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Real options valuation under technological uncertainty: A case study of investment in a post-smolt facility

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Problem description

Can real options analysis uncover additional value compared to traditional capital budgeting methods when evaluating an investment in a post-smolt facility under technology and profit uncertainty?

Preface

This thesis is conducted as part of achieving a Master of Science at the Norwegian University of Science and Technology (NTNU). The degree specialisation is in Financial Engineering at the Department of Industrial Economics and Technology Management.

The thesis is original and independent work by Jørgen Hannevik, Magnus Naustdal and Henrik Struksnæs. We would like to thank our supervisor, Associate Professor Verena Hagspiel, for stimulating discussions and rewarding counsel. We would also like to thank Senior Advisor Henning Urke at INAQ for providing valuable insights and connecting us with prominent figures within the Norwegian salmon farming industry. Further, we thank Associate Professor Torstein Kristensen at the Faculty of Biosciences and Aquaculture at the University of Nordland for helpful insight with regards to the biological aspects of salmon farming. A thankful note is also directed to CEO Alex Vassbotten at Steinvik Fiskefarm for inviting us to his fish farm in Florø. We also thank Nofima for allowing us to attend the conference "Smolt production of the future" held in Sunndalsøra in October 2014.

The study is developed in close collaboration with several industry experts. We thank Bjørn Hembre in SalMar, Anders Jon Fjellheim at Marine Harvest, Finn Christian Skjennum at Tjeldbergodden Settefisk, Ole Gabriel Kverneland at AKVA group and Kolbjørn Giskeødegård at Nordea Markets for sharing valuable industry knowledge and helping us calibrate the models. Lastly, we thank Finansavisen for publishing our article debating the practical implications of our findings. The article is enclosed on the next page.

Trondheim, June 11th, 2015

Jørgen Hannevik   Magnus Naustdal   Henrik Struksnæs

Debate article published in the June 6th edition of Finansavisen

DEBATT

Historisk høye marginer kan ha representert en hvilepute for implementering av mer relevante verdivurderingsmetoder i oppdrettsnæringen, skriver Jørgen Hannevik, Magnus Naustdal og Henrik Struksnæs.

Oppdrettsbransjen tar investeringsbeslutninger på ufullstendig grunnlag

TENK NYTT: Større usikkerhet i bransjen de senere årene medfører at realopsjonsprising er mer aktuelt, skriver artikkelforfatterne. FOTO: DREAMSTIME

Lakseoppdrettsnæringen er kapitalintensiv og preget av høy usikkerhet. Dette er et resultat av volatile laksepriser og en lang produksjonssyklus preget av varierende miljøforhold. Likevel tas investeringsbeslutninger i bransjen basert på tradisjonelle verdivurderingsmetoder som ikke tar hensyn til disse risikofaktorene.

Empirisk forskning viser at tradisjonelle verdivurderingsmetoder som diskontert kontantstrømsanalyse er unøyaktige når investeringsusikkerheten er høy. Metoden tar utgangspunkt i at investeringen er en «nå eller aldri»-mulighet, og inkluderer ikke verdien i at beslutningstaker kan handle i tråd med ny informasjon i løpet av investeringens levetid. Dette innebærer at metoden konsistent undervurderer prosjektverdi og fører til at beslutningstakere risikerer å avslå verdifulle investeringsmuligheter.

Metoder basert på opsjonsprising fanger også oppsidepotensialet av risiko. En realopsjon kan enkelt forklares som retten, men ikke forpliktelsen, til å foreta en investering. Ved å vurdere investeringsmuligheten som en realopsjon inkorporeres verdien av fleksibilitet ved usikkerhet bedre. I vår masteroppgave innen industriell økonomi ved NTNU bekrefter vi dette ved å analysere en investering i potensielt banebrytende teknologi innen lakseoppdrett ved bruk av både tradisjonell kontantstrømsanalyse og realopsjonsprising.

Produksjonsmessige utfordringer og sykdommer i sjøfasen har resultert i en gjennomsnittlig årlig dødelighet på opptil 20 prosent i norske oppdrettsanlegg. Dette har ført til at Nærings- og fiskeridepartementet er motvillige til å øke den maksimalt tillatte biomassen (MTB) pr. konsesjon. En foreslått løsning på problemet er postsmoltproduksjon. Dette innebærer å holde fisken lenger i et lukket miljø før den settes ut i tradisjonelle merder. Postsmoltproduksjon gir lavere forventet dødelighet enn tradisjonell oppdrett på grunn av en forkortet sjøperiode. Samtidig utnyttes MTB-en bedre ved at den slakteklare laksen blir erstattet med større fisk.

> **Metoder basert på opsjonsprising fanger også oppsidepotensialet av risiko**

I samarbeid med konsulentselskapet INAQ, Universitetet i Nordland, Marine Harvest og Steinvik Fiskefarm, har vi utviklet en realopsjonsmodell for evaluering av en postsmoltinvestering. Våre resultater viser at verdien av fleksibilitet i investeringen er stor på grunn av høy usikkerhet i lønnsomhet og teknologisk utvikling. Når kontantstrømsanalysen anbefaler investering, verdsetter realopsjonsmodellen merverdien av fleksibilitet i å kunne vente på optimale investeringsforhold til nær 30 prosent av investeringskostnaden.

Historisk høye marginer i oppdrettsnæringen kan ha representert en hvilepute for implementering av mer relevante verdivurderingsmetoder. Større usikkerhet i bransjen de senere årene medfører at realopsjonsprising er mer aktuelt. Vi foreslår ikke at bransjen bør slutte å basere avgjørelser på teft og tradisjonelle verdivurderingsmetoder, men dersom utvikling av nye løsninger som postsmoltproduksjon bremses ved at investeringer ikke gjennomføres som følge av et ufullstendig beslutningsgrunnlag, er det kanskje på tide å tørre å ta i bruk også mer avanserte metoder.

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Abstract

The Norwegian salmon farming industry is struggling with sea lice and diseases. This makes the authorities reluctant to allow an increase in production despite of a growing demand. A measure which can enable better utilization of the existing production limits is post-smolt production. Companies are however hesitant to invest, as post-smolt production is on an early development stage, and there is uncertainty related to the technology and the price development of post-smolt. When uncertainty is high, empirical research indicates that real options analysis is more accurate than the traditional DCF method, which is the industry standard.

In this thesis we examine if multi-factor real options analysis can uncover excess value compared to traditional DCF analysis by evaluating an investment in post-smolt production. Using dynamic programming we develop two multi-factor models enabling us to isolate and analyse the effects of two forms of technological uncertainty combined with profit uncertainty. Technological uncertainty is modelled as the arrival of an innovation that either reduces investment cost or increases production efficiency. The innovation arrival is described by a Poisson process, while profits follow a geometric Brownian motion. To solve the models we develop numerical procedures based on finite differences and algorithms solving sets of nonlinear equations.

Our results show that real options valuation uncovers significant excess value compared to the DCF method, implying that inaccurate valuations could lead salmon farming companies into rejecting sensible business opportunities such as post-smolt production. Our recommendation is however that real options valuation should be used as a complement to, and not a substitute for traditional DCF analysis as the real options models require simplifying assumptions to be mathematically tractable. Additionally we show that the salmon farming company has stronger incentives to delay investment when the benefits of technological innovations cannot be gained if they arrive after the investment is undertaken, implying that investment strategy is greatly influenced by how technological uncertainty is modelled.

Sammendrag

Den norske laksebransjen sliter med lus og andre sykdommer. Myndighetene er derfor motvillige til å tillate økt produksjon, til tross for økende etterspørsel. Et tiltak som gjør det mulig å utnytte de eksisterende produksjonsrammene bedre er postsmoltproduksjon. Likevel er selskapene motvillige til å investere, da postsmoltproduksjon er i en tidlig utviklingsfase, og det er stor usikkerhet knyttet til teknologien og utviklingen til markedsprisen på post-smolt. Når usikkerheten er høy, viser empirisk forskning at verdsettelse ved bruk av realopsjoner er mer presist enn den tradisjonelle kontantstrømsmetoden som er bransjestandarden.

I denne oppgaven undersøker vi om flerfaktor realopsjonsanalyse kan avdekke merverdi sammenlignet med kontantstrømsmetoden ved å evaluere en investering i postsmoltproduksjon. Ved hjelp av dynamisk programmering utvikler vi to flerfaktor-modeller som tillater oss å isolere og analysere effekten av to typer teknologisk usikkerhet kombinert med usikkerhet i profitt. Teknologisk usikkerhet er modellert som ankomsten av en innovasjon som enten reduserer investeringskostnaden eller øker produksjonseffektiviteten. Ankomsten av tekniske innovasjoner er modellert som en Poisson-prosess, mens profitt er modellert som geometrisk brownske bevegelser. For å løse modellene har vi utviklet numeriske metoder basert på endelig-differanse-metoden (finite difference method) og algoritmer for å løse systemer av ikke-lineære ligninger.

Resultatene viser at realopsjonsanalyse avdekker betydelig merverdi sammenlignet med tradisjonell kontantstrømsanalyse, hvilket i praksis impliserer at unøyaktige verdivurderinger kan føre til at oppdrettsselskaper avstår fra potensielt fornuftige investeringer som postsmoltproduksjon. Likevel anbefaler vi å bruke realopsjonsanalyse som et komplement og ikke et substitutt for kontantstrømsanalyse da realopsjonsmodeller krever forenklende antagelser for å være matematisk løsbare. I tillegg viser vi at oppdrettsselskapet har sterkere insentiver til å utsette en investering dersom det ikke får fordelene av teknologiske nyvinninger som ankommer etter at investeringen er foretatt. Dette tyder på at investeringsstrategien påvirkes av hvordan teknologisk usikkerhet modelleres.

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Chapter 1

Introduction

The salmon farming industry is one of Norway's largest and most important industries. Global demand for salmon is increasing, but due to biological challenges, the supply is constrained. Farmers struggle with fish diseases and sea lice, resulting in an average annual mortality rate of up to 20 percent, and at the same time forces companies to spend millions of NOK on medical treatments every year. The high mortality rate also makes the Norwegian authorities reluctant to allow an increase in the production of salmon. The authorities control production by awarding companies production licenses. Each license gives the right to keep a certain maximum volume of fish at sea at all times, referred to as the maximum allowed biomass (MAB). Companies strive to maximise the utilisation of their MAB, i.e. the utilisation of available production capacity, as this increases profits. Traditionally the salmon is slaughtered at an individual weight of 4-6 kg, and the sea cages are refilled with smolt weighing around 150 grams. To allow the smolt to reach slaughtering weight without the MAB being exceeded, the number of smolt set out approximately equals the number of fish slaughtered. As a result, total biomass standing at sea drops and the MAB utilisation decreases. Given these challenges, the Norwegian industry is desperate for measures that can enable better fish health and increased production without violating regulations.

