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Multifactor Interest Rate Models in Low-Rate Environments

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Abstract

This thesis studies a multi-factor Heath-Jarrow-Morton model and a LIBOR market model on the Norwegian, European and US interest rate market. The main concerns are the low-rate environment and exposure to negative interest rates in these models. We begin by introducing financial markets and the mathematical models explaining them. Further we discuss the problem with the current low-rate environment and the historical market practice. The focuses are implementations of two multi-factor interest rate models and the presence of negative interest rates. The historical data is provided by DNB and consists of zero coupon swap rates for several maturities in the period 2000-2012. The volatility factors are derived from historical data using principal component analysis and covariance matrices. With today's yield curve the probability of negative rates is highly significant in the HJM model, whereas it is zero in LMM because of lognormality. Monte Carlo is used on the models to compare prices of caps and floors. We show that the models do not produce the same price especially around strikes near the current 3-month rates. Further we price long butterfly spreads to show the absence of arbitrage in both models.

Sammendrag

Denne masteroppgaven studerer en multifaktor Heath-Jarrow-Morton modell og en LIBOR markedsmodell på det norske, europeiske og amerikanske rentemarkedet. Vi er spesielt interessert i lavrente regimer og risiko for negative renter i disse modellene. Vi begynner med å introdusere finansielle markeder og matematiske modeller som beskriver dem. Videre så diskuterer vi problemet med dagens lavrente regime og historisk markedspraksis. Hovedfokuset er implementeringer av to multifaktor rentemodeller og tilstedeværelsen av negative renter. De historiske dataene har vi fått fra DNB og består av swap renter på statsobligasjoner for flere sluttdatoer i perioden 2000-2012. Volatilitetsfaktorene er dratt ut fra historiske data med PCA og kovariansmatriser. Med dagens renter så er sannsynligheten for negative renter signifikant i HJM-modellen, mens den er null i LMM på grunn av lognormalitet. Vi bruker Monte Carlo på modellene for å sammenligne priser til caps og floors. Vi viser at modellene produserer ulike priser spesielt rundt innløsningspriser nær dagens 3-måneds rente. Videre priser vi long butterfly spreads for å vise at vi ikke har arbitrasje i modellene våre.

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Chapter 1

Introduction

The time value of money plays an important role in finance. One dollar today is worth more than one dollar in a year, that is why banks pay interests on bank deposits. If we deposit $M(0)$ dollars into a bank account at time 0, the value at time t is $M(t) = M(0)e^{rt}$ if the interest rate is continuously compounded. The present value of $M(t)$ dollars in t years is accordingly $M(0) = M(t)e^{-rt}$. We say that we have discounted the future value $M(t)$ with the interest rate r.

Stocks (also known as shares or equities) represent a fraction of ownership in a business or company and we hope that the reader has encountered this term before. The stocks of a public company are traded on a stock exchange. For example companies listed in Norway are quoted on the Norwegian Stock Exchange. Furthermore there have been developed stock market indices, which can measure how the stock market is doing as a whole. An index usually consists of a basket of representative stocks. A famous index is S&P 500 which consists of the top 500 most traded publicly stocks in the US and can be a good measure on how the US economy is doing as a whole.

The behaviour of quoted prices of stocks are far from predictable. We therefore model them in a probabilistic way. Section 2.3 describes a well known mathematical model for stock prices which is often called the lognormal walk. This model is not developed to predict future stock prices but is a tool to price products which depend on stock prices.

A derivative is a financial instrument whose value is derived from one or more underlying assets. It is a contract between to parties(buyer and seller) that specifies conditions under which payments are to be made between the parties. The conditions usually consist of the payment dates, underlying assets, obligations and resulting values of the payments. In this report we discuss derivatives with a stock or some kind of interest rate as the underlying asset.

Call and put options on stocks are the two simplest and most traded stock derivatives. The buyer of a European call or put option has the right but not the obligation to buy or sell the stock for a predetermined price on a predetermined future date. In Norway call and put stock options are both quoted on the Norwegian

Stock Exchange and traded over-the-counter. Options quoted on the exchange are standardized such that the conditions are known in advance. Furthermore the exchange uses a clearing house to guarantee that the contracts will be fulfilled. A clearing house stands between the two parties and its purpose is to reduce the risk between the two parties if one party fails its obligations. The clearing house requires a security deposit from both parties and it even has an own fund to cover losses above the deposits if one party defaults. Over-the-counter(OTC) means that the two parties meet directly without supervision of any exchange which opens for discussion of the conditions. OTC contracts do not necessarily use a clearing house and therefore one of the parties might fail to meet its obligations. This is called credit risk and therefore OTC contracts are more risky.

Section 3 introduces the Black-Scholes theory which is needed to prize European style call and put options. This theory is one of the elementary mathematical models for pricing financial derivatives and was introduced in 1973 by Black and Scholes[1]. The theory generates a fair price under the assumptions of the model. The trade in stock options increased heavily after the machinery for fair pricing was introduced.

The rest of this article is devoted to interest rate models and derivatives. The interest rate derivatives market is the largest derivatives market in the world. In Norway interest rate derivatives are traded over-the-counter. The underlying asset is obviously some kind of interest rate. Almost all companies have debt and are therefore exposed to interest rates. Interest rate derivatives are used by companies to control cash flows and to reduce risk. In section 4.2 we introduce an interest derivative called cap which we will investigate in this report. In short a cap is like a series of call options on an interest rate for a pre-described amount of periods. If the interest rate increases the cap will generate cash flow and is hence a security for companies against increases in interest rates. On the contrary a floor is like a series of put options on an interest rate which works as a security against declines in the interest rate.

According to the International Swaps and Derivatives Association, 80% of the world's top 500 companies as of April 2003 used interest rate derivatives to control their cash flows, whereas only 10% used stock options. Likewise with stock derivatives, the need of a fair pricing methodology is substantial for a trade in interest rate derivatives to be present. This is where the mathematical interest rate models come in. To model interest rates are much more difficult than to model stocks. The mathematical models are more complicated and we often need higher order simulations to get reasonable answers. We are going to focus on the Heath, Jarrow & Morton framework and the LIBOR market model which have become very popular. Furthermore we are going to use our models to price derivatives.

Above we argued that the price of a derivative should be fair because few companies would take an unfavourable position of a contract. In a financial mathematical perspective this means the absence of arbitrage. Arbitrage can be mathematically defined as follows. Let V_t be the value of a portfolio at time t. A portfolio is a collection of investments, in our case stocks, derivatives and bank accounts. Arbitrage is present if $V_0 = 0$ and

$$P(V_t \geq 0) = 1, \quad P(V_t \neq 0) > 0.$$

It is the possibility of a risk-free profit at zero cost. We find the arbitrage-free price of a derivative by finding the expected value of its future cash flows discounted with the risk-free interest rate. Here the risk-free interest rate can be the rate one gets in a bank account, which typically bears no risk. The models we discuss in this report possess this future and can therefore obtain fair prices for the derivatives we present.

Without the models the trade in derivatives would be negligible. The derivatives market is a place where companies can find products to help them control their cash flows and reduce their risk. Without the models companies would have to find other ways to reduce their risk. The models have also given birth and liquidity to exotic derivatives with more complex payoff structure. If one understands the dynamics and the risk of exotic derivatives they are not necessarily dangerous. However, one of the main reasons for the financial crisis in 2007-2008 were an enormous trade in a handful of such derivatives which very few actually understood. When these derivatives went from bad to worse many financial institutions and investors defaulted which resulted in a domino effect in the financial world.

After the financial crisis in 2007-2008 government bonds have been traded at negative yields in some countries, (e.g. Switzerland, Denmark, Germany, Finland, the Netherlands and Austria). This phenomenon contradicts what professors teach in introductory finance courses, i.e one dollar today is worth more than one dollar tomorrow. Suggested explanations are desire for protection against the eurozone breaking up (in which case some eurozone countries might redenominate their debt into a stronger currency).

Many interest rate models are lognormal, yielding a negative interest rate zero probability. Such models give 0 % floors zero value. However, on the street this is indeed not the case. Not before recently dealers started to seriously look at their exposure to negative rates. The possibility of generating negative interest rates has traditionally been considered a bad property for interest rate models. It contradicts common sense, you would never pay someone to lend them money. However, we need to accept reality, negative interest rates have been traded recently in the market.

In this thesis we implement a Heath-Jarrow-Morton approach and a LIBOR market model on the Norwegian, European and US interest rate market. The purpose is to study how these models behave in the current low-rate environment. HJM is Gaussian and allows negative rates whereas LMM is lognormal which means negative rates are impossible. Many interest rate models break down when interest rates go negative and therefore some adjustments have to be made. We will take a look on how practitioners have adapted models to negative rates. Furthermore we will investigate the probabilities of negative interest rates in our models. Lastly we study how derivative prices behave in the current market.

Chapter 2

Basic Stochastic Calculus

Before we present the mathematical models we need some basic stochastic calculus which is used in the modelling of financial assets. From now on a process X is stochastic if we denote its state at time t as X_t.

2.1 The Brownian Motion and Martingales

We introduce stochastic calculus by defining the Standard Brownian Motion(also called a Wiener process) which is present in many of the models we are going to use in this report.

Definition 2.1.1. *A stochastic process W_t is a Standard Brownian Motion if the following conditions hold*

1. $W_0 = 0$.
2. W_t *has independent increments, i.e. if* $r < s \leq t < u$ *then* $W_u - W_t$ *and* $W_s - W_r$ *are independent stochastic variables.*
3. *For* $s < t$, $W_t - W_s$ *is normally distributed with* $E[W_t - W_s] = 0$ *and* $VAR[W_t - W_s] = t - s$.
4. W_t *is almost surely continuous.*

Although this thesis is not about measure theory we need to introduce some definitions which will appear when we are dealing with martingales.

- The symbol $\mathcal{F}_t^X$ denotes the information generated by the stochastic process X on the interval $[0, t]$.
- The event A is $\mathcal{F}_t^X$-measurable if $A \in \mathcal{F}_t^X$.
- If the stochastic process Y satisfies $Y_t \in \mathcal{F}_t^X$ for all $t \geq 0$, we say that Y is adapted to the filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$.

Definition 2.1.2. *A stochastic process X is called a martingale with respect to the filtration $\mathcal{F}_t$ if the following conditions hold*

1. *X is adapted, i.e $X_t \in \mathcal{F}_t$.*
2. *$E|X_t| < \infty \qquad \forall t$.*
3. *For all s and t where $s \leq t$, $E[X_t|\mathcal{F}_s] = X_s$.*

2.2 Ito's Lemma

Assume that the process X is given by the stochastic differential

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.1}$$

where $\mu(t)$ and $\sigma(t)$ are adapted processes. Let f be a $C^{1,2}$-function and let the stochastic variable Z be $Z_t = f(t, X_t)$. Then dZ is given by

$$dZ_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2.$$

When we are using this formula we need some results from stochastic calculus

$$\begin{aligned}(dt)^2 &= 0,\\ dt \cdot dW_t &= 0,\\ (dW_t)^2 &= dt.\end{aligned}$$

2.3 The Lognormal Walk

One of the assumptions in the Black-Scholes model is that the stock follows a Geometric Brownian Motion (GBM)

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{2.2}$$

If σ is zero we observe that the solution will be an exponential function. This motivates us further to investigate the process $Z_t = f(t, S_t) = lnS_t$. Here we have $f(s) = lns$. By using Ito's lemma we obtain

$$\begin{aligned}dZ_t &= 0 \cdot dt + \frac{1}{S_t}dS_t - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2,\\ dZ_t &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t,\\ \int_0^T dZ_t &= \int_0^T (\mu - \frac{1}{2}\sigma^2)dt + \int_0^T \sigma dW_t,\\ Z_T - Z_0 &= (\mu - \frac{1}{2}\sigma^2)T + \sigma(W_T - W_0),\\ lnS_T &= lnS_0 + (\mu - \frac{1}{2}\sigma^2)T + \sigma W_T.\end{aligned}$$

By taking the exponential on both sides we obtain the lognormal walk

$$S_T = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T\}. \tag{2.3}$$

That is,

$$ln\frac{S_T}{S_0} \sim N((\mu - \frac{1}{2}\sigma^2)T, \sqrt{T}).$$

2.4 Feynman-Kac Formula

Before introducing a very important theorem for pricing financial derivatives we need to define the class $\mathcal{L}^2$:

Definition 2.4.1. *A process g belongs to the class $\mathcal{L}^2$ if the following conditions hold*

- $\int_0^t E[g^2(s)]ds < \infty$ *for all* $t > 0$.
- g *is adapted to the* $\mathcal{F}_t^W$*-filtration.*

Now we have enough mathematical machinery to introduce Feynman-Kac's theorem.

