may be isomorphic in $\operatorname{mod} \Lambda$, the category of finite-dimensional Λ-modules, but nonisomorphic in $\operatorname{mod} K \otimes_k \Lambda$.

When we need to specify which algebra two modules are isomorphic over, we will add a superscript to the isomorphism sign, e.g. $M \simeq^{\mathbb{R}Q} N$.

The Λ-isomorphism class of a given $K \otimes_k \Lambda$-module splits into a number of $K \otimes_k \Lambda$-isomorphism classes. In section 2 we give a complete description of these isomorphism classes.

Since isomorphism classes depend on which algebra we are working over, so do the degeneration order and the Hom-order.

Degeneration of modules is usually defined geometrically. For a natural number d and a k-algebra Λ, let $\operatorname{mod}_d \Lambda$ be the set of algebra homomorphisms from Λ to $\mathcal{M}_d(k)$, the ring of $d \times d$-matrices with entries in k. Given a homomorphism $\mu \in \operatorname{mod} \Lambda$ we can make a module structure on k^d. For any $\lambda \in \Lambda, x \in k^d$, we define $\lambda x := \mu(\lambda) \cdot x$, where x is viewed as a column vector and the multiplication on the right hand side is just matrix multiplication. This lets us identify $\operatorname{mod}_d \Lambda$ with the set of Λ-module structures on k^d. The set $\operatorname{mod}_d \Lambda$ is actually an affine variety, and we say that a module M degenerates to a module N if N is in the closure of the isomorphism class of M.

This definition only works when k is algebraically closed, and in this paper we want to look at other fields. In [9], G. Zwara showed that there is an equivalent module theoretic way to describe degeneration, and we will use this description as the definition.

Definition. Let M and N be modules in $\operatorname{mod} \Lambda$. M degenerates to N if there exists a module $X \in \operatorname{mod} \Lambda$ and an exact sequence

$$0 \longrightarrow X \longrightarrow X \oplus M \longrightarrow N \longrightarrow 0\,.$$

We denote this by $M \leq_{\text{deg}} N$. An exact sequence of the above form is called a Riedtmann sequence.

This definition works for any field. With this definition it is not obvious that $\leq_{\text{deg}}$ is a partial order, but this was shown by G. Zwara in [7].

The degeneration order does not behave nicely with respect to cancellation of common direct summands, so in [3] C. Riedtmann introduced another order.

Definition. Let M and N be Λ-modules. M virtually degenerates to N if there exists $Z \in \operatorname{mod} \Lambda$ such that $M \oplus Z \leq_{\text{deg}} N \oplus Z$. We denote this by $M \leq_{\text{vdeg}} N$.

$M \leq_{\text{deg}} N$ clearly implies $M \leq_{\text{vdeg}} N$, but for some algebras the virtual degeneration is strictly finer. This was first shown by an example due to J. Carlson (see [3]).

The last partial order we want to study in this paper is the Hom-order, which is based on the dimensions of Hom-spaces. We will denote the k-dimension of $\operatorname{Hom}_\Lambda(M, N)$ by ${}_\Lambda[M, N]$.

Definition. Given two Λ-modules M and N, $M \leq_{\text{Hom}} N$ if ${}_\Lambda[X, M] \leq {}_\Lambda[X, N]$ for all $X \in \operatorname{mod} \Lambda$ (or, equivalently, if ${}_\Lambda[M, X] \leq {}_\Lambda[N, X]$ for all $X \in \operatorname{mod} \Lambda$).

The relation $\leq_{\mathrm{Hom}}$ is clearly reflexive and transitive. In [1], M. Auslander showed that if $M \not\simeq N$ then there exists an $X \in \operatorname{mod}\Lambda$ such that ${}_\Lambda[X, M] \neq {}_\Lambda[X, N]$, which shows that $\leq_{\mathrm{Hom}}$ is also antisymmetric.

$M \leq_{\mathrm{vdeg}} N$ implies $M \leq_{\mathrm{Hom}} N$, but it is not known if $\leq_{\mathrm{Hom}}$ is strictly finer.

However, if the algebra is representation-finite, all three orders are the same. This was shown for algebras over algebraically closed fields by G. Zwara in [8] and generalized to arbitrary artin algebras by S. O. Smalø in [4].

As with isomorphisms, we add a superscript when we need to specify which algebra we are considering.

In section 3 we give several examples where $\leq^{\Lambda}_{\mathrm{deg}}$ differs from $\leq^{K\otimes_k\Lambda}_{\mathrm{deg}}$. We also give some examples of modules M, N where $M \oplus M$ degenerates to $N \oplus N$ but M does not degenerate to N. For some algebras Λ the $K \otimes_k \Lambda$-isomorphism classes are the same as the Λ-isomorphism classes. We show that in these cases $\leq^{K\otimes_k\Lambda}_{\mathrm{Hom}}$ and $\leq^{\Lambda}_{\mathrm{Hom}}$ are also the same.