One of the measures considered is *post-smolt production*, which involves growing the salmon larger in a protected environment, e.g. tanks on-land or in the sea, before moving it into traditional sea cages. This has two main benefits: First, the fish is more robust when moved into sea cages, and by shortening the seawater production period, accumulated mortality and need for expensive medical treatments is reduced. Second, replacing fish at slaughtering weight by post-smolt increases the MAB utilisation. As post-smolt production can increase both profits and fish welfare, it is expected to be beneficial from both an economic and ethical perspective. However, post-smolt technology is still in a development phase, and many of its benefits are expected rather than certain.

Companies are currently debating if an investment is economically justifiable. As post-smolt production is on an early stage in terms of R&D, there is a high level of uncertainty related to important factors such as the cost and performance level of the technology. In addition, there is uncertainty in operating profits. Traditionally, the primary method used by companies to evaluate investments is discounted cash flow analysis (DCF). However, empirical research indicates that traditional methods result in inaccurate valuations when there is high uncertainty related to an investment. Traditional methods only reflect the downside of uncertainty, while disregarding the upside potential. Consequently, decision makers risk rejecting sensible business opportunities. A method that allows to account for the upside potential of uncertainty is real options valuation (ROV), which has its roots in financial option pricing. As the investment in a post-smolt facility is subject to both price and technology uncertainty, our hypothesis is that a real options analysis will uncover additional value compared to traditional capital budgeting methods. This leads to the following problem definition:

Can real options analysis uncover additional value compared to traditional capital budgeting methods when evaluating an investment in a post-smolt facility under technology and profit uncertainty?

We aim to solve this problem by proposing real options models. The intrinsic value of the option is treated as an approximate to the net present value given by a simplistic DCF analysis with no explicitly forecast cash flows, and perpetuity growth from year one. We determine how the combination of technology and profit uncertainty affect the optimal investment strategy by developing forefront multi-factor models. In addition to a quantitative comparison, we compare DCF and ROV based on existing academic literature. Our main contribution is threefold: 1) We examine the potential of real options valuation in salmon farming; 2) We extend the real options theory within the area of theoretical multi-factor models; 3) We apply real options valuation to a specific post-smolt investment case, thereby adding to the ongoing discussion of the economic viability of post-smolt production.

First, to the best of our knowledge, we are among the first to examine the potential of real options valuation within salmon farming. The majority of academic literature focuses on the biology rather than the economics of salmon farming. The few available economic papers are based either on traditional capital budgeting methods or optimisation of operations (see e.g. Forsberg and Guttormsen (2006); Langan and Toftøy (2011); Hæreid (2011); Hannevik et al. (2014)). We have contributed to the salmon farming industry by providing study giving intuition on how uncertainty in technology and profits affects an investment in post-smolt production.

Second, multi-factor real options models are a recent development in theoretical real options literature. Hence developing these models further is an important contribution. By developing two multi-factor real options models, we have captured different aspects of investing under technology and profit uncertainty. We adapt the model for investment under technology and revenue related uncertainty proposed by Murto (2007) to the case of post-smolt investment. We extend the work by constructing a numerical procedure that allows us to solve the model. Additionally, we apply real options techniques originally proposed in papers considering investment in power generation plants under policy uncertainty, to an investment under technological uncertainty. Adkins and Paxson (2013) and Chronopoulos et al. (2015) consider an investment in a power generation plant with the possible sudden provision or retraction of a government subsidy. The subsidy is a cash payment proportional to the revenues of the firm and is only received if it is introduced before the investment is made. Adkins and Paxson (2013) consider uncertainty in both price and quantity, whereas Chronopoulos et al. (2015) only consider price uncertainty while finding optimal quantity. We find that the sudden provision of a subsidy resembles the arrival of a technological innovation, enabling us to adapt and apply the frameworks of Adkins and Paxson (2013) and Chronopoulos et al. (2015) to an investment under technological uncertainty. One important difference between a subsidy and a technological innovation is that the subsidy cannot be gained after the investment is made. We therefore extend their models so that they fit our case, where the benefit of a technological innovation is gained also after the investment is made.

Third, due to the high degree of uncertainty related to post-smolt production technology, the industry is reluctant to undertake investments. Currently the industry relies on traditional capital budgeting methods that only treats the downside potential of uncertainty. We contribute by proposing multi-factor real options models that also capture the upside potential of the uncertainty embedded in post-smolt production. As there is little available data related to post-smolt production and market prices, the models' underlying assumptions and parameters have been chosen in close collaboration with both biological researchers and representatives from the industry majors. Thus, our results represent a serious contribution to further discussions of the economical viability of post-smolt production.

The thesis is organised as follows: In Chapter 2 we present the most important risks and regulations in traditional salmon farming, and elaborate on the motivation behind post-smolt production. In Chapter 3 we compare the traditional DCF and the real options approach based on existing academic literature. Additionally we give a summary of work related to multi-factor real options modelling and technological uncertainty. In Chapter 4 we present some of the mathematics that form the basis for the models presented in Chapter 5. In Chapter 6 we quantify the parameters used in the case study. In Chapter 7 we present a post-smolt case study, and test the sensitivity of the results to changes in the input parameters. In Chapter 8 we discuss our main contributions w.r.t. related literature, as well as the model assumptions and practical applicability, before we draw conclusions and make suggestions for further research in Chapter 9.

Chapter 2

Traditional salmon farming and post-smolt production

In this chapter we present the most important sources of uncertainty and regulations in salmon farming, and elaborate on the motivation behind post-smolt production. Additionally we discuss why it is currently not implemented on a large scale by the industry. To form a basis for the chapter we illustrate the traditional value chain in Figure 2.1 below.

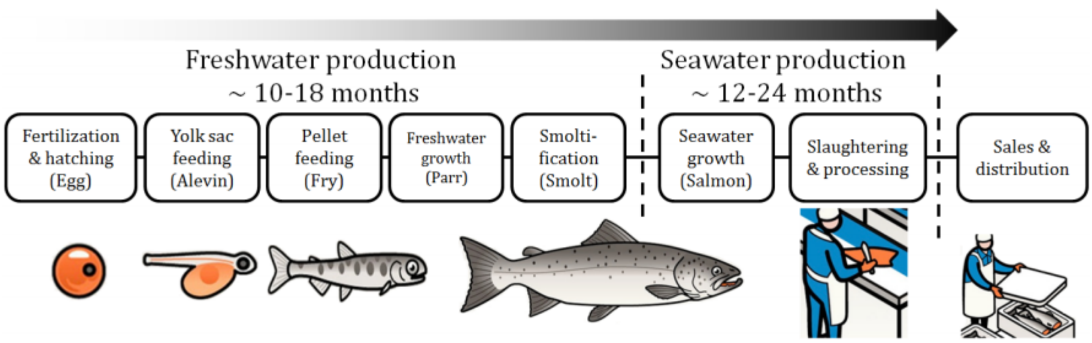


Figure 2.1: Value chain in traditional salmon farming. Source: Marine Harvest Group (2014)

Post-smolt production reduces the seawater production phase. For extended information about the traditional value chain, post-smolt production and the salmon market we refer to Hannevik et al. (2014).

2.1 Risk in traditional salmon farming

To understand the motivation behind post-smolt production it is important to look at the risk factors inherent in traditional salmon farming. In the following we distinguish between production risk and price risk. For further elaboration on risk in salmon farming we refer to Tveteraas (1999).

Production risk

- **Sea lice:** Sea lice are small parasites that live naturally in the top layer of the sea water. They stick to the salmon, and feed from its skin and flesh. The wounds may become infected by bacterias or viruses in the water, or from the lice itself. Salmon farming companies have different methods of cleaning the fish for lice, most of which are costly and involve the use of chemicals that affect the ecosystem around the sea cages. Extensive use of chemicals has also proven to lead to resistant lice. Handling sea lice is currently one of the main challenges of the salmon farming industry. It is subject to substantial investments in R&D with the objective of developing more efficient and environment friendly techniques for cleaning the fish. An additional concern is the spreading of lice to wild salmon in close proximity to facilities.
- **Diseases:** Diseases that spread via the water or via sea lice pose a big risk to farmers, and may wipe out large quantities of fish at a time. To reduce the infection risk it is necessary to control fish density, and keep a safe distance between localities.
- **Escapes:** Holes in the net pens may cause large quantities of salmon to escape. This has an impact on profitability in two ways: the farmer loses live stock and thus income, and additionally is required by the authorities to undertake measures to prevent the farmed salmon from spreading to the natural habitats of the threatened wild salmon.
- **Water temperature:** The salmon is a cold-blooded animal, and therefore the growth rate of the fish is heavily influenced by the water temperature. Temperature can also affect fish health - disease risk increases with the temperature, and temperatures below 0 °C can cause mass mortality.
- **Water quality:** The oxygen and salinity levels in the water, as well as numerous other parameters can impact fish growth. These values may vary along with the

exchange rate of the water in the sea cage.

Price risk

The price of fresh salmon is determined by a number of different factors, such as supply and demand, globalisation of the market, presence of forward contracts reducing the amount of salmon available on the spot market, and fish quality. Prices are highly volatile, and impose a high level of uncertainty to all parties involved in the market (Marine Harvest Group, 2014).

Fresh salmon can be bought through both bilateral contracts, or marketplaces such as the Oslo Stock Exchange owned Fish Pool. Fish Pool was established as late as in 2007, hence there is not enough historical price data to make accurate predictions of the price development. Dixit and Pindyck (1994) for example argue that 30-40 years of price data is required in order to draw conclusions about the structure of the price dynamics.

2.2 Regulations of salmon aquaculture

The farming of fish, shellfish and other seafood is strictly regulated by the authorities, with the main objective of ensuring sustainable operations. Conditions for obtaining a farming license and matters related to the use of land and the environment are regulated by the Aquaculture Act. In addition, the Animal Welfare Act ensures the welfare of the fish. In this section we focus on the regulations that support the prospect of post-smolt production.

- **Farming licenses:** The Norwegian Ministry of Trade, Industry and Fisheries awards licenses to companies allowing them to produce salmon on a commercial scale. The number of licenses available is limited (959 in 2013 (Marine Harvest Group, 2014)). Each license gives the right to keep a certain maximum volume of fish at sea at all times, referred to as the *maximum allowed biomass (MAB)*. Currently the limit is 780 tonnes per license. Production in closed systems currently has no similar regulations.
- **Sea lice:** In temperatures equal to or above 4 °C, the salmon farmer is legally required to count the number of sea lice per fish every seven days. Should the number exceed 0.5 adult female lice on average per fish, measures such as chemical treatment, or so-

called delicing, need to be adopted. At worst the fish would need to be prematurely harvested.

- **Fallowing localities:** After each production cycle, every production locality must be emptied and fallowed for at least two months before starting a new cycle. This is in order to minimise the infection risk between batches.