Theorem 2.4.1. *Let X satisfy the SDE in* (2.1) *with initial condition $X_t = x$ and assume that V is a solution to the boundary problem*

$$\frac{\partial V}{\partial t}(t,x) + \mu(t,x)\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 V}{\partial x^2} - rV(t,x) = 0,$$
$$V(T,x) = \Phi(x).$$

Assume furthermore that the process $e^{-rs}\sigma(s,X_s)\frac{\partial V}{\partial x}(s,X_s)$ is in $\mathcal{L}^2$. Then V is given by:

$$V(t,x) = e^{-r(T-t)}E_{t,x}[\Phi(X_T)]. \tag{2.4}$$

2.5 Change of Numeraire

We can sometimes simplify the pricing of derivatives drastically by change of numeraire. This technique is commonly used when we price interest rate derivatives under the LIBOR market model.

We assume an arbitrage free market model with asset prices S_0, S_1,...., S_n where S_i is assumed to be strictly positive.

Theorem 2.5.1. *Under the assumptions above the following hold*

- *The market model is free of arbitrage if and only there exists a martingale measure, $Q^0 \sim P$ such that the processes*

  $$\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, ..., \frac{S_N(t)}{S_0(t)}$$

  are local martingales under Q^0.

- *In order to avoid arbitrage, a T-claim X must be priced according to the formula*

$$\Pi(t;X) = E^0\left[\frac{X}{S_0(T)}|\mathcal{F}_t\right]$$

where E^0 denotes the expectation under Q^0.

2.6 Monte Carlo

The integral,

$$\alpha = \int_0^1 f(x)dx,$$

can be considered as an expectation $E[f(U)]$ where U is uniformly distributed between 0 and 1. We can estimate this integral by drawing n points independently and uniformly from $[0,1]$. Evaluating f at these points produces the Monte Carlo estimate

$$\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i).$$

The strong law of large numbers says that

$$\hat{\alpha}_n \to \alpha \text{ with probability 1 as } n \to \infty.$$

The Monte Carlo error $e_n = \hat{\alpha}_n - \alpha$ is normally distributed with mean zero and standard deviation $\frac{\sigma_f}{\sqrt{n}}$ where σ_f is given by

$$\sigma_f^2 = \int_0^1 \left(f(x) - \alpha\right)^2 dx.$$

Although σ_f is unknown we can estimate it with the sample standard deviation

$$s_f = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(f(U_i) - \hat{\alpha}_n\right)^2}.$$

Thus, when we use Monte Carlo we also get an error of the estimate. The convergence rate of Monte Carlo is $\mathcal{O}(\frac{1}{\sqrt{n}})$.

2.7 Simulation

There are several strategies when it comes to simulating a system of SDE's. A simple and popular method is the Euler scheme.

An Euler scheme for a SDE on a time grid $t_1, t_2, ..., t_n$ for the process

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

is given by

$$\hat{X}_{i+1} = \hat{X}_i + a(t_i, \hat{X}_i)(t_{i+1} - t_i) + b(t_i, \hat{X}_i)\sqrt{t_{t_i+1} - t_i}Z_{i+1}$$

where Z_{i+1} is standard normal distributed.

2.8 Interpolation Methods

In most cases we have yield points from the yield curve $y_1, y_2, \cdots, y_n$ on the nodes $T_1, T_2, \cdots, T_n$ and need to determine the yield $y(T)$ where T is not necessarily one of the T_i's. Furthermore we need the instantaneous forward rate $f(T)$ which we extract from the relation

$$f(T) = \frac{\partial}{\partial T} y(T) dT. \tag{2.5}$$

Suppose we are given some $T \in (T_1, T_n)$ which is not equal to any of the T_i's. Let us define i such that $T \in (T_i, T_{i+1})$.

Linear interpolation on spot rates :
A simple interpolation method is to use linear interpolation on the yields

$$y(T) = \frac{T - T_i}{T_{i+1} - T_i} y_{i+1} + \frac{T_{i+1} - T}{T_{i+1} - T_i} y_i.$$

We now find the instantaneous forward rate from (2.5),

$$f(T) = \frac{2T - T_i}{T_{i+1} - T_i} y_{i+1} + \frac{T_{i+1} - 2T}{T_{i+1} - T_i} y_i,$$

but sadly the forward rates are not continuous with this method.
Raw Interpolation :
This method results in piecewise continuous forward curves and is a popular and stable method. By solving (2.5) for $y(T)$ we find that the interpolating function is $y(T) = K + \frac{C}{T}$. Given the two endpoints, we get

$$f(T) = K = \frac{y_{i+1}T_{i+1} - y_i T_i}{T_{i+1} - T_i}, \qquad C = \frac{(y_i - y_{i+1})T_i T_{i+1}}{T_{i+1} - T_i}.$$

With some manipulation we also find the yield

$$y(T) = \frac{T - T_i}{T_{i+1} - T_i} \frac{T_{i+1}}{T} y_{i+1} + \frac{T_{i+1} - T}{T_{i+1} - T_i} \frac{T_i}{T} y_i.$$

Chapter 3

The Black-Scholes Model

The Black-Scholes is one of the elementary models for valuing and pricing financial derivatives. The first article about the mathematical model was published in 1973 by Fischer Black and Myron Scholes. They received the Nobel Prize in Economy for the model which resulted in a dramatical increase in the trade of options and other financial derivatives.

3.1 Assumptions

The simplest Black-Scholes model assumes that the price of an asset is in possession of the following features:

- **The asset price follows the geometric Brownian motion with constant drift and volatility**

  $$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

  The drift term, μ, is here associated with the expected return of the asset. The volatility, σ, is a measure of the variation in the price. In the equation above S_t models the asset price while W_t is a standard Brownian motion.

- **The risk-free interest rate is constant and known during the life of the derivative.** This allows traders to borrow and lend cash at a known rate.

- **The market has no transactions costs.** This is called a frictionless market and allows traders to hedge their portfolio without transactions cost.

- **The underlying asset does not pay dividends.** Dividend payouts change the asset price, so to simplify the model we assume no dividend payouts in the life of the derivative.

- **There are no arbitrage opportunities.** This means that all riskless portfolios must have the same return, i.e. the same return as an U.S. Treasury bond. There is no such thing as a free lunch.
- **Trading in the underlying asset happens continuously in time.** This is of course not possible in the real world.
- **Short-selling is possible and the assets are divisible.** That is, we can buy or sell a fraction of an asset and we can even sell assets we do not own.

3.2 European Options

We begin with introducing one of the simplest financial derivatives, European options. There are two different options associated with the European type.

- A European call option is a contract where the buyer has the right, but not the obligation, to purchase the underlying security at a fixed strike price at a predescribed date in the future.
- A European put option is a contract where the buyer has the right, but not the obligation, to sell the underlying security at a fixed strike price at a predescribed date in the future.

Hence, the buyer of a call option expects the price of the underlying to go up, whereas the buyer of a put option expects the price to go down. The predescribed date is often call the maturity of the option, T. The strike price, K, will stay constant during the time of the contract. From now on we will consider options with the underlying asset being a stock, S.

Because no person would like to loose money, the buyer of a call option would only exercise the option if the stock price, S, is greater than the strike price. No one in their right mind would purchase the stock for K dollars if the market price of the stock is less. We immediately get the following payoff for a European call option at maturity

$$\text{Payoff European Call Option } = \max(S - K, 0). \tag{3.1}$$

If $S > K$ at maturity the buyer would buy the stock for K dollars and sell it for S dollars, obtaining a risk-less profit of $S - K$ dollars.

With the same reasoning as above we can obtain the payoff of a European put option. If $K > S$ at maturity the buyer would buy the stock for S dollars in the market and sell it for K dollars, obtaining a risk-less profit of $K - S$ dollars.

$$\text{Payoff European Put Option } = \max(K - S, 0).$$

One of the assumptions of the Black-Scholes model is the absence of arbitrage. Therefore, the European-style options cannot be free, because then you can obtain a risk-less profit.

3.3 The Black-Scholes Analysis

We are now getting closer to the famous Black-Scholes equation which elegantly gives a unique price for a European option under the assumptions mentioned earlier.

Let us consider the stock to follow (2.2) with initial condition

$$S_u = y \text{ when } u < t.$$

Furthermore we assume we have an option with value $V(t, S_t)$ which depends on t and S_t. By using Ito's lemma we get the following SDE

$$dV = (\mu s \frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + \frac{\partial V}{\partial t})dt + \sigma s \frac{\partial V}{\partial s} dW. \tag{3.2}$$

In the above formula the functions' arguments are excluded to ease the notation. We now construct a portfolio consisting of one option and $-\Delta$ of the underlying stock. The value of the portfolio, Π, is obviously

$$\Pi = V - \Delta S \quad \rightarrow \quad d\Pi = dV - \Delta dS. \tag{3.3}$$

By inserting (3.2) and (2.2) into (3.3) we obtain the differential for the portfolio value

$$d\Pi = (\mu s \frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + \frac{\partial V}{\partial t} - \mu \Delta s)dt + \sigma s(\frac{\partial V}{\partial s} - \Delta)dW.$$

We now observe that we can get rid of the standard Brownian motion by choosing

$$\Delta = \frac{\partial V}{\partial s}.$$

This results in a risk-less portfolio whose differential is given by

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2})dt. \tag{3.4}$$

One of the assumptions in the Black-Scholes model is the absence of arbitrage. A risk-less portfolio thus has to generate the same return as an investment in a bank(here we also need to assume that all banks offer the same interest rate r). An investment of Π dollars in a bank has the return

$$d\Pi = r\Pi dt. \tag{3.5}$$

To avoid arbitrage we need (3.4) to be equal to (3.5),

$$r\Pi dt = (\tfrac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 s^2 \tfrac{\partial^2 V}{\partial s^2})dt.$$

By inserting (3.3) we have reached the famous Black-Scholes equation,

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial s^2} - rV = 0, \tag{3.6}$$

with the terminal condition

$$V(T, s) = \Phi(s). \tag{3.7}$$

One important fact to mention is that the PDE does not depend on the parameter μ. With other words, how rapidly/slowly the underlying stock grows does not affect the option price. We can use Feynman-Kac's formula because of the form of (3.6) and (3.7). Hence,

$$V(t, s) = e^{-r(T-t)} E^*_{t,s}[\Phi(S_T)], \tag{3.8}$$

where the stock's initial value is $S_t = s$ and the S-process has the dynamics

$$dS_u = rS_u du + \sigma S_u dW^*_u.$$

By using Feynman-Kac's formula we observe that the new S-process is not the same as the old process. The local rate of return μ is replaced by the risk-free interest rate r and W is replaced by W^*. The old process is the physical process. The new process is usually called the risk-neutral dynamics of S. The risk-neutral process is not observed but only used to price options. W is a standard Brownian motion under the physical measure $\mathcal{P}$ whereas W^* is a standard Brownian motion under the risk-neutral measure $\mathcal{P}^*$. From now on E denotes an expectation in the physical measure while E^* denotes an expectation in the risk-neutral measure.

From (2.3) we know that the terminal value S_T is lognormal

$$S_T = s \cdot \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(W^*_T - W^*_t)\},$$
$$W^*_T - W^*_t \sim N(0, \sqrt{T - t}).$$

We are now ready to price a European call option.

3.4 Pricing a European Call Option

Let the price of a European Call option at time t with initial value $S_t = s$ be given by $C(t, s)$. By inserting (3.1) into (3.8) we get

$$C(t, s) = e^{-r(T-t)} E^*_{t,s}[\max(S_T - K, 0)]$$
$$C(t, s) = e^{-r(T-t)} \int_{-\infty}^{\infty} \max(se^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma z} - K, 0) f(z) dz,$$

where $z = W^*_T - W^*_t$ to ease the notation. We observe that if $se^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma z} < K$ the integrand is zero. We can therefore change the lower limit to $z^* = \frac{1}{\sigma}\left(ln(\frac{K}{s}) - (r - \frac{1}{2}\sigma^2)(T - t)\right)$.

$$C(t, s) = e^{-r(T-t)} \int_{z^*}^{\infty} (se^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma z} - K) \frac{1}{\sqrt{2\pi(T - t)}} e^{-\frac{z^2}{2(T-t)}} dz$$
$$C(t, s) = s \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi(T - t)}} e^{\frac{(z-\sigma(T-t))^2}{2(T-t)}} dz - Ke^{-r(T-t)} \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi(T - t)}} e^{-\frac{z^2}{2(T-t)}} dz$$
$$C(t, s) = sN(d_1) - Ke^{-r(T-t)} N(d_2)$$

Here $N()$ is the cumulative distribution function of the normal distribution. d_1 and d_2 are given by

$$d_1 = \frac{ln(\frac{s}{K}) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$
$$d_2 = d_1 - \sigma\sqrt{T - t}.$$

Chapter 4

Stochastic Interest Rate

One of the assumptions in the Black-Scholes analysis was that the risk-free interest rate was constant and known during the life of the contract. This can be a reasonable assumption for a small interval of time. However, this is seldom the case for contracts with a long life. Interest rate derivatives (e.g. bonds) have a long life and we can therefore no longer make this assumption. An example is Norwegian government bonds, which are typically traded with a time to maturity of 3,5 or 10 years when the contracts are made. It should be clear that the interest rate will not be constant for such a long period of time. We therefore introduce stochastic interest rates to be able to price such contracts.