In section 4 we show that if the endomorphism ring of a module is a division ring, then the module is minimal in the degeneration- and Hom-orders.

For background on representation theory of algebras we refer the reader to [2]. For an introduction to degenerations of modules, see [4].

2. Isomorphism classes

Let k be a field, K a separable finite extension of k and Λ a k-algebra. Let $\Gamma = K \otimes_k \Lambda$. For any Λ-module M we give $K \otimes_k M$ a Γ-module structure by $(x \otimes \lambda) \cdot (y \otimes m) = xy \otimes \lambda m$. Since Λ is a subring of Γ any Γ-module is also a Λ-module.

Furthermore, any Γ-homomorphism is a Λ-homomorphism, so $X \simeq^{\Gamma} Y$ implies $X \simeq^{\Lambda} Y$. But, as Example 1 shows, the reverse implication does not hold.

In Example 1 we see that the $\mathbb{R}Q$-isomorphism class of M contains two $\mathbb{C}Q$-isomorphism classes, and one is in some sense a complex conjugate of the other. On the other hand, the $\mathbb{R}Q$-isomorphism class of the module

$$X_a: \ \mathbb{C} \underset{a}{\overset{1}{\rightrightarrows}} \mathbb{C}$$

contains only one $\mathbb{C}Q$-isomorphism class if $a \in \mathbb{R}$. If a is not real it has two $\mathbb{C}Q$-isomorphism classes. Note also that when a is real there exists a $\mathbb{R}Q$-module Y_a such that $X_a \simeq^{\mathbb{C}Q} \mathbb{C} \otimes_{\mathbb{R}} Y_a$, whereas when a is not real there is no such $\mathbb{R}Q$-module.

Similarly, for any indecomposable $\mathbb{C}Q$-module M its $\mathbb{R}Q$-isomorphism class contains either one or two $\mathbb{C}Q$-isomorphism classes. When there are two, a module in the second class can be constructed from M by complex conjugation.

More generally, if K is a normal extension of k of degree n, the Λ-isomorphism class of an indecomposable Γ-module splits into at most n Γ-isomorphism classes, and they are related by k-automorphisms of K.

Given a k-automorphism of K and a Γ-module M, we can construct a Γ-module that is Λ-isomorphic to M in the following way.

Let ϕ be a k-automorphism on K, and let M be a Γ-module. We construct a new Γ-module M^ϕ by setting $M^\phi = M$ as k-spaces, and letting the multiplication be given by $(x \otimes \lambda) \cdot_{M^\phi} m = (\phi(x) \otimes \lambda) \cdot_M m$. Now the identity on M gives us a Λ-isomorphism $\hat{\phi} : M^\phi \to M$, where for any $x \in K$ and $m \in M$ we have $\hat{\phi}(xm) = \phi(x)\hat{\phi}(m)$.

When K is a normal extension of k, let $G(K/k)$ denote its Galois group.

We are now ready to prove the main result of this section.

Theorem 1. *Let $M \in \operatorname{mod}\Gamma$. The multiplication map*

$$\mu_M : K \otimes_k M \to M$$
$$x \otimes m \mapsto xm$$

is a split epimorphism of Γ-modules.

Furthermore, if K is a normal extension of k, then we have

$$K \otimes_k M \simeq^\Gamma \bigoplus_{\phi \in G(K/k)} M^\phi.$$

Proof. We prove the first part by constructing a splitting of μ_M.

Let $\mu : K \otimes_k K \to K$ be the map given by $\mu(x \otimes y) = xy$. This is a $K \otimes_k K$-module epimorphism. Since K is separable we have by Lemma 9.2.8 and Theorem 9.2.11 in [6] that K is a projective $K \otimes_k K$-module, and thus μ splits. Let $\nu : K \to K \otimes_k K$ be a splitting of μ.