For further details on the regulations of salmon farming and aquaculture we refer to the Aquaculture Act and the Animal Welfare Act.[1]

2.3 Post-smolt production - the future of salmon farming?

The growth in the production of farmed salmon is diminishing, whereas demand is increasing (Marine Harvest Group, 2014). The market imbalance does not originate from production reluctance among the farming companies, but is a matter of regulations. Until recently, post-smolt production was not an option as government regulations stated that hatchery-reared salmon should not have an individual weight exceeding 250 grams before being set into traditional sea cages. As of 2012 however, the Ministry of Fisheries can award holders of hatchery permits licenses to produce smolt with an individual weight of up to 1000 grams in closed or semi-closed tanks on land or in the sea. This makes post-smolt production possible, and is by many industry actors and researchers highlighted as a possible solution to the market imbalance problem.

Post-smolt production has two main benefits: First, the smolt is allowed to grow larger in a protected environment before being set into traditional sea cages. This is beneficial in several ways. The time spent in seawater is shortened by approximately six months (from 18 to 12 months) (Berglihn, 2015). This leads to a reduction in the accumulated mortality, as the seawater is the most lice and disease prone environment in the production cycle. In traditional salmon farming, as much as 20 percent of the fish die before reaching slaughtering weight. In comparison, industry majors claim to have achieved mortality rates of

[1] See www.lovdata.no/dokument/NL/lov/2005-06-17-79 and www.lovdata.no/dokument/NL/lov/2009-06-19-97

1-2 percent under optimal conditions in post-smolt facilities. Given the regulations on sea lice (see Section 2.2), the farming companies could expect to save at least one delicing per production cycle, which according to Norwegian company SalMar can cost around NOK 300,000 per cage. Additionally, more time would be available for fallowing, which means less risk of vertical transmission of pathogens (Kverneland, 2011). Keeping the fish in a closed environment also eliminates the risk of escapes during the first phase of the production cycle. Finally, a higher growth rate and generally better fish welfare can be achieved by maintaining an optimal water quality.

Second, in order to maximise production efficiency and profits, salmon farmers strive to maximise their MAB utilisation. Currently post-smolt production represents a large potential for improvement of the utilisation. To exemplify this we consider a hypothetical scenario in which a company has one production license and holds 156,000 salmon at sea with an individual weight of 5 kg, amounting to a total of 780 tonnes, i.e. the MAB for one license. When the salmon is slaughtered, it is replaced by 156,000 smolt with an individual weight of 150 grams. The company's total biomass standing at sea would then be only 23.4 tonnes, hence there would be a significant decline in the total standing biomass. This decline would however be necessary to let the smolt reach an individual weight of 5 kg without the MAB being exceeded. However, by replacing the 5 kg salmon by post-smolt with an individual weight of e.g. 400 grams instead of the 150 grams smolt, the decline would be reduced.[2] Note again that this is a hypothetical example. In reality the drop is not as significant, as not all the fish is slaughtered simultaneously. Figure 2.2 illustrates the potential for better MAB utilisation.

[2] A discussion of the optimal post-smolt weight will be conducted in Chapter 6

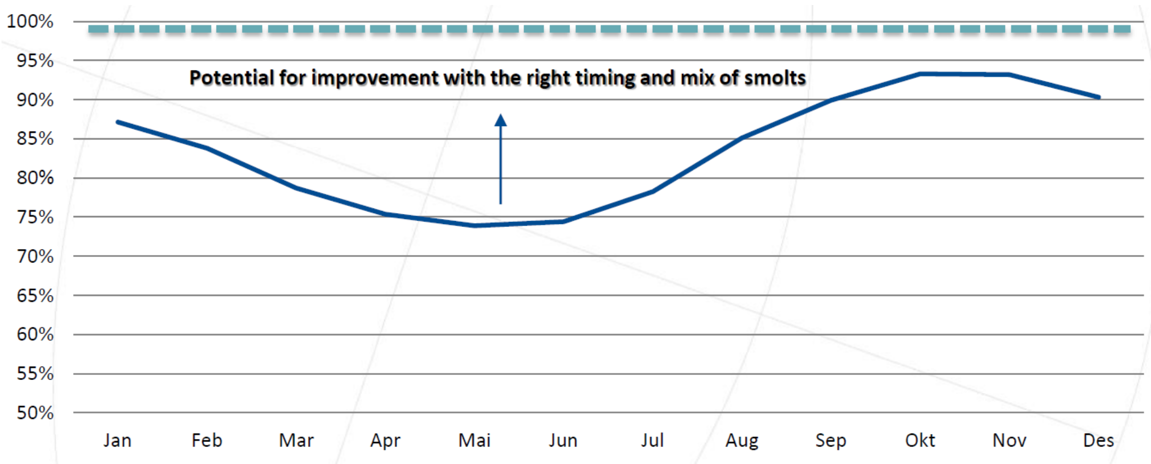


Figure 2.2: Average MAB utilisation in Norway 2010-2014. The solid line represents the average biomass standing at sea relative to the MAB. Source: Nordhammer (2015)

Even though there are strong arguments for post-smolt production in terms of reducing production risk and improving the production efficiency given by the MAB utilisation, it still involves a lot of technological uncertainty that is not inherent in traditional salmon farming. The technology used in post-smolt production is in a development phase, and many of the benefits are expected rather than certain. The performance and reliability of the current technology is for instance yet to be proved on a large scale. Additionally the acquiring cost of the equipment is subject to change. Finally, it is more expensive to produce salmon in closed systems compared to traditional sea cages.

Currently, there are two competing post-smolt production technologies. After discussing with industry representatives at the conference "Smolt production in the future" arranged by Nofima[3] we have the impression that Recirculating Aquaculture Systems (RAS) is the industry's preferred technology, as opposed to flow through systems. There are several suppliers of RAS technology, such as AKVA group, Kruger Kaldnes and Billund Aquaculture, creating high competition for the few available contracts in the market.

[3]Nofima is one of the largest institutes for applied research within the fields of fisheries, aquaculture and food research in Europe.

Recirculating Aquaculture Systems are closed-loop production systems for land-based fish farming (Figure 2.3 illustrates a typical RAS). The system recycles about 99.5 percent of the water in the system. This enables large-scale fish farming on land with minimal water usage. The main advantage is the ability to maintain optimal water quality with less effort than if the water was not recycled. Ensuring high water quality is beneficial in terms of increased growth. The main disadvantage is that the technology is fairly new, and yet to be proven on a commercial scale. Therefore there is high uncertainty related to the performance of the current technology as well as the introduction of any new and improved technology.

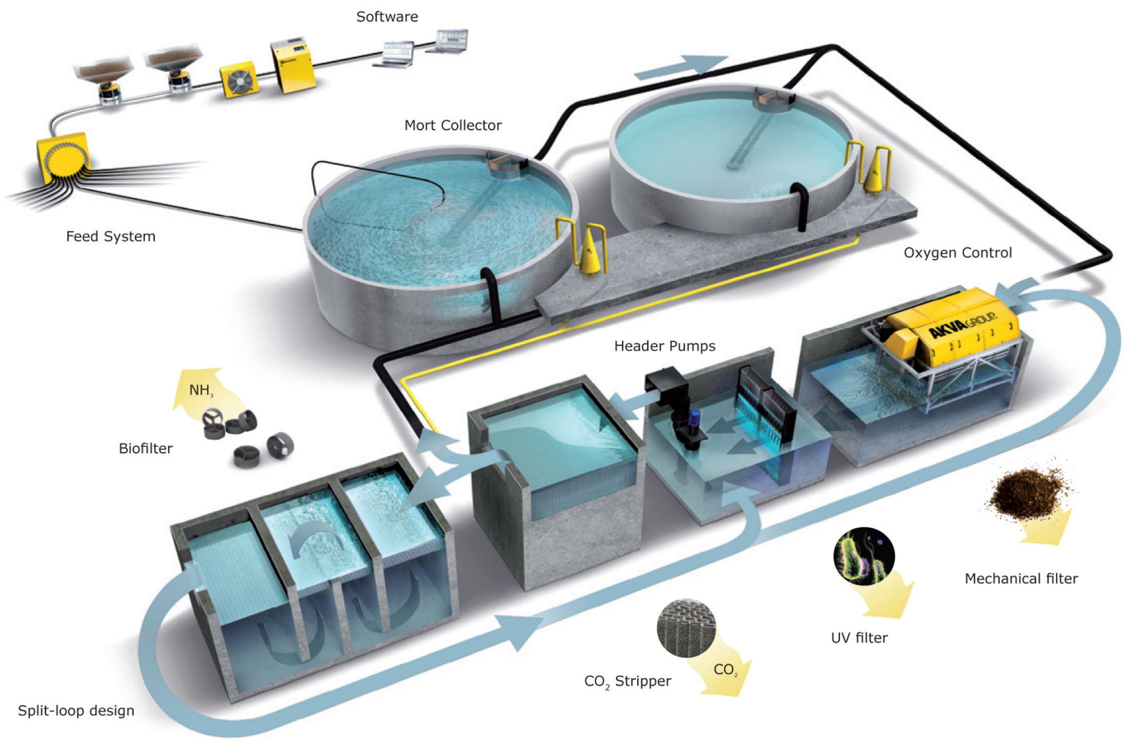


Figure 2.3: Recirculation Aquaculture System by AKVA group

Chapter 3

Related literature

In this chapter we compare the traditional DCF and the real options approach based on relevant academic literature. Additionally we give an overview of work related to multi-factor real options modelling and technological uncertainty.

3.1 Comparison of valuation methods

In this section we elaborate on the DCF and the real options approach, before comparing them based on relevant literature.

Discounted cash flow analysis

The traditional tool applied to capital budgeting is the DCF method. The DCF analysis is a static approach, i.e. the underlying assumption is that the option to invest is a now-or-never opportunity. The DCF method suggests investing if a project has a positive net present value (NPV), i.e. if the discounted expected cash flows are larger than the investment cost. The following formula displays the mathematics of a DCF valuation:

$$NPV = \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t} - K_0,$$

where CF_t is the cash flow at time t, r is the discount rate, K_0 is the initial investment cost and T is the time of the final cash flow. The company's key value drivers, such as price, quantity and gross profit should be explicitly forecast for all the years the annual cash flows are projected. However, when exceeding a certain number of years, all estimations of value drivers become imprecise. Therefore a perpetuity-based formula is applied to determine the continuation value of the investment. The continuation value is calculated following the principles of a geometric series in perpetuity:

$$Continuation\ value_T = \frac{CF_{T+1}}{r-g},$$

where g denotes the expected long-term growth rate of the firm's cash flows. The total value of the business is then simply the sum of the annual cash flows and the continuation value discounted to the present:

$$\begin{aligned} Value\ of\ operations &= \text{PV(CF during explicit forecast period)} + \text{PV(Continuation Value)} \\ &= \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t} + \frac{1}{(1+r)^T}\frac{CF_{T+1}}{r-g}. \end{aligned}$$

In our case study the intrinsic value of the option is treated as an approximate to the net present value given by a DCF analysis with no explicitly forecast cash flows. This entails that we use the continuation value with perpetuity growth from year one. Mun (2006) summarises the main advantages of the DCF method:

- Clear consistent decision criteria for all projects.
- Same results regardless of risk preferences of investors.
- Simple to explain to management: "If benefits outweigh the costs, do it!"