4.1 Short Rate Modelling

We usually model the interest rate with a short rate model. The short rate is the interest rate we can borrow money for an infinitesimally short period of time. The short rate, r_t, is given by the SDE

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t. \tag{4.1}$$

Definition 4.1.1. *A zero coupon bond with maturity date T, also called a Treasury bond, is a bond which guarantees the the holder of the contract 1 dollar to be paid on the date T. From now on $p(t, r_t; T)$ will denote the price of the bond at time t.*

It is harder to price a bond than an option because there are no underlying asset with which to hedge. We can only establish a hedged portfolio by combining two bonds with different maturity dates. Let p_1 and p_2 be the price of two different bonds with maturity dates T_1 and T_2 respectively. We construct a portfolio,

$$\Pi = p_1 - \Delta p_2,$$

where Δ is chosen such that the portfolio is risk-less. By using Ito's lemma we find the change in the portfolio in a time dt

$$d\Pi = \frac{\partial p_1}{\partial t}dt + \frac{\partial p_1}{\partial r}dr_t + \frac{1}{2}\sigma^2\frac{\partial^2 p_1}{\partial r^2}dt - \Delta\left(\frac{\partial p_2}{\partial t}dt + \frac{\partial p_2}{\partial r}dr_t + \frac{1}{2}\sigma^2\frac{\partial^2 p_2}{\partial r^2}dt\right).$$

We observe that we can obtain a risk-less portfolio, that is, remove the random component in (4.1) by choosing

$$\frac{\partial p_1}{\partial r} - \Delta\frac{\partial p_2}{\partial r} = 0 \quad \rightarrow \quad \Delta = \frac{\partial p_1}{\partial r}\bigg/\frac{\partial p_2}{\partial r}\ . \tag{4.2}$$

We now use the same reasoning as in the Black-Scholes analysis to avoid arbitrage. Because this portfolio is risk-less, it must have the same return as a bank account. That is,

$$d\Pi = r_t \Pi dt.$$

By using this relation in (4.2) we obtain

$$\frac{\frac{\partial p_1}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 p_1}{\partial r^2} - r_t p_1}{\frac{\partial p_1}{\partial r}} = \frac{\frac{\partial p_2}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 p_2}{\partial r^2} - r_t p_2}{\frac{\partial p_2}{\partial r}}. \tag{4.3}$$

We observe that the left-hand side of (4.3) is a function of T_1 while the right-hand side is a function of T_2. The only way this can be possible is if each side is independent of maturity date. We can therefore drop the subscript for the bond price and let p be the price for a bond with maturity date T

$$\frac{\frac{\partial p}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial r^2} - rp}{\frac{\partial p}{\partial r}} = \sigma(t, r_t)\lambda(t, r_t) - \mu(t, r_t). \tag{4.4}$$

In (4.4) we use the fact that the left-hand side must be independent of the maturity date. The fraction is thus a function of time and the short rate. We have introduced a new function, $\lambda(t, r_t)$, called the market price of risk, which we will talk more about later. We have now finally reached a PDE for the bond price for a Treasury bond

$$\frac{\partial p}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial r^2} + (\mu - \sigma\lambda)\frac{\partial p}{\partial r} - rp = 0, \tag{4.5}$$
$$p(r, T; T) = 1.$$

The boundary condition in (4.5) should be clear. The price of a bond at maturity should be \$1 to avoid arbitrage. We observe that the PDE is on the form we need to be able to use Feynman-Kac's formula. From (2.4) we get that the price of a zero coupon bond with maturity date T at time t is

$$p(t, T) = E^*\left[e^{-\int_t^T r_s ds}\right],$$

where the stochastic variable r_s is given by the SDE

$$dr_s = (\mu - \lambda\sigma)ds + \sigma dW_s^*,$$
$$r_t = r.$$

The reader should observe that we have once again introduced the risk-neutral measure $\mathcal{P}^*$ when pricing a contract. The new SDE is not observed but is used to price the bond. For the zero coupon bond the price at maturity is given by $\Phi(r_T) = 1$. However, for a bond with a general payoff $\Phi(r_T)$ the price at time t will be given by

$$p(t,T) = E^* \left[e^{-\int_t^T r_s ds} \Phi(r_T) \right].$$

The drift term in the risk-neutral short rate is not observed. In order to find it we have to fit the model we use to the actual prices in the market. We typically do this for all maturity dates to get the best fit. For the moment we introduce $\alpha(t, r_t)$ to ease the notation for the drift

$$\alpha(t, r_t) = \mu(t, r_t) - \lambda(t, r_t)\sigma(t, r_t). \tag{4.6}$$

Definition 4.1.2. *If the term structure $p(t,T); 0 \leq t \leq T, T > 0$ has the form*

$$p(t,T) = e^{A(t,T) - B(t,T)r} \tag{4.7}$$

where A and B are deterministic functions, then the model is said to possess an affine term structure (ATS).

If we put (4.6) and (4.7) into (4.5) we get a differential equation involving both $A(t,T)$ and $B(t,T)$

$$A_t(t,T) - (1 + B_t(t,T))\, r - \alpha(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T) = 0,$$
$$A(T,T) = 0,$$
$$B(T,T) = 0.$$

From the arguments above it should be clear that we need the risk-neutral short rate dynamics to be able to price interest rate derivatives. There are several models which possess an affine term structure. One nice property for a interest rate model is mean-reversion. A mean-reverting interest rate model will, as the name explains, tend back to the mean as it oscillates. The model is mean-reverting if α has the form

$$\alpha(t, r_t) = \eta(t) - \gamma(t) r_t.$$

An example is the CIR model, which takes the form

$$dr_t = (\eta - \gamma r_t) + \sqrt{\theta r_t} dW_t.$$

If $\eta \geq \frac{\theta}{2}$ the spot rate will always stay positive, which also has been an important property historically for an interest rate model.

4.2 Interest Rate Derivatives

Vanillas

The interest rate derivatives with simplest payoff structure are called vanillas. A cap is a vanilla interest derivative which is traded in the market. It is a contract that guarantees the holder that otherwise floating rates will not exceed a specified amount. For example, an investor might want to secure himself from high rates when taking up a loan. If the interest rate goes up, the coupon payments will increase. But, if he has bought a cap, he will receive cash flow as long as the floating interest rate is above the fixed cap rate. Thus, a cap is used to secure institutions and investors against an increase in the interest rate. A typical cap contract involves a payment at times t_i, in our case each quarter. The payment(payoff) each quarter is given by

$$\frac{E}{4}\max(r - r_c, 0). \tag{4.8}$$

Here r is the floating interest rate, r_c is the fixed cap rate and E is called the principal. The principal is just a constant telling you the size of the cash flows, a larger E corresponds to a larger price of the contract. The reason the principal is divided by 4 is because this is the quarterly payoff. In fact, each quarterly payment is called a caplet. A cap is thus the sum of many caplets. The rate which is paid on time t_i is set at time t_{i-1}.

Floors are to caps as put options are to call options. It is a contract that guarantees the holder that otherwise floating rates will not go below the fixed floor rate r_f. The payment each quarter is given by

$$\frac{E}{4}\max(r_f - r, 0).$$

One can construct one payment of a swap with a long caplet and short floorlet with the same fixed rate($r_c = r_f$). Each quarterly swap payment thus gives the holder the amount

$$\frac{E}{4}(r - r_c).$$

Exotics

Exotic derivatives are less liquid than more commonly traded instruments. They have a more complex payoff structure and are usually traded over the counter. Because the payoff structure is complex the pricing algorithms become more time-consuming and difficult. There do not exist any closed-form formulas for the prices and we therefore need simulation to get a reasonable arbitrage-free price. Monte-Carlo simulation is standard practice in the market.

Detecting Arbitrage

An easy way to detect arbitrage is to find prices of butterfly spreads of stock options type. For example a long butterfly spread in this report is constructed in the following way.

- Buy one cap with cap rate $r_c - a$.
- Sell two caps with cap rate r_c.
- Buy one cap with cap rate $r_c + a$.

Let the caplet payment dates coincide and the underlying rates be identical. The beauty of the butterfly spread is that then the payoff is bounded below by zero. Figure 4.1 displays this future. Thus, the price today of the derivative should be positive. If the price is negative (the buyer gets money today) or zero, arbitrage is possible. One could also do the same with floors instead of caps. Later we will find prices of butterfly spreads in our investigation of arbitrage.

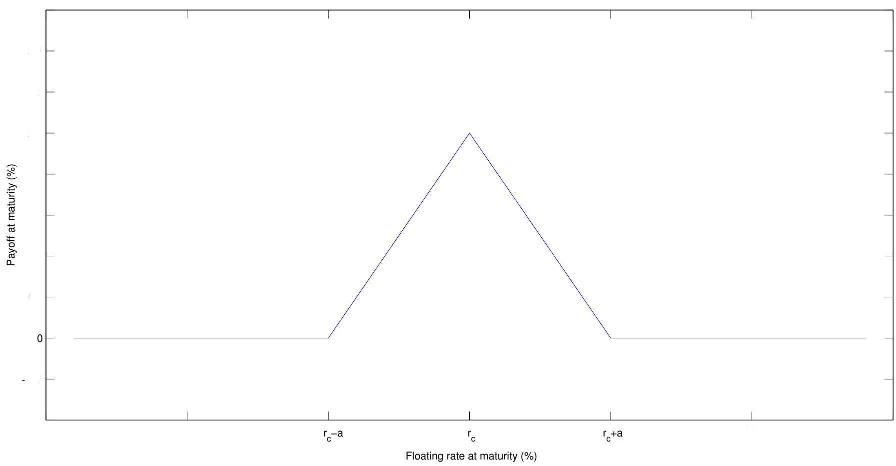

Figure 4.1: Payoff diagram of a long butterfly spread.

4.3 Black's Model

The Black model is one of the fundamental models when one comes across interest derivatives. The Black formula is to interest rate derivatives like the Black-Scholes formula is to stock derivatives. The Black model suggests that a forward interest rate F_t follows

$$dF_t = \sigma_B F_t dW_t \qquad (4.9)$$
$$F(0) = f.$$

The price today of a caplet in T years is the discounted expected payoff of (4.8) which yields the important Black's formula

$$\text{Caplet}(0, T+\delta) = \delta p(0, T+\delta)\,(fN(d_1) - KN(d_2)) \tag{4.10}$$
$$d_1 = \frac{\log(\frac{f}{K}) + \frac{\sigma_B^2}{2}}{\sigma_B\sqrt{T}}$$
$$d_2 = d_1 - \sigma_B\sqrt{T}.$$

Here δ is the length of the contract, K is the strike price and σ_B is Black's volatility. There is a one-to-one relation relation between the price of a caplet and σ_B in Black's model. Therefore, option prices are often quoted by stating the implied volatility, the unique volatility which yields the market price when used in Black's model. There are many algorithms one can use to obtain the implied volatility, for example the secant method.

The Black's model is easy and has few parameters, but this is not necessarily a good thing. There are very few people who use Black's model to price derivatives. It is first of all used for smile calibration purposes.

In practice, caplets with different strike prices K have different implied volatilities σ_B. The relation between strike price and implied volatility is called the volatility skew or smile. Fixed income and foreign exchange desks have to handle these market skews and smiles correctly because they invest across a wide range of strikes. Black's model is very time-consuming and old-fashion and better suited models have been suggested. In the next section we will introduce the SABR-model which is market practice today.

Another problem with the Black model is that the implied volatility does not exist for negative rates. In the rate environments we want to study we immediately approach a difficulty as interest rate derivatives are often quoted in Black's volatility. However, the SABR-model can treat negative rates with an elegant and easy trick.