We first consider $M = \Gamma = K \otimes_k \Lambda$. $K \otimes_k \Gamma = K \otimes_k K \otimes_k \Lambda$ is a Γ-module with multiplication $(x \otimes \lambda) \cdot (y \otimes z \otimes \kappa) = xy \otimes z \otimes \lambda\kappa$. Let $\nu_\Gamma : K \otimes_k \Lambda \to K \otimes_k K \otimes_k \Lambda$ be given by $\nu_\Gamma(x \otimes \lambda) = \nu(x) \otimes \lambda$. This is a Γ-module homomorphism and a splitting of μ_Γ. We now show that for any $f \in \operatorname{Hom}_\Gamma(\Gamma, \Gamma)$ the following diagram commutes.

$$\begin{tikzcd}
\Gamma \arrow[r, "f"] \arrow[d, "\nu_\Gamma"] & \Gamma \arrow[d, "\nu_\Gamma"] \\
K \otimes_k \Gamma \arrow[r, "K\otimes_k f"] & K \otimes_k \Gamma
\end{tikzcd}$$

The homomorphism f is given by right multiplication with an element in Γ, and it is enough to check that the diagram commutes for all generators of Γ. Let $x, y \in K$, $\alpha, \beta \in \Lambda$ and $f = - \cdot y \otimes \beta$. Then $\nu_\Gamma f(x \otimes \alpha) = \nu_\Gamma(xy \otimes \alpha\beta) = \nu(xy) \otimes \alpha\beta$ and $K \otimes_k f\nu_\Gamma(x \otimes \alpha) = K \otimes_k f(\nu(x) \otimes \alpha) = \nu(x) \otimes \alpha \cdot 1 \otimes y \otimes \beta$, and since ν is a $K \otimes_k K$-homomorphism we have $\nu(x) \otimes \alpha \cdot 1 \otimes y \otimes \beta = \nu(xy) \otimes \alpha\beta$, so the diagram commutes.

For a free Γ-module Γ^n let ν_{Γ^n} be given by $\nu_{\Gamma^n}((\gamma_1, \ldots, \gamma_n)) = (\nu_\Gamma(\gamma_1), \ldots, \nu_\Gamma(\gamma_n))$. Then for any $f \in \operatorname{Hom}_\Gamma(\Gamma^a, \Gamma^b)$ the following diagram commutes.

$$\begin{tikzcd}
\Gamma^a \arrow[r, "f"] \arrow[d, "\nu_{\Gamma^a}"] & \Gamma \arrow[d, "\nu_{\Gamma^b}"] \\
K \otimes_k \Gamma^a \arrow[r, "K \otimes_k f"] & K \otimes_k \Gamma^b
\end{tikzcd}$$

For an arbitrary M, let

$$\Gamma^a \xrightarrow{f} \Gamma^b \xrightarrow{g} M$$

be a free presentation. Then we construct ν_M from the commutative diagram

$$\begin{tikzcd}
\Gamma^a \arrow[r, "f"] \arrow[d, "\nu_{\Gamma^a}"] & \Gamma^b \arrow[r, "g"] \arrow[d, "\nu_{\Gamma^b}"] & M \arrow[d, dotted, "\nu_M"] \\
K \otimes_k \Gamma^a \arrow[r, "K \otimes_k f"] & K \otimes_k \Gamma^b \arrow[r, "K \otimes_k g"] & K \otimes_k M
\end{tikzcd} .$$

Since

$$\begin{tikzcd}
\Gamma^b \arrow[r, "g"] & M \\
K \otimes_k \Gamma^b \arrow[u, "\mu_{\Gamma^b}"] \arrow[r, "K \otimes_k g"'] & M \arrow[u, "\mu_M"']
\end{tikzcd}$$

also commutes and ν_Γ is a splitting of μ_Γ, ν_M is a splitting of μ_M.

Let $\phi \in G(K/k)$. Then we have a Γ-isomorphism $1 \otimes \hat{\phi} : K \otimes M^\phi \to K \otimes M$, so M^ϕ is a summand of $K \otimes_k M$ and the composition $\iota_\phi := 1 \otimes \hat{\phi} \circ \nu_{M^\phi}$ is the inclusion. Let $\theta \neq \phi$ be another element in $G(K/k)$, and let $m \in \operatorname{im} \iota_\phi \cap \operatorname{im} \iota_\theta$. Now we view $K \otimes_k M$ as a $K \otimes_k K \otimes_k \Lambda$-module. Since m is in $\operatorname{im} \iota_\phi$, we have for any $x \in K$ that $x \otimes 1 \otimes 1 \cdot m = 1 \otimes \phi(x) \otimes 1 \cdot m$. Thus we get

$$1 \otimes (\phi(x) - \theta(x)) \otimes 1 \cdot m = 0$$

for all $x \in K$, which means that $m = 0$. Thus $\operatorname{im} \iota_\phi \cap \operatorname{im} \iota_\theta = (0)$, so M^ϕ and M^θ are distinct summands. If K is normal it follows that

$$K \otimes_k M \simeq \bigoplus_{\phi \in G(K/k)} M^\phi.$$

□

When K is normal, this gives us a complete description of the Γ-modules that are Λ-isomorphic to a given Γ-module.