Under traditional investment methods such as DCF, uncertainty is treated as a negative factor (Carayannis and Sipp, 2010). Higher uncertainty leads to a higher discount rate, which in turn reduces the present value of the investment. In reality most of the projects or investments corporate managers face are naturally associated with high uncertainty. Yet, these projects can represent good investment opportunities. This can become more apparent when the projects are evaluated using the real options approach.

Real options analysis

A financial option is a contract that gives the buyer the right, but not the obligation, to buy or sell the underlying asset at a specific price on or before a specific date. Similarly, a real option is the right, but not the obligation, to undertake a certain investment decision on or before a specified date (Myers (1977); Bowman and Hurry (1993)). A real option is therefore not a financial instrument, but an actual option the decision maker faces (Dixit and Pindyck, 1994). The option to delay or abandon an investment opportunity are examples of real options. Like its financial counterpart, the main value driver of a real option, is uncertainty. As opposed to traditional valuation methods where uncertainty is treated solely as a negative factor, real options valuation captures both the upside and the downside potential of uncertainty. The uncertainty is determined by the volatility of the option's underlying and the time to maturity. However, unlike for financial options, there is usually no straightforward options formula one can apply to evaluate real options.

Empirical research indicates that the DCF method result in inaccurate valuations when there is high uncertainty related to the value of an investment. This inaccuracy can be reduced by using real options valuation, however critics pose several challenges associated with its application. In the following we discuss the methods based on relevant literature.

Choosing valuation method

Van Putten and MacMillan (2004) presents the notion of an "option zone" in projects with uncertainty (see Figure 3.1), which is described as the area where the DCF value is modestly positive. This "zone" represents an area where application of real options analysis is particularly suited to determine whether projects are attractive investment opportunities, or should be discarded.

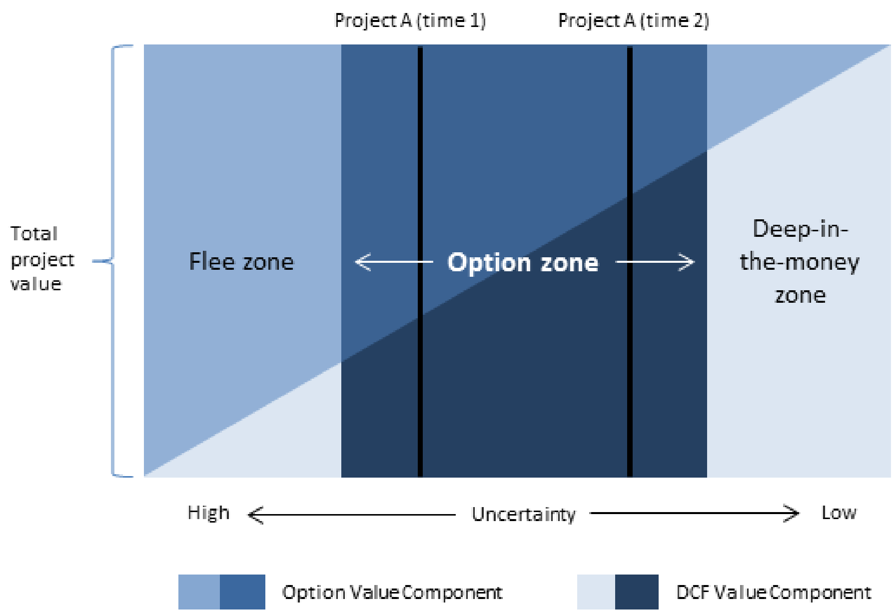


Figure 3.1: The "option zone": In time 1 there is high uncertainty, hence the option value component of Project A dominates. In time 2 there is less uncertainty, hence the DCF value component dominates.

Figure 3.1 presents the project value as consisting of two parts: an option value component, and a DCF value component. For a project with even a modest degree of uncertainty, as much as 50 percent of the total project value can stem from the option value component. As the salmon farming company in our case is considering an investment under uncertainty w.r.t. both profits and technology, the Van Putten and MacMillan (2004) model makes a strong argument for the use of a real options approach. Koller et al. (2010) share this idea of the "option zone", by claiming that including flexibility in a project valuation is most important when the project's NPV is close to zero. Koller et al. (2010) also choose to distinguish between the flexibility and uncertainty inherent in a real option. They claim that in order to exploit the value of uncertainty, one needs managerial flexibility, i.e. ability to respond to new information. This is illustrated in Figure 8.1

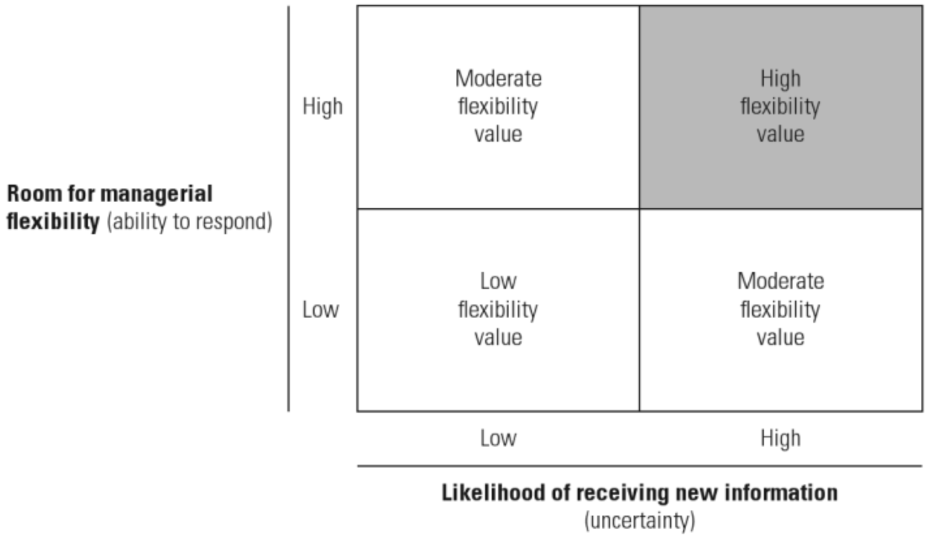


Figure 3.2: The value of a real option is high when uncertainty is high, and the management have the ability to respond to new information

We acknowledge that the advocates of real options valuation make compelling arguments for its use. Not only does it provide more accurate project value than traditional methods when there is uncertainty, but it also tells managers how to act in the future in response to new information. But, however much praised by academics, the real options approach has not had a breakthrough in corporate valuation (Graham and Harvey, 2001). Lander and Pinches (1998) propose three possible reasons:

1. The required modeling assumptions are often violated in a practical real options application,
2. The necessary additional assumptions required for mathematical tractability limit the scope of applicability,
3. The types of models currently used are not well known or understood by corporate managers and practitioners.

Van Putten and MacMillan (2004) state several additional challenges for applying current option valuation models to real projects. One problem is that it is hard to find good proxies for the input variables the real options models require. One example is the estimation of volatility. For financial options it is quite straightforward to calculate volatility using the historical prices of the underlying assets. For real projects however, data about historical prices rarely exist, and consequently calculating a realistic volatility is difficult.

Bodén and Åhlén (2007) perform a study of implementation impediments of ROV through literature research and interviews. They find that there does not seem to be a common standard for which assumptions can be made when applying ROV. Moreover, there exists little knowledge on the accuracy of ROV under different modeling assumptions. This fact seem to decrease the willingness of real-world practitioners to adopt the real options framework. Furthermore, Bodén and Åhlén (2007) argue that companies are not searching for capital budgeting methods that value flexibility to such a large extent as many academics propose.

In terms of choosing valuation method Van Putten and MacMillan (2004) and Luehrman (1998) emphasize that DCF and real options analysis are not mutually exclusive approaches to valuation. They conclude that real options valuation should be used as a complement to, and not a substitute for DCF analysis. Guthrie (2009) takes a firmer stance than Luehrman (1998) and Van Putten and MacMillan (2004) by claiming that a real options approach should replace a static DCF approach fully in cases where the decision-making process will be carried out over time.

3.2 Technological uncertainty and multi-factor modelling

We consider an investment in a facility where there is uncertainty about the underlying technology. Hence it is important to understand how technological uncertainty can affect investment decisions. As we wish to model profit uncertainty in addition to technological uncertainty, we also need to develop an understanding of multi-factor modelling. In this section we therefore review literature written on real options analysis under technological uncertainty and multi-factor modelling.

Farzin et al. (1998) use real options analysis to determine the optimal timing of technology adoption for a competitive firm which faces a stochastic innovation process with uncertainty in both the arrival rate and the magnitude of technological innovations. They argue that the technology adopter must weigh the cost of investing prematurely given the possible arrival of further improved technology, against the opportunity cost of foregone cash flows incurred while waiting for improvements. They show that even in the absence of other kinds of uncertainty, e.g. uncertainty about market conditions, a firm's optimal timing of adoption is greatly influenced by technological uncertainties. This supports the prospect of including technology as an important source of uncertainty in our real options analysis.

The random arrival of technological innovations has been modelled in different ways in the literature. Grenadier and Weiss (1997) assume a variable $X(t)$, denoting the state of technological process, following a geometric Brownian motion. When this variable rises to an upper boundary X_h, the innovation arrives and is ready for adoption. By observing the difference between the current level of the Brownian motion process and the upper boundary, the firm has an idea about how long it will be until the new technology arrives. In Huisman and Kort (2004) the arrival of a new technology is assumed to follow a Poisson process so that at every point in time the probability that a new technology arrives is the same. Hence, their approach assumes that the firms have no information about the progress made within the research and development process of the technology. We adopt this modelling approach, and thus the assumption that the probability of an arrival is constant throughout the period considered.

With regards to the impact of technological progress, Doraszelski (2004) introduces a distinction between innovations and improvements. Innovations are technological breakthroughs, while improvements denote the engineering refinements following an innovation. He uses three scenarios for the arrival of technological improvements: diminishing, learning and time invariant. Diminishing describes the situation where improvements arrive at a higher rate just after an innovation as the easiest fixes are done first. Learning describes the case where one needs to understand the technology introduced by the innovation, before one can make improvements to it. In a time invariant scenario the arrival rate of improvements is not affected by the time since last innovation. As the effect of technological improvements add up, Doraszelski (2004) find that firms do not necessarily wait for a technological breakthrough, but rather have an incentive to delay the adaption of a new technology until it is sufficiently advanced. We adopt the convention of distinguishing between innovations and improvements by modelling technological improvement as a steady drift due to 'learning by doing' in the industry, and technological innovation as a time invariant Poisson jump process.