4.4 SABR Model

The SABR model is a stochastic volatility model, the name stands for "stochastic alpha, beta, rho". In Black's model the volatility is a constant. Under the SABR model, the forward interest rate and volatility follow the dynamics

$$dF_t = \alpha_t F_t^{\beta} dW_1, \qquad F(0) = f$$
$$d\alpha_t = \nu\alpha_t dW_2, \qquad \alpha(0) = \alpha_0$$
$$dW_1 dW_2 = \rho dt.$$

The skewness parameter, β, is constant and can vary between zero and one. The price of a caplet is given by Black's formula. Hagan [8] uses singular perturbation theory to show that the implied volatility is given by the following relation

$$\sigma_B(K,f) = \frac{\alpha}{(fK)^{\frac{1-\beta}{2}}\left\{1+\frac{(1-\beta)^2}{24}\log^2 f/K + \frac{(1-\beta)^4}{1920}\log^4 f/K + ...\right\}} \cdot \left(\frac{z}{x(z)}\right)$$
$$\cdot\left\{1+\left[\frac{(1-\beta)^2\alpha^2}{24(fK)^{1-\beta}} + \frac{\rho\beta\nu\alpha}{4(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]t_{ex} + ...\right\}, \tag{4.11}$$

where

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log f/K$$
$$x(z) = \log\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right).$$

That is, (4.11), returns the implied volatility we insert into (4.10) to get the corresponding Black price of the caplet. However, there are also flaws with the SABR model in connection with negative rates. The model can only generate negative rates with $\beta = 0$, which results in the normal model. Sadly we get no lower bound on the interest rate. Few traders would for example give a -10% interest rate a nonzero probability.

Furthermore, regardless of the value of β, the SABR model breaks down in the current low-rate environment. For low strikes, the model implies negative probability densities which have no meaning. From (4.11) it should also be clear that there exists no implied volatility if the strike price and the rate today have different signs. This will be a problem when we are dealing with negative/low rates.

One way to solve the discussed problems is to use the shifted SABR model. Consider the dynamics

$$dF = \alpha(F-S)^\beta dW_1, \qquad F(0) = f - S$$
$$d\alpha = \nu\alpha dW_2, \qquad \alpha(0) = \alpha_0$$
$$dW_1 dW_2 = \rho dt.$$

Under these dynamics the corresponding implied Black's volatility is

$$\sigma_B(K) = \frac{\alpha\log(\frac{f-S}{K})}{\int_K^{f+S}\frac{df'}{(f'-S)^\beta}} \cdot \left(\frac{\zeta}{\hat{x}(\zeta)}\right) \cdot \tag{4.12}$$
$$\left\{1+\left[\frac{2\gamma_2-\gamma_1^2+1/f_{av}^2}{24}\alpha^2(f_{av}-S)^{2\beta} + \frac{1}{4}\rho\nu\alpha\gamma_1(f_{av}-S)^\beta + \frac{2-3\rho^2}{24}\nu^2\right]T\right\}$$

where

$$
\begin{aligned}
f_{av} &= \sqrt{(f-S)K}, \\
\gamma_1 &= \frac{\beta}{f_{av}-S}, \\
\gamma_2 &= \frac{\beta(\beta-1)}{(f_{av}-S)^2}, \\
\zeta &= \frac{\nu(f-S-K)}{\alpha(f_{av}-S)^\beta}, \\
\hat{x}(\zeta) &= \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right).
\end{aligned}
$$

Although the formula seems massive, the computation is actually simple and fast. Because the use of SABR to obtain volatility smiles has been market practice for some time, many practitioners use the shifted SABR model when dealing with low rates and strikes. Another approach has been suggested by Andreasen and Huge [10], which involves a method for expanding for all strikes simultaneously using finite differences. It is important to repeat that neither the Black model or the SABR model are used to price derivatives. They are only used for smile calibration purposes. Derivatives are priced with term-structure models and the next chapter is dedicated to one of these models.

Chapter 5

The Heath, Jarrow & Morton Framework

The HJM approach models the forward rate curve and is quite different from the previous mentioned methods. Instead of modelling the short rate and then derive the forward rates, the HJM approach builds a model for the entire forward rate curve. The forward rates are known today(from the yield curve) which means that the yield curve-fitting is contained naturally in the model. The relation between the forward rate $f(t, T)$ and a zero coupon bond with maturity T is

$$p(t;T) = e^{-\int_t^T f(t;s)ds} \quad \rightarrow \quad f(t;T) = -\frac{\partial}{\partial T}\log p(t;T). \tag{5.1}$$

Now let us assume that all zero coupon bonds satisfy the one-factor model

$$dp(t;T) = \mu(t,T)p(t;T)dt + v(t,T)p(t;T)dW_t \tag{5.2}$$

where the maturity date T is fixed. If we use Ito's lemma on (5.1) and (5.2) we get a SDE for the forward rate under $\mathcal{P}$

$$df(t;T) = \frac{\partial}{\partial T}\left(\frac{1}{2}v^2(t,T) - \mu(t,T)\right)dt - \frac{\partial}{\partial T}v(t,T)dW_t.$$

When pricing derivatives we need to return to the risk-neutral world. Let the risk-neutral forward rate curve satisfy

$$df(t;T) = \alpha(t,T)dt + \sigma(t,T)dW_t^*.$$

From before we know that the diffusion term is identical under the risk-neutral measure and the physical measure

$$\sigma(t,T) = -\frac{\partial}{\partial T}v(t,T).$$

The drift term is easily obtained to be

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds,$$

which is commonly known as the no-arbitrage condition for the HJM model. Hence, the equation for the evolution of the forward rate curve is

$$df(t;T) = \sigma(t,T)\int_t^T \sigma(t,s)ds + \sigma(t,T)dW_t^*,$$
$$f(0,T) = f^*(0,T)$$

where $f^*(0,T)$ is the observed forward rate in the market today.

Above the forward rate curve was only described by one Brownian motion, which is the case for one-factor interest rate modelling. The HJM framework becomes much more powerful if we model the forward rate with several Brownian motions. This is where the multifactor HJM comes into play. A N-factor HJM model of the forward rate curve satisfies the N-dimensional SDE

$$df(t,T) = \left(\sum_{i=1}^{N}\sigma_i(t,T)\int_t^T \sigma_i(t,s)ds\right)dt + \sum_{i=1}^{N}\sigma_i(t,T)dW_i^*. \tag{5.3}$$

In the above representation the Brownian motions are uncorrelated. We can also interpret the volatility and the Brownian motion as N-dimensional vectors such that (5.3) takes the form

$$df(t,T) = \left(\boldsymbol{\sigma}(t,T)^T\int_t^T \boldsymbol{\sigma}(t,s)ds\right)dt + \boldsymbol{\sigma}(t,T)^T\mathbf{dW}^*.$$

5.1 Simulation under the HJM Framework

The simulation procedure we use in this article is based on Glasserman[3]. Ideally we could build a model of an infinite number of bonds with different maturities. However, the market does not consist of all these bonds. There are of course only a finite set of bonds sold in the market. Therefore we have to introduce a discrete approximation of (5.3) to simulate the forward rate dynamics. We start with a discrete time grid:

$$0 = t_0 \le t_1 \le t_2 \le, \ldots.., \le t_{M-1} \le t_M.$$

Without loss of generality we are going to let the maturity grid be exactly the same. That is, we are going to simulate the forward rate for the same set of dates. First of all we need the forward rate curve today. Let $\hat{f}(t_i,t_j)$ be the discretized forward rate for maturity t_j as of time t_i. The corresponding bond price is given by

$$\hat{p}(t_i,t_j) = \exp\left(\sum_{l=i}^{j-1}\hat{f}(t_l,t_j)[t_{l+1}-t_l]\right).$$

Because we introduce discretization, we also introduce discretization error. We minimize this error by choosing initial forward rates such that our estimated bond prices equal the true bond prices in the market for all the maturities on our time grid

$$\hat{p}(0,t_j) = p(0,t_j) \quad j = 1, ..., M.$$

Accordingly, our initial forward rates satisfy

$$\hat{f}(0,t_j) = \frac{1}{t_{j+1} - t_j} \log\left(\frac{p(0,t_j)}{p(0,t_{j+1})}\right). \tag{5.4}$$

When we have decided the initial discretized forward rates the multifactor simulation follows

$$\hat{f}(t_i,t_j) = \hat{f}(t_{i-1},t_j) + \hat{\mu}(t_{i-1},t_j)\left[t_i - t_{i-1}\right] + \sqrt{t_i - t_{i-1}}\boldsymbol{\sigma}(t_{i-1},t_j)^T \mathbf{Z}_i \ , \tag{5.5}$$
$$\text{for} \ \ j = i, ..., M,$$

where $\mathbf{Z}_i$ and $\boldsymbol{\sigma}(t_{i-1},t_j)$ are N-dimensional vectors. Here the discretized drift term is given by

$$\hat{\mu}(t_{i-1},t_j) = \sum_{k=1}^{N} \hat{\mu}_k(t_{i-1},t_j)$$

and

$$\hat{\mu}_k(t_{i-1},t_j)[t_{j+1}-t_j] = \frac{1}{2}\left(\sum_{l=i}^{j} \hat{\sigma}_k(t_{i-1},t_l)[t_{l+1}-t_l]\right)^2 - \frac{1}{2}\left(\sum_{l=i}^{j-1} \hat{\sigma}_k(t_{i-1},t_l)[t_{l+1}-t_l)\right]^2.$$

This is the discrete counterpart of the no-arbitrage condition for a multifactor HJM model. With the iteration scheme in (5.5) we can simulate the forward rates in the future. What remains to be decided are the initial forward rates and the volatility structure. We decide the initial forward rates from today's yield curve. From (5.4) we see that we can decide the discretized forward rates if we can observe the bond prices for each maturity on our time grid. There might be some bonds maturing on some of the dates on our time grid in the market, but seldom all. Therefore we have to estimate some of the bond prices to get our initial forward rates. We estimate the unknown bond prices by interpolating the bond prices observed in the market. By doing this, we get estimates for the initial discretized forward rates. In this report we have used linear interpolation. When analysing the volatility in the forward rate curve one usually assumes that the volatility only depends on the time to maturity. This is called Musiela parametrization,

$$\sigma(t,T) = \sigma(T-t) = \sigma(\tau).$$

When it comes to developing the volatility structure, we can use Principal Component Analysis(PCA) on historical forward rates. Suppose we have historical time series of forward rates for several maturities. It is intuitive that many of the daily movements are common between the rates. For example, a downgrading of Norway's creditworthiness would probably affect all Norwegian government bonds regardless of the time to maturity. The purpose of PCA is to find common movements between these rates. First of all we need to find the daily changes for each rate. This is easily done when we are in possession of the historical rates. We give a loose description of how PCA is done in the next section.

5.2 Principal Component Analysis

Suppose we have observations of daily changes in the rates for N different maturities. PCA is a procedure that uses an orthogonal transformation to convert a set of observations of N possibly correlated variables into a set of linearly uncorrelated variables called principal components. This transformation is defined such that the first principal component has the largest possible variance(it accounts for as much variability in the data as possible). Furthermore the succeeding principal components have the largest possible variance with the constraint that they are orthogonal to the preceding components(that is, they are uncorrelated). Firstly we find the covariance matrix, $\mathbf{C}$, for the changes in the rates for the N different maturities. This will be a $N \times N$ matrix where $\mathbf{C}_{ij}$ represents the covariance between the daily changes for maturity i and j. Furthermore we find the eigenvalues and eigenvectors of $\mathbf{C}$,

$$\mathbf{C} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-\mathbf{1}}.$$

Here $\mathbf{V}$ is a matrix whose columns are the eigenvectors to $\mathbf{C}$. $\boldsymbol{\Lambda}$ is a diagonal matrix with the corresponding eigenvalues. If $\mathbf{V}$ and $\boldsymbol{\Lambda}$ are sorted in order of decreasing eigenvalue, the j^{th} column of $\mathbf{V}$ is the j^{th} principal component of $\mathbf{C}$. The first eigenvector gives the dominant part in the movement of the rate curve. Its i^{th} entry represents the movement of the i^{th} maturity. Furthermore the volatility factors are given by

$$\sigma_j(\tau_i) = \sqrt{\lambda_j}\mathbf{V}_{ij}. \tag{5.6}$$

If we want the volatility for other maturities we need to interpolate between the ones we already know. In this report we have used linear interpolation. If we choose n principal components, they describe

$$\frac{\sum_{j=1}^{n} \lambda_j}{\sum_{j=1}^{N} \lambda_j} \times 100\%$$

of the variability in the rate curve.

Chapman and Pearson[7] find that three factors can explain 99 % of the variation in the American yield curve. They found that the first principal component describes a parallel shift(its entries have the same sign) in the yield curve. The second and third component represent respectively a twisting and a bending of the curve.

However, we do principal component analysis on forward rates. The decay of the eigenvalues is generally considerably slower so we need to include more factors to be able to explain enough variability. Furthermore the intuitive economical meaning of the factors are not necessarily clear. Kletskin [9] showed that three factors only explain 73.81 % with PCA on US forward rates in the period 1982-2003.

5.3 Pricing Derivatives

In this thesis we will consider the 3-month rate as the short rate, i.e. the rate on the bond with shortest possible maturity in the market. The relationship between the instantaneous forward rate and the short rate is by definition

$$r(t) = f(t,t).$$

Hence, we can use (5.5) to simulate the short rate on our time grid. We furthermore set the payment dates for the derivative we want to price to the dates on our time grid. Then it is a straightforward procedure to price a derivative using Monte Carlo simulation. After one simulation of (5.5) we get the short rate $\hat{r}(t_i)$ for the dates $i = 0, 1, ..., M-1$. From now on let $\hat{r}(t_i) = \hat{r}_i$.