Corollary 2. *Let K be a normal extension of k, and let $M_1, \dots, M_r$ be indecomposable $K \otimes_k \Lambda$-modules. If $M \simeq^\Lambda M_1 \oplus \dots \oplus M_r$, then there exist $\phi_1, \dots, \phi_r \in G(K/k)$ such that $M \simeq^{K \otimes_k \Lambda} M_1^{\phi_1} \oplus \dots \oplus M_r^{\phi_r}$.*

When K is not normal, this does not hold. Then Λ-isomorphisms do not even preserve the number of indecomposable Γ-summands.

Example 2. Let $K = \mathbb{Q}(\alpha)$ where α is a root of $X^3 - 2$. K is not a normal extension of $\mathbb{Q}$, and it has no nontrivial $\mathbb{Q}$-automorphisms. $K \otimes_\mathbb{Q} K$ as a module over itself decomposes to $K \oplus L$, and $L \simeq K^2$ as K-modules, but not as $K \otimes_\mathbb{Q} K$-modules. In fact L is an indecomposable $K \otimes_\mathbb{Q} K$-module.

3. Partial orders

Given two Γ-modules M and N, we can ask if M degenerates to N as a Γ-module, but also if M degenerates to N as a Λ-module.

If we have $M \leq_{\deg}^{\Gamma} N$, then there is an exact sequence of Γ-modules

$$0 \longrightarrow X \longrightarrow X \oplus M \longrightarrow N \longrightarrow 0\ .$$

This is also an exact sequence of Λ-modules, so we also have $M \leq_{\deg}^{\Lambda} N$.

We have already seen examples where $M \simeq^\Lambda N$ but $M \not\simeq^\Gamma N$. These examples also show that Λ-degeneration does not imply Γ-degeneration. There are also proper Λ-degenerations that are not Γ-degenerations.

Example 3. Consider the algebra

$$\Lambda = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} \end{pmatrix} \subseteq \mathcal{M}_2(\mathbb{C}).$$

This is a hereditary $\mathbb{R}$-algebra corresponding to the Dynkin graph B_2. We have that $\mathbb{C} \otimes_\mathbb{R} \Lambda \simeq \mathbb{C}Q$ as $\mathbb{C}$-algebras, where Q is the quiver

$$Q:\ 1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3\ ,$$

via the isomorphism $f : \mathbb{C} \otimes_\mathbb{R} \Lambda \to \mathbb{C}Q$ given by $f(1 \otimes (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})) = (e_1 + e_3)$, $f(1 \otimes (\begin{smallmatrix} i & 0 \\ 0 & 0 \end{smallmatrix})) = i(e_1 - e_3)$, $f(1 \otimes (\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix})) = e_2$, $f(1 \otimes (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})) = (\alpha + \beta)$ and $f(1 \otimes (\begin{smallmatrix} 0 & i \\ 0 & 0 \end{smallmatrix})) = i(\alpha - \beta)$.

The simple $\mathbb{C}Q$-modules S_1 and S_3 are isomorphic as Λ-modules. The $\mathbb{C}Q$-modules

$$I_1 : \ \mathbb{C} \xleftarrow{\ 1\ } \mathbb{C} \longrightarrow 0 \ , \qquad I_3 : \ 0 \longleftarrow \mathbb{C} \xrightarrow{\ 1\ } \mathbb{C}$$

are also isomorphic as Λ-modules. I_1 degenerates to $S_2 \oplus S_3$ as a Λ-module, but not as a $\mathbb{C}Q$-module, and the same holds for $I_3 \leq_{\mathrm{deg}}^{\Lambda} S_2 \oplus S_1$.

In the above example all Λ-degenerations of Γ-modules can be decomposed into Γ-degenerations and Λ-isomorphisms. That is, for any modules $M, N \in \operatorname{mod}\Gamma$ such that $M \leq_{\mathrm{deg}}^{\Lambda} N$, there exist $M', N' \in \operatorname{mod}\Gamma$ such that $M \simeq^{\Lambda} M' \leq_{\mathrm{deg}}^{\Gamma} N' \simeq^{\Lambda} N$. This does not hold for all algebras.

Example 4. Let Q be the quiver

$$Q : \ \bullet \rightrightarrows \bullet \rightrightarrows \bullet$$

and let $\Lambda = \mathbb{R}Q$ and $\Gamma = \mathbb{C}Q$. Consider the modules given by the following representations:

$$A : \ 0 \rightrightarrows \mathbb{C} \underset{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)}{\rightrightarrows}} \mathbb{C}^2 \ , \quad B : \ \mathbb{C}^3 \underset{\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)}{\rightrightarrows}} \mathbb{C}^2 \underset{\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\rightrightarrows}} \mathbb{C}^2$$

$$C : \ \mathbb{C}^3 \underset{(0\ 1\ 0)}{\overset{(1\ 0\ 0)}{\rightrightarrows}} \mathbb{C} \rightrightarrows 0 \ , \quad X : \ 0 \rightrightarrows \mathbb{C} \underset{(i)}{\overset{(1)}{\rightrightarrows}} \mathbb{C}$$