The development of multi-factor real options models is at an early stage. Therefore the amount of relevant literature is limited. Bosetti and Tomberlin (2004) present a two-factor real options model of fishing fleet dynamics, where the sources of uncertainty are salmon price and quantity of salmon caught. The model extends the work of Tomberlin (2001), who only considers one stochastic process. The results show that this extension significantly improves the predictive power of the model. This indicates that including several factors of uncertainty in real options models contributes to improve the explanatory power of the models. We contribute to this work by adding technological uncertainty in the form of a single Poisson arrival, and by improving the solution procedure through the use of similarity methods.

de Magalhaes Ozorio et al. (2013) consider the issue of choosing stochastic processes in multi-factor real options models. Since most real options can be exercised like American options, they claim that multi-factor models have to be solved numerically, e.g. by using finite difference or tree methods. Furthermore, they state that in the case of models with more than two factors, one has to resort to special Monte Carlo approaches such as the least square Monte Carlo method suggested in Longstaff and Schwartz (2001). This is the approach taken by Jafarizadeh et al. (2012) who consider an optimal abandonment of an oil field, where the oil price follows a two-factor process.

Murto (2007) examines the conjoined effects of technological and revenue uncertainty on the timing of the investment decision. Technological uncertainty is assumed exogenous to the firm and is represented by a Poisson process where an arrival is characterised by a reduction of the investment cost. Revenues are assumed to follow a geometric Brownian motion. While revenues are subject to both up and down moves, a characteristic feature of technological progress is that it only moves in one direction. To derive analytical solutions, Murto (2007) only solves the model for three special cases: (1) Deterministic price process ($\sigma = 0$), (2) deterministic technological progress ($\lambda \to \infty$ and $\phi \to 1$) and (3) full collapse of the investment cost ($\phi = 0$). Doing so he is able to give some intuition on how revenue and technology related uncertainty affects the problem, when the relationship between the two factors affecting technological uncertainty, arrival rate λ and investment cost reduction factor ϕ, is fixed. However, he is not able to give intuition on the effect of independent changes in λ and ϕ, nor is he able to determine the option value $f(p)$ or the investment threshold p^* for arbitrary input parameters.

The paper of Himpler and Madlener (2014) is closely based on the work of Murto (2007). They propose a two-factor model to find the optimal timing of a wind farm repowering. They consider uncertainty in profit and investment cost, both assumed to follow geometric Brownian motions. The investment cost is subject to technological uncertainty and is assumed to have negative drift due to a continuous learning rate of using new equipment. Himpler and Madlener (2014) make these assumptions in order to achieve mathematical tractability, as they claim that Murto (2007)'s model cannot be solved analytically. We extend Himpler and Madlener (2014) and Murto (2007) by providing a general numerical procedure that allows us to solve the optimal stopping problem for the entire solution space, where we follow Murto (2007) and Huisman and Kort (2004) by modelling technological uncertainty as a Poisson jump process. In addition we apply the model to a specific case study of a post-smolt investment.

Adkins and Paxson (2013) and Chronopoulos et al. (2015) consider investments in power generation plants under policy uncertainty. More specifically the uncertainty is related to the possible sudden provision or retraction of a constant government subsidy. Adkins and Paxson (2013) consider both a subsidy in the form of a cash payment proportional to the quantity produced and the revenues, whereas Chronopoulos et al. (2015) consider a subsidy in the form of a cash payment proportional to the revenues of the firm. Adkins and Paxson (2013) assumes that price and quantity follow geometric Brownian motions, while the introduction of the subsidy follows a Poisson jump process. Chronopoulos et al. (2015) only consider uncertainty in revenues. Both papers assume that the firm will only get the benefit of the subsidy if the decision to invest is made after its arrival. However, their solution procedures differ. Adkins and Paxson (2013) solve their model by value matching and smooth pasting between two value regions, allowing them to derive a fully analytical solution. Chronopoulos et al. (2015) solve their model by value matching and smooth pasting between three value regions, thus they can only derive a quasi-analytical solution. After examining the arguments and mathematics of both papers, we are convinced that

Chronopoulos et al. (2015) have chosen the appropriate solution approach. Having found that the sudden provision of a subsidy resembles the arrival of a technological innovation, we extend Adkins and Paxson (2013) and Chronopoulos et al. (2015) by adapting and applying their frameworks to the post-smolt facility investment case, allowing for the salmon farming company to get the benefit of the innovation regardless of when the investment is made.

We choose not to elaborate on literature treating uncertainty in salmon farming, as the majority of available papers focus on the biology rather than the economics of salmon farming. The few available economic papers are based either on traditional capital budgeting methods or optimisation of operations. For further information we refer to the work of Forsberg and Guttormsen (2006), Langan and Toftøy (2011), Hæreid (2011) and Hannevik et al. (2014).

Chapter 4

Mathematical background

In this chapter we provide a brief explanation of some of the mathematical methods that form the basis of the models presented in Chapter 5.

4.1 Dynamic programming

There are two main approaches to solving real options problems: Dynamic programming and contingent claims analysis. The two methods are quite similar, but differ in the discount rates that firms use to value future cash flows. However, Dixit and Pindyck (1994) show that the methods give the same result. In this thesis we use dynamic programming, which is the method used in the majority of the academic real options literature. Dynamic programming breaks a whole series of decisions into two components: the immediate decision, and a valuation function that captures the consequence of all subsequent decisions. At each decision point t, the set of available decisions is denoted by u_t. A firm should choose u_t to maximise the expected value of their investment opportunity $F_t(x_t)$. The value of $F_t(x_t)$ is equal to the sum of the immediate profit flow $\pi(x_t, u_{,t})$ and the discounted expected value of $F(x_{t+1})$

$$F_t(x_t) = max_{u_t} \left\{ \pi(x_t, u_{,t}) + \frac{1}{1+r} \mathcal{E}_t[F_{t+1}(x_{t+1})] \right\}$$

The rationale behind this equation is stated in Bellman's principle of optimality: *An optimal policy has the property that, whatever the initial action, the remaining choices constitutes an optimal policy with respect to the sub problem starting at the state that results from the initial actions.*

We want to determine the conditions under which an investment in a post-smolt facility should be undertaken. In mathematics and real options literature, this is known as an optimal stopping problem. In any decision point there is a binary choice; invest and receive the corresponding termination payoff (stopping), or defer the investment and face a similar binary choice at the next decision point (continuing). We let $\Omega(x_t)$ denote the termination payoff, and the Bellman equation becomes:

$$F_t(x_t) = max\{\Omega(x_t), \pi(x_t) + \frac{1}{1+r}\mathcal{E}_t[F_{t+1}(x_{t+1})]\}$$

If the investment under consideration has a finite time horizon T, the optimal decision at each point can be found by starting at the end and working backwards: $F_{T-1}(x_{t-1}) = max\left\{\Omega(x_{t-1}), \pi(x_{t-1}) + \frac{1}{1+r}\mathcal{E}_{T-1}[\Omega_T(x_T)]\right\}$. Else, if the considered investment is a perpetual option, we do not have a finite time horizon T and a known final value function from which we can work backwards. Instead we end up with a recursive structure that is independent of t: $F(x) = max\left\{\Omega(x), \pi(x) + \frac{1}{1+r}\mathcal{E}[F(x^{'})|x]\right\}$. Both models presented in Chapter 5 consider perpetual options.

By arguments of continuity one can impose what is known as a value matching condition. The value matching condition links the continuation region with the stopping region, and states that for a certain value x^* we are indifferent between taking the termination payoff $\Omega(x)$ or deferring the investment:

$$F(x^*) = \Omega(x^*)$$

To find the value of the investment opportunity $F(x)$ along with the optimal investment threshold x^*, we need to introduce a second condition known as the smooth pasting condition. The smooth pasting condition ensures that the continuation and stopping region meets tangentially in x^*:

$$F_x(x^*) = \Omega_x(x^*)$$

4.2 Itô's lemma

In order to describe how the value of the investment changes in continuous time, we can use It$\hat{o}$'s lemma. It$\hat{o}$'s lemma enables integration and differentiation of functions of It$\hat{o}$ processes. An It$\hat{o}$ process is a process where a variable x changes according to the following expression: $dx = a(x,t)dt + b(x,t)dz$, where dz is the increment of a Wiener process. Both $a(x,t)$ and $b(x,t)$ are known functions. It$\hat{o}$'s lemma is derived through the Taylor expansion of $F(x,t)$, neglecting higher order terms:

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2$$

$$dF = \left[\frac{\partial F}{\partial t} + a(x,t)\frac{\partial F}{\partial x} + \frac{1}{2}b(x,t)\frac{\partial^2 F}{\partial x^2}\right]dt + b(x,t)\frac{\partial F}{\partial x}dz$$

In both models presented in Chapter 5 the option value is dependent on two It$\hat{o}$ processes, hence we need to use the multivariate It$\hat{o}$'s lemma. It$\hat{o}$'s lemma can be generalised and applied to m It$\hat{o}$ processes: $dx_i = a_i(x_1,...,x_m,t)dt + b_i(x_1,...,x_m,t)dz_i$, with $\mathcal{E}[dz_i dz_j] = p_{ij}dt$

$$dF = \frac{\partial F}{\partial t}dt + \sum_i \frac{\partial F}{\partial x_i}dx + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 F}{\partial x_i \partial x_j}dx_i dx_j$$

Substituting for dx_i, the expanded form becomes

$$\begin{aligned} dF =& \Bigg[\frac{\partial F}{\partial t} + \sum_i a_i(x_1,...,x_m,t)\frac{\partial F}{\partial x_i} + \frac{1}{2}\sum_i b_i^2(x_1,...,x_m,t)\frac{\partial^2 F}{\partial x_i^2} \\ &+ \frac{1}{2}\sum_{i\neq j}\rho_{ij}b_i(x_1,...,x_m,t)b_j(x_1,...,x_m,t)\frac{\partial^2 F}{\partial x_i^2}\Bigg]dt \\ &+ \sum_i b_i(x_1,...,x_m,t)\frac{\partial F}{\partial x_i}dz_i \end{aligned}$$