The floating rate in a caplet is usually based on discrete compounding, for example a LIBOR rate. For the interval $[t_i, t_{i+1}]$ the corresponding discretely compounded rate $\hat{F}_i$ satisfies

$$\frac{1}{1 + \hat{F}_i(t_{i+1} - t_i)} = e^{-\hat{r}_i(t_{i+1}-t_i)}.$$

Let $g(\hat{F}_i)$ denote the simulated cash flow on date t_i. We need to find the present value of the payments to find the fair price of the derivative. The present value of $g(\hat{F}_i)$ is given by

$$PV(g(\hat{F}_i)) = g(\hat{F}_i) \cdot \prod_{n=1}^{i} e^{\hat{r}_{n-1}(t_n - t_{n-1})}.$$

The price for the derivative is then the sum of the present values of the cash flows. We do many simulations and take the average to get a price estimate $\hat{P}$ of the derivative. $\hat{P}$ will have unit currency and gives us little information about what one actually pays for the contract. Because we deal with interest rates it is more beneficial to present the price as a rate because we can compare this rate with the rates in the market. Consider a bond with price $\hat{P}$ paying a fixed coupon rate c on the same dates as the quarterly paying cap

$$\hat{P} = \frac{c}{4} \sum_{i=1}^{M} e^{-y_i t_i}. \tag{5.7}$$

The coupon rate is easily obtained from the above equation and tells us what rate we pay for the cap. The y_i's are the yields to the future payment dates and is obtained from today's yield curve.

Chapter 6

The LIBOR Market Model

Until now we have considered interest rate models based on infinitesimal interest rates like the instantaneous short rate. Although they are nice to handle from a mathematical point of view they can never be observed in the market. LIBOR rates are discrete market rates which can be observed in the market. LIBOR stands for London Inter-Bank Offered Rate and is calculated daily through an average of rates offered by banks in London. LIBOR rates are based on simple interest.

We need to introduce some new definitions in the pursuit of the zero coupon bond price. If we have a fixed set of maturities in increasing order T_0, T_1,...., T_N we define δ_i by

$$\delta_i = T_i - T_{i-1}.$$

The number δ_i is called a tenor and is typically equal to a quarter of a year.

Definition 6.0.1. *Let $p_i(t)$ denote the zero coupon price $p(t, T_i)$, that is, the price of a zero coupon bond at time t which matures at T_i. Let $L_i(t)$ denote the LIBOR forward rate contracted at t for the period $[T_{i-1}, T_i]$,*

$$L_i(t) = \frac{p_{i-1}(t) - p_i(t)}{\delta_i p_i(t)} \qquad i = 0, 1, ..., N \tag{6.1}$$

If we solve (6.1) for the bond price, we get that the bond price at a tenor date T_n which matures at T_i ($T_i > T_n$) is given by

$$p_i(T_n) = \prod_{j=n}^{i-1} \frac{1}{1 + \delta_j L_j(T_n)}.$$

It should be clear that the above bond price is at a tenor date T_n. But what happens if we want the price of a zero coupon bond at an arbitrary date t? Suppose that $T_n < t < T_{n+1}$ and we want to find the price $p_i(t)$ for some $i > n + 1$. The factor,

$$\prod_{j=n+1}^{i-1} \frac{1}{1 + \delta_j L_j(t)},$$

discounts the cash flow at T_i back to T_{n+1} but not back to t. Define the function $\eta : \{0, T_{M+1}\}$ by taking $\eta(t)$ to satisfy

$$t_{\eta(t)-1} \leq t \leq T_{\eta(t)}.$$

Using this notation, we get the zero coupon bond price at t

$$p_i(t) = p_{\eta(t)}(t) \prod_{j=\eta(t)}^{i-1} \frac{1}{1+\delta_j L_j(t)} \qquad 0 \leq t \leq T_i.$$

The next step is to formulate the evolution of the forward LIBOR rates. Under the spot measure the forward LIBOR rates follow a system of SDE's of the form

$$\frac{dL_n(t)}{L_n(t)} = \mu_n(t)dt + \sigma_n(t)^T dW(t), \quad 0 \leq t \leq T_n, \quad n = 1, ..., M. \tag{6.2}$$

Here $W(t)$ is a d-dimensional standard Brownian motion. The coefficients $\mu_n(t)$ and $\sigma_n(t)$ can also depend on the current vector of rates $(L_1(t), ..., L_M(t))$. As in the HJM model, also the LIBOR market model has a drift condition which eliminates arbitrage. This condition is

$$\mu_n(t) = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t)\sigma_n(t)^T \sigma_j(t)}{1+\delta_j L_j(t)}. \tag{6.3}$$

Furthermore this transforms (6.2) to

$$\frac{dL_n(t)}{L_n(t)} = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t)\sigma_n(t)^T \sigma_j(t)}{1+\delta_j L_j(t)} dt + \sigma_n(t)^T dW(t), \quad 0 \leq t \leq T_n, \quad n = 1, ..., M. \tag{6.4}$$

The reader should observe that under the spot measure the LIBOR rate is lognormal and hence cannot go negative. Before we introduce simulation algorithms we need to specify the volatility structure.

6.1 Volatility

6.1.1 Implied Volatility

The LIBOR market model is often calibrated to actively traded derivatives, for example caps. Assume that the forward rate in (4.9) is a LIBOR forward rate $L_n(t)$. If the price of a caplet for the interval $[T_n, T_{n+1}]$ is given we can find the unique implied volatility v_n by inverting (4.10). We choose σ_n to be any deterministic R^d-valued function satisfying

$$\frac{1}{T_n} \int_0^{T_n} ||\sigma_n(t)||^2 dt = v_n^2.$$

By imposing this constraint on all the σ_j's the model is calibrated to all caplet prices.

6.1.2 Historical Volatility

Because we do not have quoted prices for interest rate derivatives in our possession we need to find a volatility structure somehow differently. Another possibility is to find the historical volatility. This process has some similar elements as the PCA we did in the HJM framework. We assume an instantaneous volatility structure specified by the matrix

Inst. Vol.	$t \in (T_0, T_1\,]$	$(T_1, T_2\,]$	$(T_2, T_3\,]$	$\cdots$	$(T_{M-1}, T_M\,]$
$L_1(t)$	$\sigma_{1,1}$	expired	$\cdots$	$\cdots$	expired
$L_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	expired	$\cdots$	$\vdots$
$L_3(t)$	$\sigma_{3,1}$	$\sigma_{3,2}$	$\sigma_{3,3}$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$L_i(t)$	$\sigma_{i,1}$	$\sigma_{i,2}$	$\sigma_{i,3}$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	expired
$L_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\cdots$	$\sigma_{M,M}$

Table 6.1: Historical volatility structure

The LIBOR rate $L_i(t)$ does not vary after T_i and any volatility with this rate equals zero after this maturity. The next paragraph explains how we can find such a volatility structure.

We begin by finding the covariance matrix, $\mathbf{C}$, for the daily changes in historical LIBOR rates for N different maturities. $\mathbf{C}$ will be a $N \times N$ matrix where $\mathbf{C}_{ij}$ represents the covariance between the daily changes for maturity i and j. We construct a lower triangular matrix, $\hat{\mathbf{C}}$, by setting the entries above the diagonal equal to zero. This matrix will be of order percent squared but we want our volatility structure to be of order percent. A proper volatility structure is therefore the square root matrix of $\hat{\mathbf{C}}$. We find the eigenvalues and eigenvectors of $\hat{\mathbf{C}}$,

$$\hat{\mathbf{C}} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}.$$

Now note that $\mathbf{Y} = \mathbf{V}\mathbf{\Lambda}^{0.5}\mathbf{V}^{-1}$ will be a square root of $\hat{\mathbf{C}}$ because

$$\mathbf{Y}\mathbf{Y} = \mathbf{V}\mathbf{\Lambda}^{0.5}\mathbf{V}^{-1}\mathbf{V}\mathbf{\Lambda}^{0.5}\mathbf{V}^{-1} = \mathbf{V}\mathbf{\Lambda}^{0.5}\mathbf{\Lambda}^{0.5}\mathbf{V}^{-1} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} = \hat{\mathbf{C}},$$

where simply $\mathbf{\Lambda}^{0.5}$ is a diagonal matrix with the square root of the eigenvalues on the diagonal. What remains now is to specify the volatility vectors $\sigma_j(t)$ in (6.4). We assume that the vectors are constants and define them in the following way

$$\sigma_n(t_i) = \mathbf{Y}_{in}. \tag{6.5}$$

6.2 Simulation

We begin by fixing a time grid $0 = t_0 \leq t_1 \leq, ..., \leq t_{m+1}$ over which to simulate. To ease the simulations the tenor dates $T_1, T_2, ..., T_{M+1}$ will fully coincide with the time grid. An Euler scheme on (6.2) yields an explicit simulation algorithm

$$\hat{L}_n(t_{i+1}) = \hat{L}_n(t_i) + \mu_n(\hat{L}_n(t_i), t_i)\hat{L}_n(t_i)(t_{i+1} - t_i) + \hat{L}_n(t_i)\sqrt{t_{t_i+1} - t_i}\boldsymbol{\sigma}(t_i)^T \mathbf{Z}_{i+1}, \tag{6.6}$$

where $\mu_n(\hat{L}_n(t_i), t_i)$ is given by (6.3) and Z_1, Z_2,... are independent $N(0, I)$ vectors in R^d. We assume that we have today's yield curve ($t = 0$) and can therefore initialize the LIBOR forward rates with

$$\hat{L}_n(0) = \frac{p_n(0) - p_{n+1}(0)}{\alpha_n p_{n+1}(0)}, \quad n = 1, ..., M. \tag{6.7}$$

6.3 Pricing Derivatives

The pricing algorithm is straightforward. With (6.6) we get simulated paths of the discretized variables $\hat{L}_1, \hat{L}_2, .., \hat{L}_M$. Assume that we want to price a derivative with a cash flow $g(L(T_n))$ at time T_n. We simulate to time T_n and then calculate the present value of the payment

$$PV(g(\hat{L}(T_n)) = g(\hat{L}(T_n)) \cdot \prod_{j=0}^{n-1} \frac{1}{1 + \delta_j \hat{L}_j(T_j)}.$$

The simulated price of the derivative is then the sum of the present values of all the cash flows. By averaging over the simulations we get an estimate of the price at time 0. We denote the price of the derivative as the coupon rate c in (5.7).

Chapter 7

Data Description

The three data sets are provided by DNB. The first data set represents the daily quoted yields for Norwegian zero coupon bonds for several maturities in the period 01.03.2000 to 11.29.2012. The second and third data sets represent daily quoted yields for the European and US zero coupon bonds in the period 01.08.2006 to 15.02.2013. The available maturities for each country were 3 months, 1, 2, 3, 5, 7 and 10 years. Under the Heath-Jarrow-Morton framework we model the instantaneous forward rate and then derive the bond prices. In the LIBOR Market Model we model simple forward rates. The volatility in (5.5) corresponds to the volatility in the instantaneous forward rate curve. Therefore, we need somehow to find the volatility of the instantaneous forward rate curve to be able to simulate under (5.5). In other words, we need to derive the historical instantaneous forward rate curve from the historical yield curve. Likewise we need the historical LIBOR forward rates to find the appropriate volatilities in (6.5).

7.1 Inversion from Yield Curve to Forward Rate Curve

7.1.1 Instantaneous Forward Rate Curve

The shortest maturity in the data is three months. We will therefore consider the three-month rate as the short rate. The short rate is defined to represent the yield on a bond with infinitesimal maturity, but in practice one should take this rate to be the yield on a liquid finite-maturity bond. We will in the same way take the instantaneous forward rate to be the three-month rate forward in time.

We can obtain an estimate of the instantaneous forward rate curve by the following procedure. Firstly we set the instantaneous forward rate today equal to the 3-month rate. Secondly we use interpolation on the initial data to find the yields for bonds with maturity 1.25, 2.25, 3.25, 5.25 and 7.25 years. Lastly we use (5.4) to find the 3-month rate in 1, 2, 3, 5 and 7 years.

Figure 7.1 shows the evolution of the Norwegian instantaneous forward rates in the period 2000-2012 and the European and US instantaneous forward rates in the period 2006-2013. We instantly observe that the assumption of a constant interest rate in the Black-Scholes model must be set aside for long time horizons. Table 7.1 shows the means and standard deviations of the forward rate curves. We observe that the mean increases with maturity while the standard deviation decreases with maturity. However, the mean and standard deviation for the daily changes are much more similar over different maturities. The original time series is clearly not stationary. This is the reason why we use PCA on the daily changes in the rates rather than on the rates themselves. Further we note that the US interest rate market is more volatile than the two other markets.