Now there is an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\operatorname{mod}\Lambda$, so we have $B \leq_{\mathrm{deg}}^{\Lambda} A \oplus C$. However, we have ${}_{\Gamma}[X, A \oplus C] = 1 < {}_{\Gamma}[X, B] = 2$, so $B \not\leq_{\mathrm{deg}}^{\Gamma} A \oplus C$. Letting ϕ denote complex conjugation we also have $A^{\phi} \simeq^{\Gamma} A$, $C^{\phi} \simeq^{\Gamma} C$ and ${}_{\Gamma}[X, B^{\phi}] = 2$, so there are no modules M and N such that $M \simeq^{\Lambda} B \leq_{\mathrm{deg}}^{\Gamma} A \oplus C \simeq^{\Lambda} N$.

Λ-isomorphisms do not preserve Γ-degenerations, and two Λ-isomorphic modules can behave quite differently in the Γ-degeneration order. For example, minimality is not preserved.

Example 5. Let Q be the Kronecker quiver and let $\Lambda = \mathbb{R}Q$ and $\Gamma = \mathbb{C} \otimes_{\mathbb{R}} \Lambda$. Let M, N and N' be the modules given by

$$M : \ \mathbb{C}^2 \underset{\left(\begin{smallmatrix} i & 0 \\ 1 & i \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\rightrightarrows}} \mathbb{C}^2$$

$$N:\ \mathbb{C}^2 \underset{\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\rightrightarrows}} \mathbb{C}^2\ .$$

$$N':\ \mathbb{C}^2 \underset{\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}{\rightrightarrows}} \mathbb{C}^2$$

We have that $M \leq_{\deg}^{\Gamma} N$, but N' is minimal in the degeneration order of $\operatorname{mod}\Gamma$ and $N \simeq^{\Lambda} N'$.

For some algebras, e.g. $\Lambda = kQ$ where Q is a simply laced Dynkin quiver, the isomorphism classes in $\operatorname{mod}\Lambda$ and $\operatorname{mod} K \otimes_k \Lambda$ are the same. It seems likely that in these cases the degeneration order should also be the same. The Hom-order is indeed the same.

Theorem 3. *Let Λ be a k-algebra and $\Gamma = K \otimes_k \Lambda$. The following are equivalent:*

(1) $M \simeq^{\Lambda} N \iff M \simeq^{\Gamma} N$ *for all* $M, N \in \operatorname{mod}\Gamma$.
(2) $M \leq_{\mathrm{Hom}}^{\Lambda} N \iff M \leq_{\mathrm{Hom}}^{\Gamma} N$ *for all* $M, N \in \operatorname{mod}\Gamma$.

Proof. We always have that $M \simeq^{\Gamma} N \implies M \simeq^{\Lambda} N$. If $M \simeq^{\Lambda} N$, then we have $M \leq_{\mathrm{Hom}}^{\Lambda} N$ and $N \leq_{\mathrm{Hom}}^{\Lambda} M$. Assuming that 2 holds, we then have $M \leq_{\mathrm{Hom}}^{\Gamma} N$ and $N \leq_{\mathrm{Hom}}^{\Gamma} M$, and thus $M \simeq^{\Gamma} N$. This shows that 2 implies 1.

Now assume that 1 holds.

For any k-algebra R and R-modules A and B we have $\operatorname{Hom}_{K\otimes_k R}(K\otimes_k A, K \otimes_k B) \simeq_K K \otimes_k \operatorname{Hom}_R(A, B)$. Thus for any Γ-modules X and M we have ${}_{\Gamma}[K \otimes_k X, K \otimes_k M] = n \cdot {}_{\Lambda}[X, M]$, where n is the degree of K. But given 1 we also have ${}_{\Gamma}[K\otimes_k X, K\otimes_k M] = {}_{\Gamma}[X^n, M^n] = n^2 \cdot {}_{\Gamma}[X, M]$, so we get $n \cdot {}_{\Gamma}[X, M] = {}_{\Lambda}[X, M]$. It follows that $M \leq_{\mathrm{Hom}}^{\Lambda} N$ implies $M \leq_{\mathrm{Hom}}^{\Gamma} N$.

Assume that $M \leq_{\mathrm{Hom}}^{\Gamma} N$. For every Λ-module X we have $n \cdot {}_{\Lambda}[X, M] = {}_{\Gamma}[K \otimes_k X, M] \leq {}_{\Gamma}[K \otimes_k X, N] = n \cdot {}_{\Lambda}[X, M]$, and thus $M \leq_{\mathrm{Hom}}^{\Lambda} N$.