4.3 Optimal investment timing

We now illustrate how to use dynamic programming and Itô's lemma to find the optimal investment threshold V^*, by considering a simplified investment case with value $F(V) = V - I$. The models presented in Chapter 5 follow the same reasoning and structure, but the underlying processes are more complex. V is the project value, i.e. the present value of the future cash flows, and I is the investment cost, which is assumed to be constant. V is assumed to evolve according to a geometric Brownian motion (GBM) given by:

$$dV = \alpha V dt + \sigma V dz,$$

where α represent the drift term, and σ the volatility of the project value. The investment opportunity can be considered a perpetual American option, i.e it can be exercised at any point in time. The question which then arises is: *When is it optimal to invest?* Mathematically the problem can be formulated as follows:

$$max_T \mathcal{E}_{V_0}[(V_T - I)e^{-rT}],$$

where r is the discount rate, and T is the time at which the investment is made. $V_T - I$ is the intrinsic value of the option at time T, i.e. it represents the net present value of the project at time T. In Chapter 7 we compare real options value and traditional DCF value by looking at the difference between the option value $F(V)$ and the option's intrinsic value $V(R) - I$. We require that $r > \alpha$, otherwise it would always be better to delay the investment. The investment opportunity $F(V)$ yields no cash flows before the investment is undertaken, i.e. $\pi(t < T) = 0$. Hence, the return for holding the investment opportunity rF is equal to its capital appreciation dF. This gives us the following Bellman equation:

$$rFdt = \mathcal{E}[dF]. \tag{4.1}$$

dF can be expanded using Itô's lemma. As $F(V)$ is not directly dependent on t, the dt term vanishes and we get:

$$\begin{aligned}\mathcal{E}[dF] &= \mathcal{E}[F'(V)dV + \frac{1}{2}F''(V)(dV)^2] \\ &= \mathcal{E}[\alpha V F'(V)dt + \sigma V F'(V)dz + \frac{1}{2}F''(V)(\alpha^2V^2dt^2 + 2\alpha\sigma V^2dzdt + \sigma^2V^2dz^2)].\end{aligned}$$

Using the fact that $\mathcal{E}(dz) = 0$, $\mathcal{E}(dz^2) = dt$, and eliminating higher order terms of dt, the expression for dF can be substituted back into Eq. (4.1):

$$rFdt = \alpha V F'(V)dt + \frac{1}{2}\sigma^2V^2F''(V)dt \quad \text{for} \quad V \leq V^*.$$

Dividing all terms by dt, and rearranging, we get the following differential equation:

$$\frac{1}{2}\sigma^2V^2F''(V) + \alpha V F'(V) - rF = 0 \quad \text{for} \quad V \leq V^*. \tag{4.2}$$

This differential equation must satisfy the following boundary conditions

1. $F(0) = 0$,
2. $F(V^*) = V^* - I$,
3. $F'(V^*) = 1$.

Condition 1 arises from the fact that V follows a GBM, hence if $V = 0$, $F(V)$ will stay at zero forever. Condition 2 is the value-matching condition. It says that when the firm chooses to invest, it will receive a payoff of $V^* - I$. Condition 3 is the smooth-pasting condition. It ensures that $F(V)$ is continuous and smooth at the exercise point V^*. Solving the differential equation with respect to the boundary condition yields the following solution (see Dixit and Pindyck (1994) for derivation):

$$F(V) = AV^{\beta_1},$$

$$V^* = \frac{\beta_1}{\beta_1 - 1} I,$$

$$A = \frac{V^* - I}{(V^*)^{\beta_1}},$$

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(\frac{\alpha}{\sigma^2} - \frac{1}{2})^2 + \frac{2\rho}{\sigma^2}} > 1.$$

Unlike Eq. (4.2), not all differential equations can be solved analytically. In the next two sections, we present numerical solution techniques necessary for solving the models presented in Chapter 5.

4.4 Solving differential equations numerically

In order to solve the model presented in Section 5.1 we need to solve a differential equation numerically. To choose an appropriate numerical algorithm for solving a differential equation, one has to study the characteristics of the equation. One important characteristic is "stiffness". For stiff equations, implicit methods often provide better algorithm stability[1] than explicit methods. On the other hand, explicit methods are in general easier to implement and computationally faster. We illustrate the difference between implicit and explicit methods by the two versions of the Euler algorithm, where h denotes the step size and $f(x_n, y_n)$ denotes the derivative of the function $y(x)$:

[1] In a stable algorithm, small changes in initial data cause only small changes in the final result.

$$\text{Explicit Euler method:} \quad y_{n+1} = y_n + hf(x_n, y_n), \tag{4.3}$$

$$\text{Implicit Euler method:} \quad y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}). \tag{4.4}$$

In the explicit method given by Eq. (4.3), y_{n+1} is computed based on the known values y_n and $f(x_n, y_n)$. In the implicit Euler method given by Eq. (4.4), the right hand side takes the unknown derivative $f(x_{n+1}, y_{n+1})$ as input. Hence Eq. (4.4) must be solved as an equation for y_{n+1}. Euler's method is a part of a family of numerical methods known as *Finite difference methods*. Like Euler's method they are based upon the principle of replacing the derivatives in the equation by differential quotients. The differential quotients are found through Taylor expansion of $f(x)$ at the neighbouring points determined by the chosen grid spacing h:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f^{(3)}(\varepsilon). \tag{4.5}$$

The most common approximations to use are

$$\textit{Forward difference}: f'(x) \approx \frac{f(x+h) - f(x)}{h}, \tag{4.6}$$

$$\textit{Backward difference}: f'(x) \approx \frac{f(x) - f(x-h)}{h}, \tag{4.7}$$

$$\textit{Central difference}: f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}. \tag{4.8}$$

4.5 Solving systems of algebraic equations numerically

Calculating investment thresholds by applying value matching and smooth pasting conditions often require finding n unknowns $x_1, x_2, ..., x_n$, satisfying a set of n nonlinear equations:

$$
\begin{aligned}
f_1(x_1, x_2, ..., x_n) &= 0 \\
f_2(x_1, x_2, ..., x_n) &= 0 \\
&\vdots \\
f_n(x_1, x_2, ..., x_n) &= 0
\end{aligned}
$$

This is the case for the model presented in Section 5.2. Several mathematical software packages such as MATLAB have built in support for performing such numerical procedures. In order to be able to adopt and modify the advanced built in methods, it is important to understand the basic principles these algorithms are based upon.

To solve an equation of the form $f(x) = 0$, when there is no formula for the exact solution, we can employ an approximation method based upon iteration. We start with an initial guess x_0 and compute approximations x_1,x_2... for the unknown solution iteratively. That is, we find a sequence of values $x_1, x_2, ..., x_n, x_{n+1}, ...$ such that $f(x_n)$ gets closer and closer to 0 as n increases. Methods with *global convergence* will produce the correct solution regardless of the initial guess x_0, while methods with *local convergence* require an initial guess near the actual solution to $f(x) = 0$. Since methods with *global convergence* tend to have a faster convergence rate (quadratic or higher) than methods with *local convergence*, one often first use a *"global"* method to find adequate starting values as input to a *"local"* method.

MATLAB's built-in method for solving systems of equations is *fsolve*. It is possible to choose different underlying algorithms, but by default *fsolve* is set to use the *trust region dogleg* method. As this method has its roots in Newton's method, we include the simplistic Newton's method, which is based upon linearisation around the current guess using Taylor series:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We have now revised many of the fundamental techniques necessary for modelling real options as continuous time optimal stopping problems. In the next chapter we build on these techniques and the literature presented in Chapter 3.

Chapter 5

Model

In this chapter we present two multi-factor models for evaluating optimal investment timing in a post-smolt production facility, under profit and technology uncertainty, using a real options approach. Both models consider perpetual options, i.e. options that can be exercised any time from now until infinity. The models are parametrised in Chapter 6 and case studies based on the parameters are presented in Chapter 7.

We assume that technological evolution is exogenous to the salmon farming company considered. A technological innovation is assumed to improve the efficiency of either the production processes of the technology suppliers, or the post-smolt production equipment. We have developed two separate models in order to isolate each of the possible outcomes in combination with uncertain profits. We assume that per unit profits follow a GBM. A similar assumption is made by Himpler and Madlener (2014). The models are adapted to a small company that is assumed to only have the financial power to invest once. Tax is neglected as it has a similar effect on both net present value and real options value, and will therefore not affect our ability to compare the values. In both real options models we assume that profit generation starts instantaneously after investment, i.e. there is no time lag in setting up the facility (a similar assumption is made by for example Bernanke (1980) and Cukierman (1980)).

5.1 Multi-factor model with stochastic profit and investment cost

In this section we present a two-factor model where we isolate the effect of technological innovations improving the efficiency of the technology suppliers' production processes. The competition among the suppliers of post-smolt technology is high, which entails pressure on selling prices. Therefore we assume that a reduction of the suppliers' production costs, will lead to a lower investment cost for the salmon farming company. To model this we choose to adopt the work of Murto (2007) who proposes a model for investment timing subject to technological and revenue-related uncertainties. In his model the arrival of a technological innovation is assumed to be Poisson distributed, and to reduce the investment cost. To derive analytical solutions, Murto (2007) only considers three special cases: (1) Deterministic price process ($\sigma = 0$), (2) deterministic technological progress ($\lambda \to \infty$ and $\phi \to 1$) and (3) full collapse of the investment cost ($\phi = 0$). We extend his work by adapting it to a post-smolt investment problem, and by proposing a numerical procedure which allows us to solve the model for the entire solution space.

We consider a risk neutral salmon farming company which is considering to undertake a single irreversible investment in a post-smolt facility. The time t is continuous and infinite. The value of the investment opportunity $F(\pi, K)$ evolves according to the stochastic processes of the per unit profits denoted by π, and investment cost denoted by K. The processes are assumed to be independent Markov processes that are time-homogeneous. The time-homogeneity and Markov properties entail that the decision to wait or stop depends only on the current value of $F(\pi, K)$, not on the historical development of the process or calendar time. Furthermore, this means that the solution space can be divided into two regions; the waiting region, where it is optimal to wait, and the stopping region, where it is optimal to invest.

The unit profit at time $t \geq 0$, π_t, is stochastic and assumed to follow a GBM such that

$$d\pi = \alpha_\pi \pi dt + \sigma_\pi \pi dZ_\pi, \qquad \text{where} \qquad \mathcal{E}[\pi_t] = \pi_0 e^{\alpha_\pi t}.$$

α_π represents the drift of π, σ_π the volatility and dZ_π the increment of a standard Wiener process. The total annual profit at time t denoted by R_t is given by:

$$R_t(\pi_t) = \pi_t Q,$$

where Q is the annual production quantity, which is assumed to be constant in this model. This relationship entails that R_t follows a GBM with the same properties as π_t, such that:

$$dR = \alpha_R R dt + \sigma_R R dZ_R, \quad \text{where} \quad \mathcal{E}[R_t] = R_0 e^{\alpha_R t},$$

where $\alpha_R = \alpha_\pi$, $\sigma_R = \sigma_\pi$ and $dZ_R = dZ_\pi$. The expected discounted value of the project at time t, denoted by $V(R_t)$, is:

$$V(R_t) = \mathcal{E}_R[\int_{s=t}^{\infty} (R_s)\, e^{-r(s-t)} ds] \ ,$$

$$V(R_t) = \int_{s=t}^{\infty} \left(R_t e^{\alpha_R (s-t)} \right) e^{-r(s-t)} ds,$$

$$V(R_t) = \frac{R_t}{r - \alpha_R}.$$

The salmon farming company faces an investment cost at time t denoted by K_t, which is assumed to be a one-time, sunk cost. The arrival of an innovation reducing K_t is assumed to be Poisson distributed and always improve upon the best-available technology. Hence, K_t follows a strictly decreasing Poisson jump process given by:

$$dK_t = K_t dq_1,$$

where dq_1 is given by:

$$dq_1 = \begin{cases} 0, & \text{with probability } 1 - \lambda dt, \\ 1 - \phi, & \text{with probability } \lambda dt, \end{cases}$$

where $\phi \in [0, 1]$ is a constant reflecting the size of the investment cost reduction and λ is the Poisson intensity, i.e. the technology innovation arrival rate. ϕ is defined such that a large value indicates a small innovation, while a small ϕ indicates a large innovation. This entails that a large arrival rate, λ, and a small ϕ constitutes frequent innovations with high reducing impact on the investment cost and thus, results in the greatest option value. The investment cost at any time $t \geq 0$ is equal to:

$$K_t = K_0 * \phi^{N_t}, \quad \text{where} \quad \mathcal{E}[K_t] = K_0 e^{-\lambda t(1-\phi)}.$$

K_0 denotes the investment cost at time $t = 0$ and N_t is a Poisson random variable with mean λt counting the number of innovations.