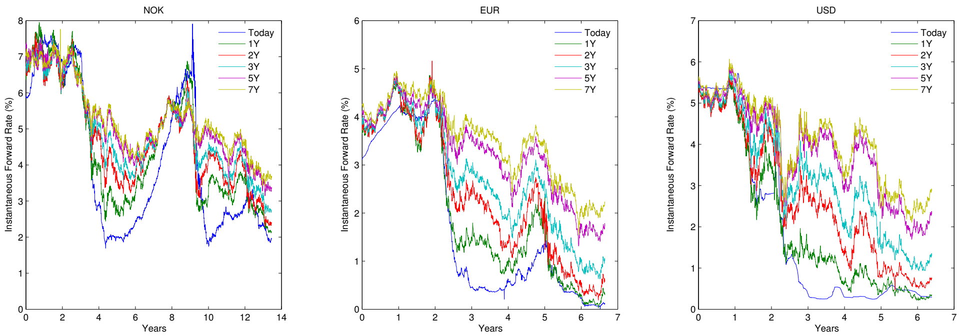

Figure 7.1: Instantaneous forward 3-month rate curve in the period 2000-2012 for NOK rates and for EUR and USD rates in the period 2006-2013.

Norway 2000-2012				
Maturity	Mean (%)	Std. (%)	Mean daily change (%)	Std. daily change (%)
3M	4.13	2.01	-0.001183	0.05135
1.25Y	4.54	1.68	-0.001246	0.06132
2.25Y	4.77	1.38	-0.001326	0.06466
3.25Y	4.94	1.23	-0.001178	0.05590
5.25Y	5.21	1.06	-0.001073	0.07048
7.25Y	5.32	1.00	-0.001023	0.06171
Europe 2006-2013				
Maturity	Mean (%)	Std. (%)	Mean daily change (%)	Std. daily change (%)
3M	1.74	1.61	-0.001805	0.02628
1.25Y	2.06	1.50	-0.002010	0.06382
2.25Y	2.40	1.36	-0.001954	0.08037
3.25Y	2.75	1.18	-0.001714	0.05296
5.25Y	3.26	0.93	-0.001408	0.05346
7.25Y	3.53	0.84	-0.001220	0.06326
US 2006-2013				
Maturity	Mean (%)	Std. (%)	Mean daily change (%)	Std. daily change (%)
3M	1.88	2.03	-0.003242	0.03232
1.25Y	2.04	1.75	-0.003153	0.07256
2.25Y	2.67	1.56	-0.002875	0.08151
3.25Y	3.15	1.38	-0.002593	0.08029
5.25Y	3.84	1.13	-0.002030	0.08519
7.25Y	4.09	1.02	-0.001744	0.08740

Table 7.1: Statistics for the Norwegian, European and US 3-month instantaneous forward rates.

7.1.2 LIBOR Forward Rate Curve

The idea behind the LIBOR market model is to simulate the points on the yield curve we already have. In other words, we simulate something which is in fact quoted in the market. In comparison the instantaneous forward rate is something mathematicians found up to model interest rates, it is completely abstract. Given the zero coupon bond prices from DNB we can produce seven historical LIBOR forward rates. Let $L_i(t)$ denote the forward LIBOR rate at time t for the interval (T_i, T_{i+1}) and let $T = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\} = \{\frac{1}{4}, 1, 2, 3, 5, 7, 10\}$. By using (6.7) we get $\hat{L}_i(0)$ for $i = 1, 2.., 6$. Figure 7.2 displays the historical LIBOR forward

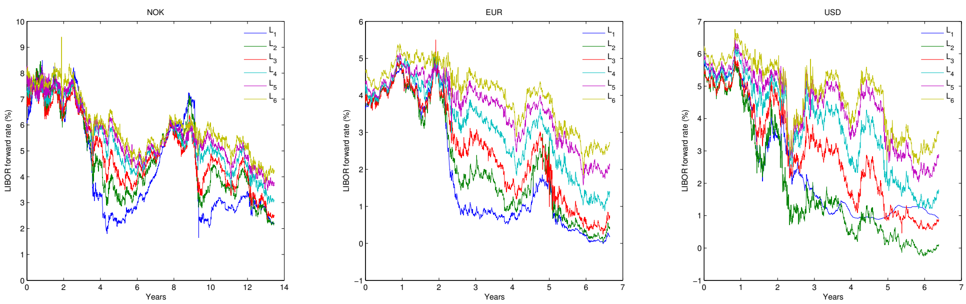

Figure 7.2: LIBOR forward rate curve for NOK rates in the period 2000-2013 and for EUR and USD rates in the period 2006-2013.

rates for NOK, EUR and USD. We observe that these are far from constant as well. Table 7.2 consists of the means and standard deviations for the curves. They have the same structure as the statistics for the instantaneous forward rates and we make the same conclusion. The volatility structure should be constructed from the daily changes in the rates.

7.2 Volatility Structures

7.2.1 PCA on the Instantaneous Forward Rate Curves

We continue our investigation of the instantaneous forward rate curve by finding the covariance matrix for the daily changes between different maturities. Because we have 6 different maturities, the covariance matrix will be a 6×6 matrix. PCA on the daily changes will leave us with a volatility structure which we can use in the Heath-Jarrow-Morton framework. Figure 7.3 shows the three most dominant principal components for the different markets. The first principal component is flat compared to the other components for all markets. This indicates that a parallel shift in the forward rate curves is the dominant movement, as it should be. The second principal component moves in a linear fashion in the NOK and USD rates. It only changes sign once, and indicates that a twisting of the curve is the second

Norway 2000-2012				
Rate	Mean (%)	Std. (%)	Mean daily change (%)	Std. daily change (%)
L_1	4.43	1.98	-0.001155	0.07206
L_2	4.82	1.65	-0.001371	0.09598
L_3	4.99	1.35	-0.001427	0.07981
L_4	5.40	1.20	-0.001245	0.07504
L_5	5.64	1.10	-0.001141	0.10670
L_6	5.94	1.10	-0.001151	0.09241
Europe 2006-2013				
Maturity	Mean (%)	Std. (%)	Mean daily change (%)	Std. daily change (%)
L_1	1.91	1.65	-0.002074	0.05252
L_2	2.21	1.48	-0.002066	0.09349
L_3	2.57	1.35	-0.001983	0.11090
L_4	3.17	1.13	-0.001654	0.07264
L_5	3.64	0.88	-0.001353	0.06499
L_6	4.04	0.83	-0.001169	0.06831
US 2006-2013				
Maturity	Mean (%)	Std. (%)	Mean daily change (%)	Std. daily change (%)
L_1	2.44	1.67	-0.002978	0.05145
L_2	1.93	1.85	-0.003339	0.11936
L_3	2.91	1.57	-0.002880	0.09502
L_4	3.73	1.38	-0.002544	0.10140
L_5	4.36	1.13	-0.001945	0.10370
L_6	4.74	1.06	-0.001681	0.11581

Table 7.2: Statistics for the Norwegian, European and US LIBOR forward rates.

most dominant movement in the NOK and USD instantaneous forward rate curves. The second PC in the EUR rates changes sign twice and indicates that the second most important feature in the forward rate curve is a bending. The third principal component has different structure in the three different markets. The Norwegian and US third PC changes sign three times and their economical meaning is not clear. The European third PC changes sign once and reflects a twisting of the curve. These findings suggest that PCA on forward rates give less intuitive results as with the same procedure on yield curves.

Table 7.3 shows the eigenvalues for the PCA. We observe that the decay of the eigenvalues is slowest for the Norwegian market and fastest for the US market. This suggests that the Norwegian interest market is more illiquid than the others.

The volatility structure is easily obtained from (5.6). Figure 7.4 displays the volatility factors in terms of time to maturity. We observe that the first factors have more volatility.

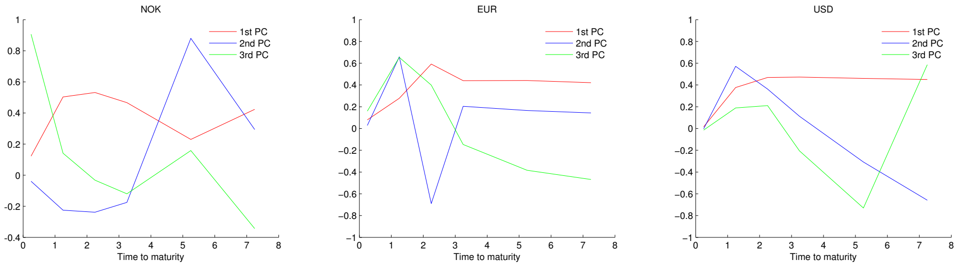

Figure 7.3: The three largest principal components for Norway, Europe and the US.

	Norway	
Eigenvalues	Value (in 10^{-3})	Cumulative variation Explained (%)
λ_1	0.2546	45.3
λ_2	0.1313	68.7
λ_3	0.0660	80.4
λ_4	0.0507	89.4
λ_5	0.0369	96.0
λ_6	0.0226	100.0
	Europe	
Eigenvalues	Value (in 10^{-3})	Cumulative variation Explained (%)
λ_1	0.2582	52.4
λ_2	0.1292	78.6
λ_3	0.0576	90.3
λ_4	0.0189	94.1
λ_5	0.0156	97.3
λ_6	0.0135	100.0
	US	
Eigenvalues	Value (in 10^{-3})	Cumulative variation Explained (%)
λ_1	0.6499	75.8
λ_2	0.0874	86.0
λ_3	0.0610	93.1
λ_4	0.0261	96.1
λ_5	0.0183	98.3
λ_6	0.0147	100.0

Table 7.3: The eigenvalues of the covariance matrix and amount of cumulative variation explained for the Norwegian, European and US 3-month instantaneous forward rates 2000-2012.

7.2.2 Historical Volatility in the LMM

We do the same procedure with our historical LIBOR forward rates and find the covariance matrix for the daily changes between the maturities, which also will be

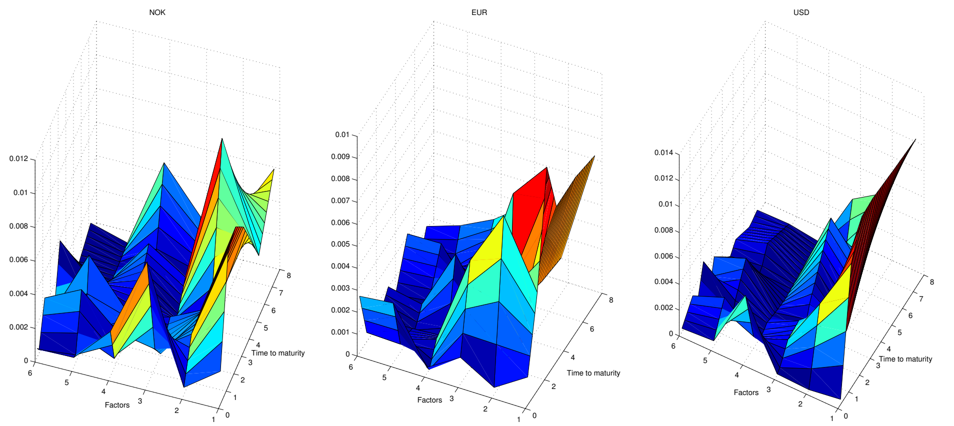

Figure 7.4: Volatility Structure in HJM framework.

a 6 × 6 matrix. The next step is to set the entries above the diagonal to zero, motivated by the discussion in section 6.1. Lastly we find our volatility structure by using (6.5). Figure 7.5 shows the volatility structure for the three different markets. Without doubt the diagonals contribute with most volatility.

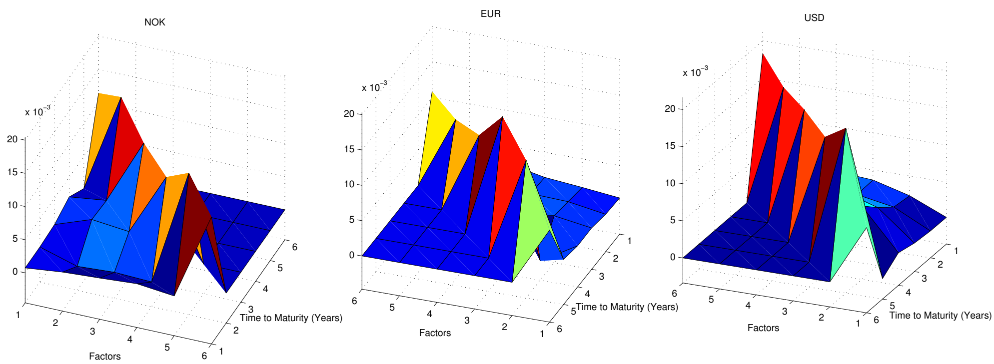

Figure 7.5: Volatility Structure in LMM.