□

This leaves the question of whether the same result holds for degenerations and virtual degenerations. If Λ satisfies the statements of Theorem 3, do we also have that $\leq_{\deg}^{\Lambda}$ and $\leq_{\deg}^{\Gamma}$ are the same?

For representation-finite algebras all three orders are the same, so in that case the answer is yes. It looks like all algebras that satisfy the statements of Theorem 3 are representation-finite, so one option is to try to prove that.

Another possible way to prove Theorem 3 for degenerations is to use a Riedtmann sequence in $\operatorname{mod}\Lambda$ to construct a Riedtmann sequence in $\operatorname{mod}\Gamma$.

When isomorphism classes are the same, we have that for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ there is a short exact sequence

$$0 \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow 0$$

in $\operatorname{mod}\Gamma$, obtained by applying $K \otimes_k -$. It seems like this should imply that there is a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\operatorname{mod}\Gamma$ as well, and thus that $M^n \leq_{\mathrm{deg}} N^n$ implies $M \leq_{\mathrm{deg}} N$. Unfortunately this is not true in general.

The next example, which is a variant of the Carlson example mentioned in the introduction, shows that $M^n \leq_{\mathrm{deg}} N^n$ does not imply $M \leq_{\mathrm{deg}} N$.

Example 6. Let Λ be the exterior k-algebra in two variables X and Y. Let $f \in \Lambda$ be an element of degree 1, i.e. $f = aX + bY$ for some $a, b \in k$, and let (f) be the submodule of Λ generated by f. There is an exact sequence of Λ-modules

$$0 \longrightarrow (f) \longrightarrow \Lambda \longrightarrow (f) \longrightarrow 0\ ,$$

which shows that $\Lambda \leq_{\mathrm{deg}} (f)^2$. If $g \in \Lambda$ is another element of degree 1, then we have $\Lambda^2 \leq_{\mathrm{deg}} ((f) \oplus (g))^2$. However, by Theorem 5.4 in [5] we have $\Lambda \leq_{\mathrm{deg}} (f) \oplus (g)$ if and only if $(f) \simeq (g)$. As in the original Carlson example, we have $\Lambda \leq_{\mathrm{vdeg}} (f) \oplus (g)$ for all f, g.

Adding a suitable K-structure, this shows that we may have $M \leq^{\Lambda}_{\mathrm{deg}} N$ and $M \leq^{\Gamma}_{\mathrm{vdeg}} N$ without having $M \leq^{\Gamma}_{\mathrm{deg}} N$.

We give one more example of this, due to S. Oppermann and S. O. Smalø. Here we also see that we can have a monomorphism from A^2 to B^2 without having any monomorphisms from A to B.

Example 7. Let Λ be the exterior k-algebra in three variables X, Y and Z. Let $\mathfrak{r}$ be its radical and S the simple Λ-module. The Λ-homomorphism $f : (\Lambda/\mathfrak{r}^2)^2 \to (\mathfrak{r}/\mathfrak{r}^3)^2$ given by right multiplication with the matrix $\left(\begin{smallmatrix} X & Y \\ Y & Z \end{smallmatrix}\right)$ is a monomorphism, but there are no Λ-monomorphisms from $\Lambda/\mathfrak{r}^2$ to $\mathfrak{r}/\mathfrak{r}^3$. Since $\operatorname{coker} f$ is semisimple, we have an exact sequence

$$\eta :\ 0 \longrightarrow (\Lambda/\mathfrak{r}^2)^2 \longrightarrow (\mathfrak{r}/\mathfrak{r}^3)^2 \longrightarrow (S^2)^2 \longrightarrow 0\ ,$$

but there is no exact sequence

$$0 \longrightarrow \Lambda/\mathfrak{r}^2 \longrightarrow \mathfrak{r}/\mathfrak{r}^3 \longrightarrow S^2 \longrightarrow 0\ .$$

The exact sequence η shows that $(\mathfrak{r}/\mathfrak{r}^3)^2 \leq_{\mathrm{deg}} (\Lambda/\mathfrak{r}^2 \oplus S^2)^2$, and thus $\mathfrak{r}/\mathfrak{r}^3 \leq_{\mathrm{Hom}} \Lambda/\mathfrak{r}^2 \oplus S^2$.