The problem the salmon farming company faces is to choose the investment timing that maximises the expected net value of the project $V(R_t) - K_t$, and thereby the value of the investment opportunity, denoted by $F(R_t, K_t)$. This value is given by

$$F(R_\psi, K_\psi) = Sup_\psi \mathcal{E}_R[e^{-\psi r}(V(R_\psi) - K_\psi)],$$

$$F(R_\psi, K_\psi) = Sup_\psi \mathcal{E}_R[e^{-\psi r}(\frac{R_\psi}{r - \alpha_R} - K_\psi)], \tag{5.1}$$

where R_ψ and K_ψ refer to the total annual profit and the investment cost at a time ψ which represents the optimal investment timing. The company's discount rate is denoted by r. As $V(R_\psi)$ is continuous, the optimal stopping region can be expressed as a closed set Ω in the (R, K)-space. Ω, i.e. the stopping region, can be entered either by diffusion of R or by a sudden jump of K.

Since the option is perpetual, calendar time does not affect the problem and we omit the subscript "t" from here on. By examining Eq. (5.1) we observe that $F(kR, kK) = kF(R, K)$, entailing that F is homogeneous of degree one in (R, K). Thus, by setting $k = 1/K$, we get $F(R, K)/K = F(R/K, 1)$. The ratio of total annual profits to the investment cost, $\frac{R}{K}$, can be understood as a benefit-cost ratio. Both R and K are subject to stochastic development and when their ratio reaches a specified level, investment should be undertaken by the salmon farming company. We introduce a new variable p, to simplify notation of the ratio:

$$p = \frac{R}{K}. \tag{5.2}$$

The option value can then be simplified to:

$$F(R, K)/K = F(R/K, 1) = F(p, 1) = f(p), \tag{5.3}$$

$$\Downarrow$$

$$F(R, K) = Kf(p),$$

Hence the problem is reduced from two dimensions to one, simplifying the solution procedure of the model considerably. The new variable p follows the combined GBM jump process:

$$dp = p\mu dt + p\sigma dz + pdq_2,$$

where dq_2 is given by:

$$dq_2 = \begin{cases} 0, & \text{with probability } 1 - \lambda dt, \\ \frac{1}{\phi} - 1, & \text{with probability } \lambda dt. \end{cases}$$

The optimal solution to the stopping problem is defined by the threshold level p^*, which is the ratio of total annual profits R to the investment cost K signalling an economically justifiable investment. To determine this threshold, we use the fact that K drops to ϕK during the next time period of length dt with probability λdt, and get the following Bellman equation:

$$rF(R,K)dt = \mathcal{E}[dF(R,K)] + \lambda[F(R,\phi K) - F(R,K)]dt \quad \text{for} \quad (R,K) \notin \Omega.$$

We apply Itô's lemma to the first term on the right hand side of the equation, divide by dt and rearrange terms, ending up with the following partial differential equation (PDE):[1]

$$\frac{1}{2}\sigma_R^2 R^2 F_{RR}(R,K) + \alpha_R R F_R(R,K) - rF(R,K) + \lambda[F(R,\phi K) - F(R,K)] = 0. \tag{5.4}$$

Substituting the expressions for p and $f(p)$ found in Eqs. (5.2) and (5.3), we can express $F(R,K)$, $F(R,\phi K)$ and the partial derivatives of $F(R,K)$ in one dimension:

$$F(R,K) = Kf(p),$$

[1] See Appendix A for derivation.

$$F(R, \phi K) = \phi K F(\frac{R}{\phi K}, 1) = \phi K f(\frac{R}{\phi K}) = \phi K f(\frac{p}{\phi}), \tag{5.5}$$

$$F_R(R, K) = K f'(p) \frac{\partial p}{\partial R} = K f'(p) \frac{1}{K} = f'(p), \tag{5.6}$$

$$F_{RR}(R, K) = f''(p) \frac{\partial p}{\partial R} = \frac{f''(p)}{K}. \tag{5.7}$$

Inserting Eqs. (5.3), (5.5), (5.6) and (5.7) into (5.4) and dividing by K, we obtain the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma_R^2 p^2 f''(p) + \alpha_R p f'(p) - (r + \lambda) f(p) + \lambda \phi f(\frac{p}{\phi}) = 0 \quad \text{for} \quad p \leq p* . \tag{5.8}$$

The value of the investment opportunity is thus a function $F(R, K) = K f(p)$ such that $f(p)$ satisfies Eq. (5.8) whenever p is in the continuation region, i.e. lower than the threshold level, p^*. At the threshold level $F(R, K)$ must satisfy the following value matching and smooth pasting conditions:

$$F(R, K) = \frac{R}{r - \alpha_R} - K, \quad \text{when} \quad \frac{R}{K} = p^*,$$

$$F_R(R, K) = \frac{1}{r - \alpha_R}, \quad \text{when} \quad \frac{R}{K} = p^*.$$

These conditions can be rewritten in terms of f and p as:

$$f(p^*) = \frac{p^*}{r - \alpha_R} - 1, \tag{5.9}$$

$$f'(p^*) = \frac{1}{r - \alpha_R}. \tag{5.10}$$

Finally, as p follows a combined GBM-jump process, we know that if p becomes equal to zero, it will stay at zero forever. Hence, p=0 ($R = 0$) is an absorbing barrier, which constitutes the initial boundary condition:

$$f(0) = 0. \tag{5.11}$$

To arrive at a well-defined problem, we need to determine the expression for the stopping region, i.e for $p > p^*$. The value of the investment opportunity expressed in terms of $f(p)$ and p for this region is:

$$f(p) = \frac{p}{r - \alpha_R} - 1, \quad \forall \quad p \geq p^*.$$

To solve the problem we need to determine the function $f(p)$ and the threshold value p^* such that $f(p)$ satisfies Eq. (5.8) when $p < p^*$. This problem cannot be solved analytically (Murto, 2007) so we construct a numerical procedure to find the solution. We can use either an explicit or an implicit method (see Section 4.4). An algorithm based on an explicit Euler scheme produces correct results, but is highly unstable, indicating that Eq. (5.8) is a stiff equation (see Appendix B for mathematical derivation and algorithm). Therefore a numerical procedure based on an implicit finite difference scheme was developed.[2] A similar approach is applied by Pinto (2014) to solve an ODE free boundary problem. We discretise the domain $\Omega = [0, p^*]$ by introducing the equidistributed grid points $(p_i)_{1 \leq i \leq n}$

[2]We thank Espen R. Jakobsen, Professor in Mathematics at NTNU, for guidance on the development of this method.

given by $p_i = (i-1)h$, where h is the grid spacing and n is an integer given by $n = \frac{p^*}{h} + 1$.[3] The unknowns of the problem are the values $f(p_1), ..., f(p_n)$ and we introduce the vector $\mathbf{F} = [f_1, ..., f_n]^{\mathrm{T}}$.[4] The following differential quotients are used as approximations for the derivatives:

$$f'(p_i) \approx \frac{f_{i+1} - f_i}{h}, \tag{5.12}$$

$$f''(p_i) \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}. \tag{5.13}$$

As Eq. (5.8) is non-local to the point $f(p_i)$, we have to approximate the value $f(\frac{p_i}{\phi})$. We do this through linear extrapolation, using the fact that Eq. (5.12) approximates $f'(p_i)$:

$$f(\frac{p_i}{\phi}) \approx f_i + (\frac{p_i}{\phi} - p_i)\frac{f_{i+1} - f_i}{h}. \tag{5.14}$$

We substitute Eqs. (5.12)-(5.14) into Eq. (5.8) and find the following relationship for the continuation region:

$$\begin{aligned} &f_{i-1}\left(\frac{1}{2}\sigma_R^2 p_i^2\right) + f_i\left(- \sigma_R^2 p_i^2 + h^2(\lambda\phi - r - \lambda) - h(\alpha_R p_i + \lambda\phi(\frac{p_i}{\phi} - p_i))\right) \\ &+ f_{i+1}\left(\frac{1}{2}\sigma_R^2 p_i^2 + h(\alpha_R p_i + \lambda\phi(\frac{p_i}{\phi} - p_i))\right) = 0 \qquad \text{for} \quad p_i \leq p^*. \end{aligned} \tag{5.15}$$

As stated in Eq. (5.15), f_i can be expressed as a function of f_{i-1} and f_{i+1}. We let the coefficients of f_{i-1}, f_i and f_{i+1} be denoted by a_i, b_i and c_i respectively:

[3] h is set such that n becomes an integer.

[4] The first node is $i = 1$ in order to be compatible with MATLAB arrays which are "1-indexed"

$$a_i = \frac{1}{2}\sigma_R^2 p_i^2, \tag{5.16}$$

$$b_i = -\sigma_R^2 p_i^2 + h^2(\lambda\phi - r - \lambda) - h(\alpha_R p_i + \lambda\phi(\frac{p_i}{\phi} - p_i)), \tag{5.17}$$

$$c_i = \frac{1}{2}\sigma_R^2 p_i^2 + h(\alpha_R p_i + \lambda\phi(\frac{p_i}{\phi} - p_i)). \tag{5.18}$$

Using the notation presented in Eqs. (5.16-5.18), Eq. (5.15) becomes:

$$a_i f_{i-1} + b_i f_i + c_i f_{i+1} = 0 \qquad \text{for} \quad p_i \leq p^*. \tag{5.19}$$

Combining Eq. (5.19) with the initial condition given by Eq. (5.11) and the boundary condition given by Eq. (5.9) results in the following set of equations:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & b_2 & c_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2} & b_{n-1} & c_n \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{p^*}{r-\alpha} - 1 \end{bmatrix} \tag{5.20}$$

This matrix equation can be solved for $\mathbf{F} = [f_1, ..., f_n]^{\mathrm{T}}$ in MATLAB. As p^* is not known, we have to make an initial guess. In order to test if we have found the correct threshold p^* we use the smooth pasting condition given by Eq. (5.10) to see if the following relationship holds:

$$f'(p^*) = \frac{f_n - f_{n-1}}{h} = \frac{1}{r - \alpha_R}. \tag{5.21}$$

If Eq. (5.21) holds, the correct p^* is found. If not, the guess for p^* is updated, and the matrix equation (5.20) is solved again. The procedure of solving Eq. (5.20) by guessing p^* is known as a shooting method, and can be performed efficiently through bisection. This is conducted iteratively in MATLAB using the code embedded in Appendix C until a solution is found. For purpose of clarification and illustration we include a graphical description of the problem in Figure 5.1.