Chapter 8

Results

With the volatility structure in our possession we are ready to simulate in the HJM framework and the LMM.

8.1 Simulation

8.1.1 Simulation in the HJM Framework

In the HJM framework we have simulated under (5.5) using the algorithms in Glasserman[3]. Because we want to incorporate all historical volatility into our model we use all six factors when we simulate forward in time. Figure 8.1 shows the distribution of the simulated 3-month rate in seven years for the Norwegian, European and US market respectively. We observe that the distributions look normal. The distributions' means are 3.8 %, 2.32 % and 3.14 % respectively and the standard deviations are 2.33 %, 2,20 % and 3.15 %.

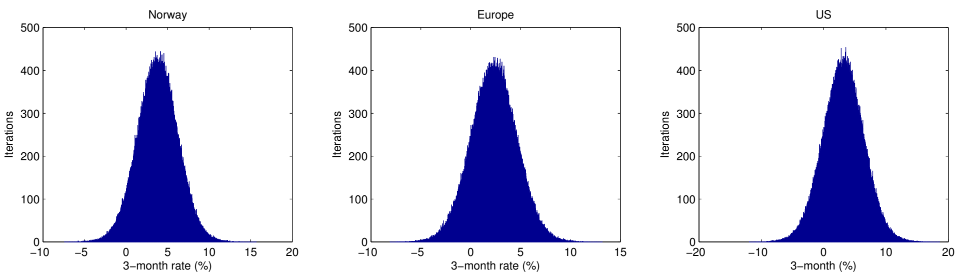

Figure 8.1: Distribution of simulated 3-month rate curve in seven years with 1,000,000 Monte Carlo simulations. Means (3.8 %, 2.31 % and 3.14 %)

Because the HJM model uses Gaussian variables it cannot guarantee positive rates. We observe clearly that the probability of a negative 3-month rate in seven years is highly significant for all three markets. The probabilities of a negative

3-month rate in seven years are 5,10 %, 14,65 %, 15.91 % respectively. These results clearly contradict the classical concept of interest rates. Introductory economics books state that one dollar is worth more today than tomorrow but these distributions suggest otherwise.

In our Heath-Jarrow-Morton model we simulate the short rate forward in time and there are three months between each node. Later when we price derivatives the payment dates will typically be equivalent to these nodes and the payoffs depend on the short rate. The probabilities of a negative short rate on the nodes are therefore interesting and have the possibility to strongly affect the price. Figure 8.2 clearly shows the significance of negative rates in the Heath-Jarrow-Morton framework in the current low-rate environment. We know that negative rates are allowed in the HJM framework but can we really translate this to the real world? Many argue that negative rates are impossible. If this indeed is the truth we do a horrible mistake by pricing interest rate derivatives with our model, for example floors will get overpriced with our model. We also observe that the closer today's short rate is to zero, the higher the probability is of a negative short rate in the future (there is a much bigger chance for a negative short rate in EUR or USD rates than in NOK rates).

Figure 8.2 also shows the probability of the short rate in the future being lower than the current level. We clearly see the market expectation of a rise in interest rates in the future because the probability is decreasing with time. With the current yield curves it is satisfying that the EUR and USD probabilities are generally lower than the NOK probabilities.

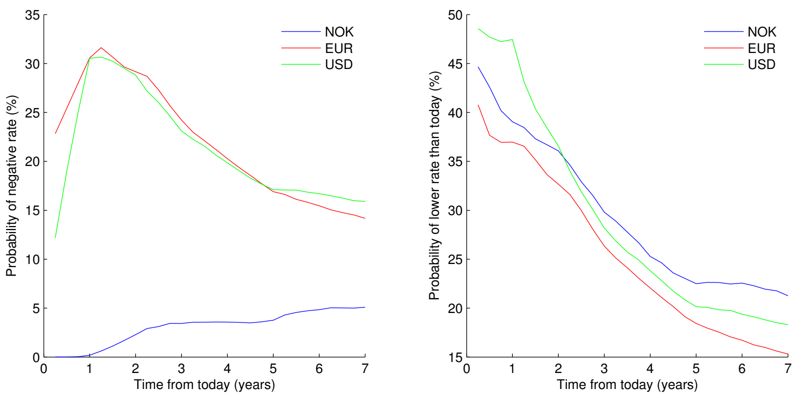

Figure 8.2: a) Probability of negative 3-month rate forward in time for Norway, Europe and the US. The probabilities are clearly significant. b) Probability of a lower rate than the current level. The graphs decline which means the market expects rises in interest rates.

8.1.2 Simulation in the LIBOR Market Model

In the LMM we use an Euler scheme and simulate under (6.6) using all six factors. Figure 8.3 illustrates the distributions for simulated NOK, EUR and USD LIBOR 3-month rates in three months and seven years time. We see that the distributions for the LIBOR rates in seven years have more volatility. This makes sense, it is harder to foresee the rate in seven years than in three months. The distributions are unbiased, i.e. the distributions' means are equal to the LIBOR forward rates today. In LMM we only model the the nodes on the yield curve we have today. For maturities between these nodes we need to interpolate. In this thesis we assumed a piecewise constant forward rate. For example the LIBOR forward rate simulated in one year also covers maturities up to two years. We observe that the distributions do not cover negative interest rates. This is due to the fact that LIBOR rates in LMM are modelled as lognormal and hence cannot go negative.

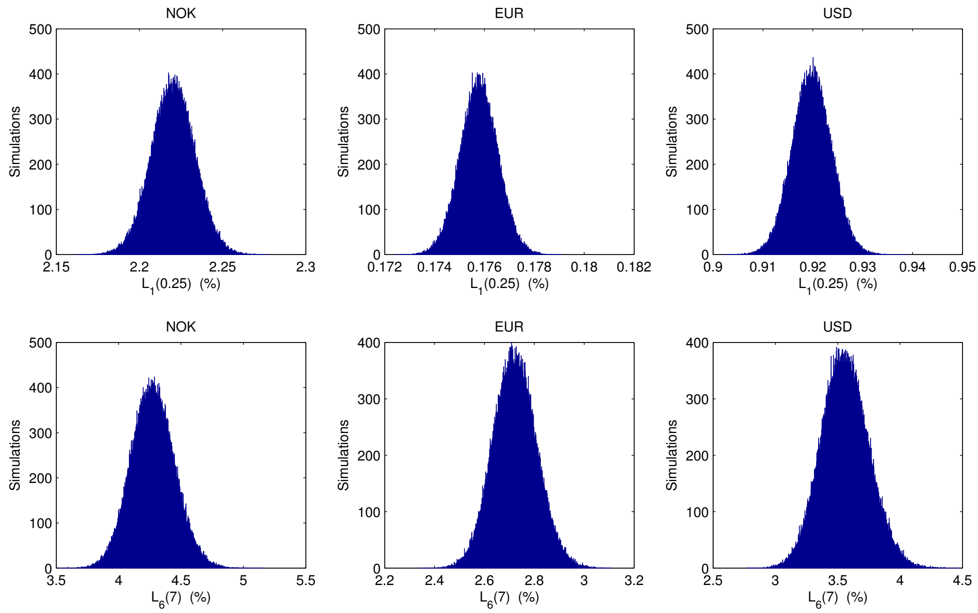

Figure 8.3: Distributions of $L_1(0.25)$ and $L_6(7)$ for NOK, EUR and USD rates.

8.2 Pricing Derivatives

We now want to investigate the prices our models generate. Rather than quoting the price in the actual currency we quote the price as the coupon rate in (5.7). A rate is of course much easier to compare to today's yield curve. Figure 8.4 shows the current yield curves, instantaneous forward rates and LIBOR forward rates in the three different markets we are investigating. We observe that NOK rates are a level above the two other markets. The NOK and EUR yield curves represent a normal yield curve i.e. the yield is an increasing function of maturity. This means that market participants expect the interest rates to rise in the future. This is also reflected in figure 8.2. The USD yield curve is normal except that it declines between 1Y and 2Y. We see that we have discretized away the decline between 1Y and 2Y in the USD instantaneous forward rate curve.

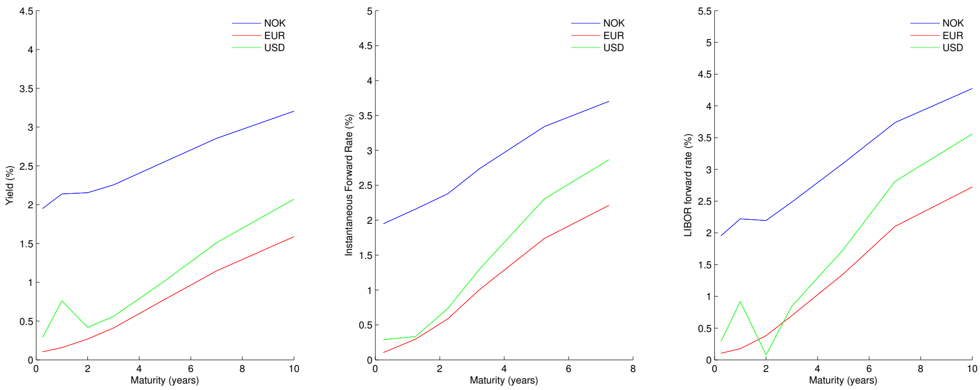

Figure 8.4: Current yields, instantaneous forward rates and LIBOR forward rates for Norway, Europe and the US.

8.2.1 Cap

We begin by pricing a cap with quarterly paying caplets. We use the 3-month LIBOR rate as the floating rate. Because a cap is a security against rises in interest rates the price of a cap should decrease with strike price and increase with number of caplets and volatility factors. Figure 8.5 displays the cap price versus the strike price r_c for NOK, EUR and USD rates.

The price does indeed decrease with the fixed cap rate. For negative strikes the HJM and LMM models generate approximately the same price. For $r_c = -5\%$ the prices differ about 2 basis points. Simulated rates in the HJM model can go negative and OTM for negative strikes which leads to a smaller Monte Carlo price than in the LMM. As the strike approaches the current 3-month rate the HJM price is clearly greater. The reason for this can be explained by the simulated rate distributions in the two models. Whereas the rate distributions in HJM have more standard deviation and cover a large range of interest rates the rate distributions in the LMM are narrow around the LIBOR forward rates today. We observe for example in figure 8.1 that HJM simulations can end up in the money for strike prices above $r_c = 10\%$, which eventually leads to larger prices than in the LMM.

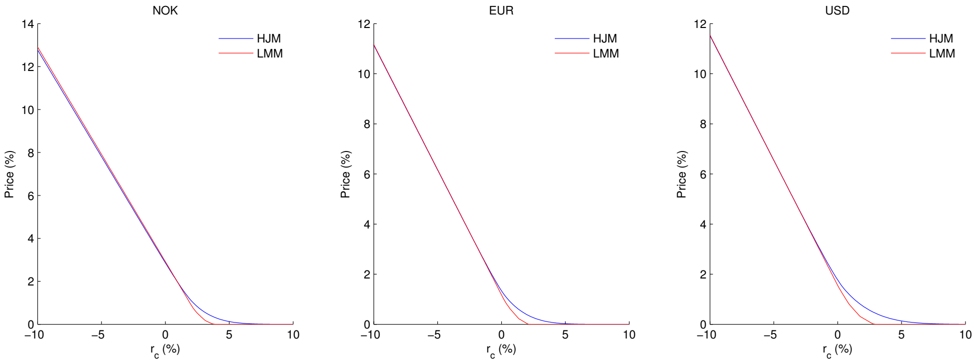

Figure 8.5: Cap prices versus strike price on NOK, EUR and USD rates. Number of caplets is 29 and number of factors is 6 in both HJM and LMM. The prices clearly differ with strikes near the current 3-month rate.

With the same strike price a cap on NOK rates is most expensive followed by USD and EUR rates in both HJM and LMM. This is no surprise as the current NOK rates are a level above USD and EUR rates. This price difference is explained by dissimilarities in the macroeconomics and not by the differences between HJM and LMM.

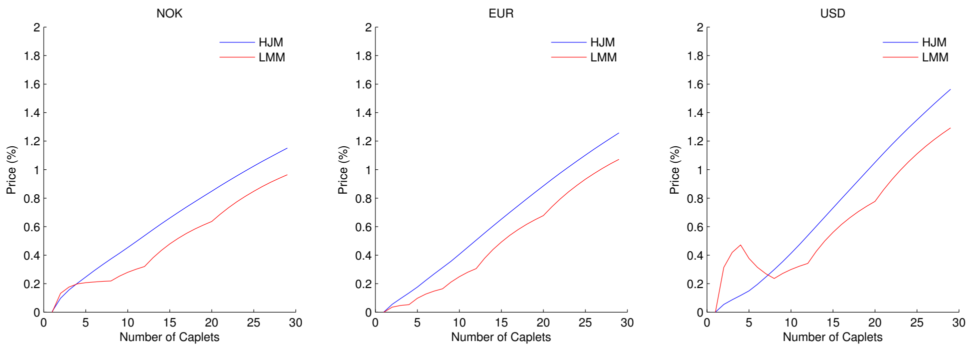

Figure 8.6: Cap prices versus number of caplets on NOK, EUR and USD rates. The number of factors is 6 and the strike price is today's 3-month rate i.e. 1.95%, 0.1% and 0.29% respectively in both HJM and LMM.