We also have $\mathfrak{r}/\mathfrak{r}^3 \leq_{\mathrm{vdeg}} \Lambda/\mathfrak{r}^2 \oplus S^2$. There are exact sequences

$$0 \longrightarrow \Lambda/\mathfrak{r}^2 \xrightarrow{\left(\begin{smallmatrix} X \\ Z \end{smallmatrix}\right)} \mathfrak{r}/\mathfrak{r}^3 \oplus (Z)/\mathfrak{r}^3 \longrightarrow \mathfrak{r}/(XZ) \longrightarrow 0\ ,$$

$$0 \longrightarrow (XZ) \longrightarrow \mathfrak{r}/(XZ) \longrightarrow (XZ, YZ) \longrightarrow 0\ ,$$

$$0 \longrightarrow (YZ) \longrightarrow (XZ, YZ) \longrightarrow S \longrightarrow 0\ ,$$

$$0 \longrightarrow (Z)/\mathfrak{r}^3 \longrightarrow (XZ) \oplus (YZ) \longrightarrow S \longrightarrow 0\ ,$$

which show that $\mathfrak{r}/\mathfrak{r}^3 \oplus (Z)/\mathfrak{r}^3 \leq_{\mathrm{deg}} \mathfrak{r}/(XZ) \oplus \Lambda/\mathfrak{r}^2 \leq_{\mathrm{deg}} (XZ) \oplus (XZ, YZ) \oplus \Lambda/\mathfrak{r}^2 \leq_{\mathrm{deg}} (XZ) \oplus (YZ) \oplus S \oplus \Lambda/\mathfrak{r}^2 \leq_{\mathrm{deg}} \Lambda/\mathfrak{r}^2 \oplus S^2 \oplus (Z)/\mathfrak{r}^3$.

There is no degeneration though, as we will see from Proposition 4 below.

In both examples we have a virtual degeneration, so it is possible that $M^n \leq_{\mathrm{deg}} N^n$ implies $M \leq_{\mathrm{vdeg}} N$. Note also that the exterior algebras do not satisfy the statements of Theorem 3. Thus it is still possible that in this more restricted case $M^n \leq_{\mathrm{deg}} N^n$ also implies $M \leq_{\mathrm{deg}} N$.

To see that $\mathfrak{r}/\mathfrak{r}^3$ does not degenerate to $\Lambda/\mathfrak{r}^2 \oplus S^2$ in Example 7, we will look at their submodules. The 4-dimensional submodule $\Lambda/\mathfrak{r}^2 \subseteq \Lambda/\mathfrak{r}^2 \oplus S^2$ is generated by one element. In $\mathfrak{r}/\mathfrak{r}^3$, on the other hand, any submodule generated by one element is at most 3-dimensional. This turns out to be impossible if we have a degeneration.

For a Λ-module M and a natural number i, let $\mathrm{Sub}_i\, M$ be the set of submodules of M that are generated by i elements. We have a function $f_i : \mathrm{mod}\,\Lambda \to \mathbb{N}$ for each i, given by

$$f_i(M) = \max_{N \in \mathrm{Sub}_i M} \dim_k N.$$

Proposition 4. *Let k be an algebraically closed field and Λ a finite-dimensional k-algebra. Let M and N be Λ-modules such that $M \leq_{\mathrm{deg}} N$. Then $f_i(M) \geq f_i(N)$ for all i.*

Proof. We want to show that for any $m, d, i \in \mathbb{N}$, the set $\{X \in \operatorname{mod}_d \Lambda | f_i(X) \leq m\}$ is closed in $\operatorname{mod}_d \Lambda$. Let $\{\lambda_1, \dots, \lambda_n\}$ be a basis for Λ. For any i-tuple $\mathbf{x} = (x_1, \dots, x_i)$ of elements in k^d, we have a function $\phi_{\mathbf{x}} : \operatorname{mod}_d \Lambda \to \mathcal{M}_{d\times ni}(k)$ given by

$$Y \mapsto \begin{pmatrix} Y(\lambda_1)x_1 & \cdots & Y(\lambda_n)x_1 & Y(\lambda_1)x_2 & \cdots & Y(\lambda_n)x_i \end{pmatrix}.$$

The columns of $\phi_{\mathbf{x}}(\rho)$ span the submodule of Y generated by $\{x_1, \cdots, x_i\}$, thus the dimension of that submodule equals the rank of $\phi_{\mathbf{x}}(Y)$. Let $Z_m \subseteq \mathcal{M}_{d\times ni}(k)$ be the set of matrices with rank at most m. Then the set of modules where $\{x_1, \dots, x_i\}$ generates an at most m-dimensional submodule is the inverse image of Z_m, and we have

$$\{X \in \operatorname{mod}_d \Lambda | f_i(X) \leq m\} = \bigcap_{\mathbf{x}\in(k^d)^i} \phi_{\mathbf{x}}^{-1}(Z_m),$$

which is closed since the maps $\phi_{\mathbf{x}}$ are continuous and Z_m is closed.

Hence the closure of the isomorphism class of M is contained in $\{\rho \in \operatorname{mod}_d \Lambda | f_i(\rho) \leq f_i(M)\}$, so if $M \leq_{\text{deg}} N$ we have $f_i(N) \leq f_i(M)$. □

It follows immediately that we cannot have a degeneration in Example 7 if the field is algebraically closed. Even if the field is not closed, a degeneration is not possible. If there were a degeneration, applying $\overline{k}\otimes_k -$ to its Riedtmann-sequence would show that $N = \overline{k}\otimes_k(\Lambda/\mathfrak{r}^2 \oplus S^2)$ is a degeneration of $M = \overline{k} \otimes_k \mathfrak{r}/\mathfrak{r}^3$. But $f_1(M) = 3$ and $f_1(N) = 4$, so by Proposition 4 we have $M \not\leq_{\text{deg}} N$ and consequently $\mathfrak{r}/\mathfrak{r}^3 \not\leq_{\text{deg}} \Lambda/\mathfrak{r}^2 \oplus S^2$.

4. Endomorphism rings

Let k be a field, Λ a k-algebra and M and N Λ-modules such that $M \leq_{\text{deg}} N$. Since also $M \leq_{\text{Hom}} N$, we have ${}_\Lambda[M, M] \leq {}_\Lambda[M, N]$ and ${}_\Lambda[M, N] \leq {}_\Lambda[N, N]$, and thus ${}_\Lambda[M, M] \leq {}_\Lambda[N, N]$. If $M \not\simeq N$, then this is a strict inequality. If k is algebraically closed, this can be shown geometrically. For arbitrary fields it can be seen from the following lemma, which is Lemma 5.3 from [4].

Lemma 5. *Let M and N be two nonisomorphic Λ-modules such that $M \leq_{\text{Hom}} N$. Then we have ${}_\Lambda[N, M] < {}_\Lambda[N, N]$.*

It follows from Lemma 5 that if N is a proper Λ-degeneration of M, then $\operatorname{End}_\Lambda M$ must have strictly smaller dimension than $\operatorname{End}_\Lambda N$. Thus if $\operatorname{End}_\Lambda M$ is one-dimensional, M cannot be a proper degeneration of anything.

If k is algebraically closed, the only finite extension of k is k itself. In this case it is obvious that if $\mathrm{End}_\Lambda M$ is a field, then M must be minimal in the Hom-order, and thus also in the degeneration order. When k is not algebraically closed, $\mathrm{End}_\Lambda M$ might be a field different from k. In this case, ${}_\Lambda[M,M]$ is greater than one, so it is not immediately obvious that M should be minimal. However, it is.

Proposition 6. *Let M be a Λ-module such that $\mathrm{End}_\Lambda(M)$ is a division ring. Then M is minimal in the Hom-order, and also in the degeneration order.*

Proof. Assume there exists a module $N \not\simeq M$ such that $N \leq_{\mathrm{Hom}} M$. By Lemma 5 we have $0 < {}_\Lambda[M,N] < {}_\Lambda[M,M]$. On the other hand, $\mathrm{Hom}_\Lambda(M,N)$ is a right $\mathrm{End}_\Lambda(M)$-module, which is free since $\mathrm{End}_\Lambda(M)$ is a division ring. Thus ${}_\Lambda[M,M]$ divides ${}_\Lambda[M,N]$, which is a contradiction.

Hence M is minimal in the Hom-order, and since the degeneration order is coarser than the Hom-order, M is also minimal in the degeneration order.

$\square$

References

[1] Auslander, M. Representation theory of finite dimensional algebras. Contemp. Math. 13 (1982), 27–39.

[2] Auslander, M., Reiten, I. and Smalø, S. O. Representation Theory of Artin Algebras. Cambridge University Press, 1994.

[3] Riedtmann, C. Degenerations for representations of quivers with relations. Ann. Sci. Ecole Nomale Sup. 4 (1986), 275–301.

[4] Smalø, S. O. Degenerations of representations of associative algebras. Milan Journal of Math. 76, 1 (2008), 135–164.

[5] Smalø, S.O. and Valenta, A. Finitely presented functors and degenerations. Comm. Algebra 34, no. 5, (2006), 1861–1889.

[6] Weibel, C. An Introduction to Homological Algebra. Cambridge University Press, 1995.

[7] Zwara, G. A degeneration-like order for modules. Arch. Math. 71 (1998), 437–444.

[8] Zwara, G. Degenerations for modules over representation-finite algebras. Proc. Amer. Math. Soc. 127, 5 (1999), 1313–1322.

[9] Zwara, G. Degenerations of finite-dimensional modules are given by extensions. Composito Math. 121 (2000), 205–218.