Figure 8.6 illustrates how the cap price depends on the number of caplets when we use today's 3-month rate as the fixed cap rate. We expect the cap price to increase strictly with number of caplets and it might seem wrong at first sight that the LMM price on USD rates is decreasing from 3 to 8 caplets. However, the LMM price on USD rates reflects the LIBOR forward rates in figure 8.4. By looking at (5.7) we see that the price in currency does not necessarily decrease even though the coupon price decreases. This is the case here, it is just a reflection of the current yield curve. The prices on USD rates generated by HJM do not have the same structure. In this model we simulate the USD instantaneous forward rates which do not have the same trend as the LIBOR forward rates in figure 8.4.

Lastly we include a graph of the HJM prices versus number of volatility factors. Figure 8.7 shows this relation. As expected the prices are increasing in terms of factors. The prices increase most from 1 to 3 factors which is the case when one uses principal component analysis to obtain a volatility structure. Further we note that the USD price is largest followed by the EUR price.

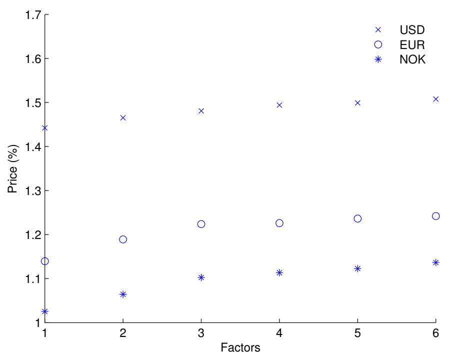

Figure 8.7: HJM cap prices versus number of volatility factors on NOK, EUR and USD rates. The number of factors is 6 and the strike price is today's 3-month rate i.e. 1.95%, 0.1% and 0.29% respectively. We see that the first factors contribute most to the price.

8.2.2 Floor

A more interesting derivative in the current rate environment might be the floor. The buyer of a floor secures himself from a drop in the interest rate, typically a lender would find floors necessary to reduce interest rate risk. We let the floor pay quarterly floorlets and use the 3-month LIBOR rate as the floating rate. The floor price should increase with the fixed floor rate r_f. Figure 8.8 shows that the floor prices increase with the strike price as expected. The most important observation without doubt is that the HJM model generates a nonzero price for floors with strike prices below zero. The HJM prices for floors with $r_f = 0\%$ are 2, 16 and 21 basis points for NOK, EUR and USD rates respectively. In the LIBOR market model the rate is modelled lognormal and cannot be zero or negative by definition. Hence, the LMM prices for 0 % floors are zero. Further we observe that the LMM and HJM prices differ most in the region around today's 3-month LIBOR rate. As the fixed floor rate increases the prices approach each other.

With the same fixed floor rate a floor on EUR rates is more expensive than on

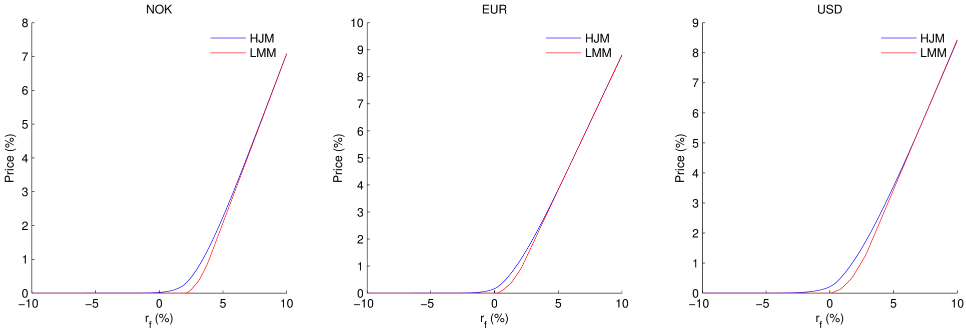

Figure 8.8: Floor prices versus strike price on NOK, EUR and USD rates. Number of caplets is 29 and number of factors is 6 in both HJM and LMM.

USD or NOK rates in both models. This is again explained by the current yield curves and not by the differences between our models. The EUR yield curve is on a lower level and hence a floor on EUR rates should be more expensive.

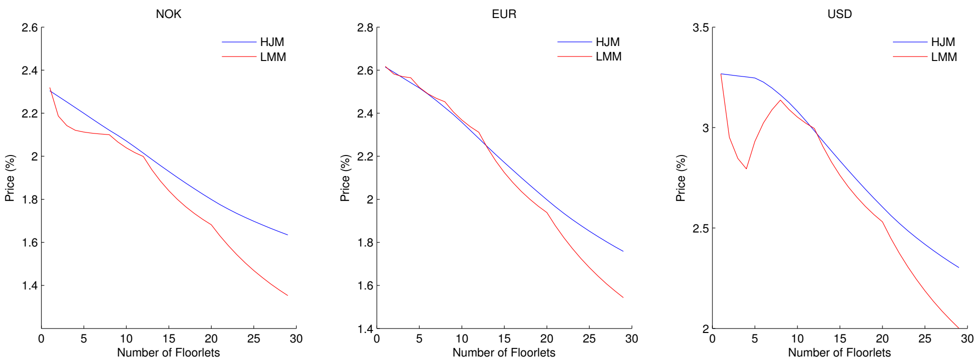

Figure 8.9: HJM floor prices versus number of caplets on NOK, EUR and USD rates. Number of factors is 29 and strike price is equal to the LIBOR forward 3-month rate in seven years, i.e. 4.27%, 2.72% and 3.56% for NOK, EUR and USD rates respectively.

Figure 8.9 illustrates how the floor depends on the number of floorlets in the different markets. Again the LMM price for USD rates reflects the yield curve in figure 8.4. We observe that the coupon price generally decreases with the number of floorlets. This observation reflects that the market expects the interest rates to rise in the future, and hence the relative price of the floor decreases with the length of the contract. We again observe that the Monte Carlo prices in the Heath-Jarrow-Morton framework are larger because of the exposure to negative rates.

The HJM prices in terms of number of volatility factors are illustrated in figure 8.10. Also the floor prices are strictly increasing functions of volatility factors. We observe that the USD price is largest.

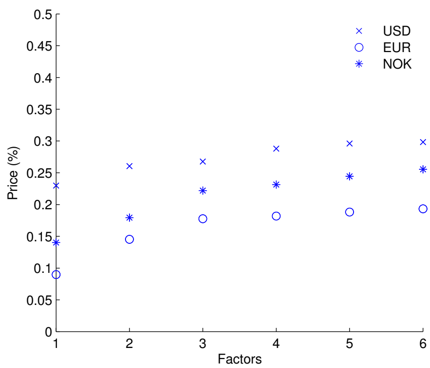

Figure 8.10: HJM floor prices versus number of factors on NOK, EUR and USD rates. Number of caplets is 29 and strike price is equal to today's 3-month rate i.e. 1.95%, 0.1% and 0.29% respectively. The first factors contribute the most to the price.

8.2.3 Long Butterfly Spread

An easy way to detect arbitrage is to find prices of butterfly spreads. The long butterfly spread price is the same regardless if it is constructed with caps or floors. Figure 8.11 displays the long butterfly spread prices constructed by caps. We observe that the prices are nowhere negative or zero which means that our models do not generate arbitrage possibilities. The HJM and LMM prices have the same profile but they also differ. First of all the HJM prices have fatter tails. The prices stay significant even though the strike price goes negative or very high. Another difference is that the LMM prices are less smooth than the HJM prices. The reason for this can be traced back to how we calibrated our models. The caps are paying quarterly. Hence, the HJM model discretizes the whole yield curve into points with 3 months in between and simulate all these points. On the contrary LMM simulates the real points we have in our data set (3M, 1Y, 2Y, 3Y, 5Y, 7Y and 10Y) and then we have to choose how to estimate the points in between. In this thesis we assumed a piecewise constant LIBOR rate. In other words, our forward LIBOR rate is not continuous which result in the non-smooth butterfly prices.

The prices peak when r_c is above today's 3-month rate. For example the HJM prices peak at $r_c = 2.15\%$, $r_c = 0.33\%$ and $r_c = 0.40\%$ for NOK, EUR and USD rates. This is no surprise. A long butterfly spread is a bet on the interest rate staying inside a region centred at r_c with length $2a$. Figure 8.4 shows that the market expectation is that interest rates will rise in all markets in the future. Hence, the long butterfly spread prices should peak on strikes above the current 3-month rates. The LMM prices peak on larger strikes because the LMM does not cover negative rates.

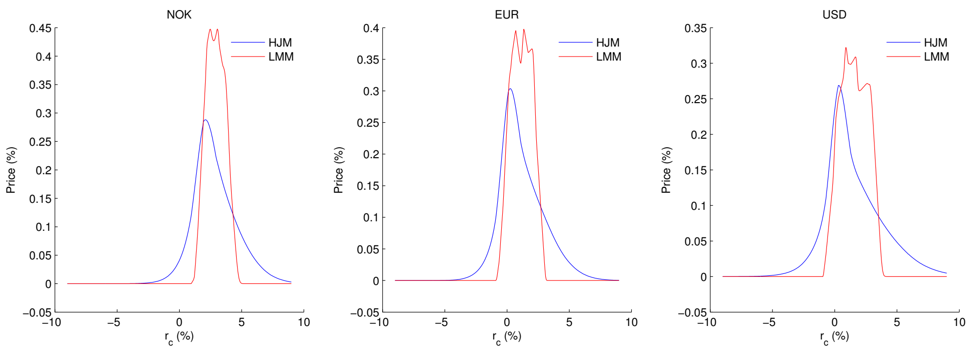

Figure 8.11: Long butterfly spread price versus strike price when $a = 1\%$.

Chapter 9

Conclusion

The current low-rate environment is very interesting because it challenges the fundamental rule in finance that the rate is a positive variable. This motivated us to implement two advanced term-structure models, Heath-Jarrow-Morton and LIBOR market model, to highlight differences under the current market conditions in Norway, Europe and the US. In the HJM model we used principal component analysis on historical daily changes in instantaneous forward rates to get an expression for the volatility structure. The LMM used the square root of the covariance matrix of historical daily changes in LIBOR forward rates as the volatility structure. We showed that USD rates were more volatile than NOK and EUR rates. Furthermore the eigenvalue decay in the PCA showed that the Norwegian market is more illiquid than the European and US market.

The main differences between the two models were the exposure to negative rates. The HJM model is Gaussian and we showed that under this framework a negative 3-month rate was significant in the future. In the LMM the LIBOR forward rates are modelled lognormal and cannot go negative. The probability of a negative 3-month rate in the future in HJM depends heavily on the current 3-month rate level. The simulated NOK 3-month rate was negative in the future with a much lower probability than the USD and EUR 3-month rate, which is not very surprising.

The differences between the two models became apparent when we used Monte Carlo simulation to price derivatives. Although the prices only differed with a few basis points for negative and large strikes the prices started to seriously deviate around strikes near the current 3-month rate level. Floors with 0% strike price were priced to 2bp, 16bp and 21bp with underlying NOK, EUR and USD rates respectively under the Heath-Jarrow-Morton framework. The LMM obviously priced these derivatives to zero. Further the long butterfly spreads showed that both models excluded arbitrage.

Although the two models used the same data sets as starting point, the results showed that different model assumptions and dynamics lead to different prices. Firstly we calibrated each model to the current market conditions and constructed a volatility structure of historical forward rates. The next step was to simulate forward rates under these conditions. Using historical volatility is beneficial because it gives us an insight in the possible risk we take by trading in the rates historically. However, if these models were to be used for trading they would need to be market consistent, i.e. produce the same price as the market. An interest rate model built on historical volatility will not be market consistent. A recalibration to market prices would suffice, for example in the LIBOR market model we could incorporate an implied volatility structure.

In low-rate environments the results show that one should avoid using the log-normal LMM to price low-strike derivatives. Zero strike floors have a value in certain markets and if a practitioner uses a model which assumes positive rates he will loose a lot of money. On the contrary one should also be careful when one uses the HJM under regular market conditions. A 0% floor might be worthless in some markets. However, as 0% floors are illiquid objects in low-rate environments they are even more illiquid under regular market conditions. Thus, this might not be problematic from a practitioner's perspective.

Further work would be to try to match our models to market prices by recalibrating the volatility structures. It would also be interesting to develop a model based on LIBOR rates which allows negative interest rates and calibrates to market prices with for example the shifted SABR model.

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