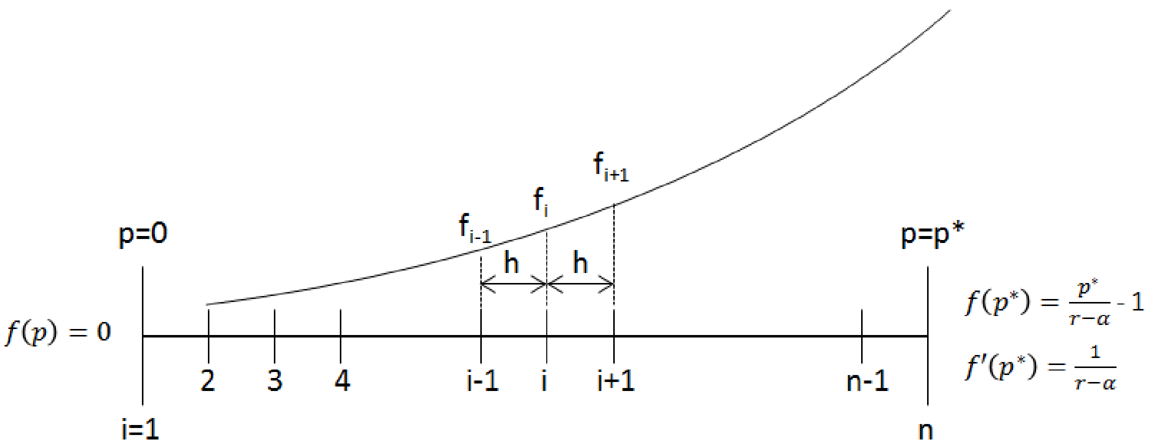


Figure 5.1: Finite difference grid with node spacing h and associated boundary conditions. A grid point i has an associated function value f_i. The function value f_i is computed based on solving an equation set including the neighbouring points f_{i-1} and f_{i+1}

Error of numerical method

In discretising Eq. (5.8) we incur a truncation error when approximating the derivatives of $f(p)$ with Taylor expansions. The error is denoted ϵ_1 and its magnitude is given by the following expression where C is a constant (Frey, 2008).

$$\epsilon_{f'(p)} = \left| \frac{f_{i+1} - f_i}{h} - f'(p) \right| \leq Ch, \tag{5.22}$$

$$\epsilon_{f''(p)} = \left| \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} - f''(p) \right| \leq Ch^2. \tag{5.23}$$

Eq. (5.22) shows that the error associated with a forward difference approximation to $f'(x)$, $\epsilon_{f'(p)}$, is proportional to the step size h. Eq. (5.23) shows that the error associated with a central difference approximation to $f''(x)$, $\epsilon_{f''(p)}$, is proportional to h^2. We use a step size of $0 \leq h \leq 1$ and thus the total error ϵ_1 is bounded by the lowest order h-term. Hence, the error arising from the finite difference approximation is proportional to h. Choosing a low h will therefore increase accuracy, but at the same increase computational running time.

5.2 Multi-factor model with stochastic profit and quantity, and sudden arrival of an innovation

In this section we present a multi-factor model where we isolate the effect of technological innovations improving the efficiency of the post-smolt production equipment. According to industry experts, such an innovation will likely arrive in the form of research describing best-practice for operating a post-smolt facility. As there is a high degree of cooperation in the salmon farming industry when it comes to post-smolt R&D, information leading to significantly more efficient production will likely be shared between companies. We therefore assume that the improvement can be adopted at no cost, and that the benefit is gained regardless of investment timing. Reduced mortality is found to be the most probable outcome of improved production processes, and as there is limited potential for improvement in this area (see Chapter 2), we allow for the arrival of only one innovation. Additionally we extend Doraszelski (2004) by distinguishing a large innovation from minor improvements. We incorporate the effect of minor improvements as a steady increase in produced quantity resulting from continuous 'learning by doing' in the industry.

Pertinent to our analysis is the work of Adkins and Paxson (2013) and Chronopoulos et al. (2015), who consider investments in power generation plants under policy uncertainty. More specifically the uncertainty is related to the possible sudden provision or retraction of a constant government subsidy in the form of a cash payment proportional to the revenues of the investing firm. In addition to the policy uncertainty, Chronopoulos et al. (2015) consider uncertainty in price, while Adkins and Paxson (2013) consider uncertainty in both price and produced quantity. We have found that the sudden provision of a subsidy resembles the arrival of a technological innovation, enabling us to base our work on their models. To the best of our knowledge, we are the first to adapt and apply real options techniques originally proposed in papers considering investment in power generation plants under policy uncertainty, to an investment under technological uncertainty. One important difference between a subsidy and a technological innovation is that the subsidy cannot be gained after the investment is made. We therefore extend their models so that they fit our case, where the benefit of a technological innovation is gained also after the investment is made.

As in Model 1 we consider a risk neutral salmon farming company which is considering to undertake a single irreversible investment in a post-smolt facility. The time t is continuous and infinite. The value of the investment opportunity $F(\pi, Q)$ evolves according to the stochastic processes of the per unit profits denoted by π, and annual production volume denoted by Q. The processes are assumed to be correlated Markov processes that are time-homogeneous. The time-homogeneity and Markov properties entail that the decision to wait or stop depends only on the current value of $F(\pi, Q)$, not on the historical development of the process or calendar time.

The unit profit at time $t \geq 0$, π_t, is stochastic, and assumed to follow a GBM such that:

$$d\pi = \alpha_\pi \pi dt + \sigma_\pi \pi dZ_\pi, \qquad \text{where} \qquad \mathcal{E}[\pi_t] = \pi_0 e^{\alpha_\pi t}.$$

α_π represents the drift in π, σ_π the volatility and dZ_π the increment of a Wiener process. In addition we assume that the quantity sold at time $t \geq 0$, Q_t, is stochastic, following a GBM such that:

$$dQ = \alpha_Q Q dt + \sigma_Q Q dZ_Q, \qquad \text{where} \qquad \mathcal{E}[Q_t] = Q_0 e^{\alpha_Q t}.$$

The drift term, α_Q, represents small costless technological improvements allowing for a slight improvement in facility efficiency. The volatility term, σ_Q, stems from oscillating mortality rates. Strictly speaking, Q_t is a discrete variable, but since a facility would produce at high volumes, we model quantity as a continuous process.

Investment in a post-smolt facility can be made at a fixed, one-time and irreversible cost K. Note that this assumption is made in order to isolate the effect of a technological innovation improving the efficiency of the post-smolt production equipment. Holding K fixed irrespective of technology efficiency is also assumed by Doraszelski (2004). Since the option is perpetual, calendar time does not affect the problem and we omit the subscript "t". The prevailing total annual profit is denoted by $T = \pi Q$. The introduction of a technological innovation will boost the total annual profits by a factor $\tau \geq 0$, such that the annual profits will increase from T to $T(1+\tau)$.

Let us now look at the value of the investment opportunity, denoted by $F(\pi, Q)$, in two scenarios. We let the value of the investment opportunity when a technological innovation has not been introduced, but will be introduced in the next period of length dt with probability λdt, be denoted by F_0. The total annual profit that signals an economically justified investment is denoted by $T_0^* = (\pi_0 Q_0)^*$. When an innovation already has been introduced, the value of the investment opportunity is denoted by F_1. In this scenario the total annual profit that signals an economically justified investment is denoted by $T_1^* = (\pi_1 Q_1)^*$. Both thresholds are hit either by achieving a high unit profit π, high quantity Q, or both. The thresholds, T_0^* and T_1^*, being products of two variables, are surfaces in $\mathbb{R}^3$.

In mathematical terms the total option value the salmon farming company holds is equal to the following:

$$F = F_0(1-\delta) + F_1\delta, \tag{5.24}$$

where δ is a binary variable, i.e.

$$\delta = \begin{cases} 1, & \text{if innovation has arrived} \\ 0, & \text{otherwise} \end{cases}$$

First we will evaluate the option value in the latter scenario, F_1, and find the associated threshold profit, denoted by $T_1^* = (\pi_1 Q_1)^*$. We expect this threshold to be lower than the threshold in the first scenario denoted by $T_0^* = (\pi_0 Q_0)^*$ because of the increase in project value caused by innovation. Hence, the salmon farming company is expected to have weaker incentives to postpone an investment after the innovation has arrived.

Assuming the innovation has arrived ($\delta = 1$), the salmon farming company is holding the option with a value denoted by F_1. The option will be exercised at T_1^*, and the Bellman equation becomes:

$$rF_1 dt = \mathcal{E}(dF_1), \tag{5.25}$$

which says that the capital appreciation of holding the option over some infinitely small period of time dt must equal the expected drift in the option value. Expanding the R.H.S. using multivariate Itô's lemma gives:

$$\mathcal{E}(dF_1) = \mathcal{E}\left(\frac{\partial F_1}{\partial \pi}d\pi + \frac{\partial F_1}{\partial Q}dQ + \frac{1}{2}\left[\frac{\partial^2 F_1}{\partial \pi^2}(d\pi)^2 + 2\frac{\partial^2 F_1}{\partial \pi \partial Q}d\pi dQ + \frac{\partial^2 F_1}{\partial Q^2}(dQ)^2\right]\right)$$

$$= \frac{\partial F_1}{\partial \pi}\alpha_\pi \pi dt + \frac{\partial F_1}{\partial Q}\alpha_Q Q dt + \frac{1}{2}\frac{\partial^2 F_1}{\partial \pi^2}\sigma_\pi^2 \pi^2 dt + \frac{\partial^2 F_1}{\partial \pi \partial Q}\sigma_\pi \sigma_Q \rho \pi Q dt + \frac{1}{2}\frac{\partial^2 F_1}{\partial Q^2}\sigma_Q^2 Q^2 dt.$$

ρ represents the correlation between π and Q. Substituting this back into the Bellman equation given by Eq. (5.25) and dividing both sides by dt results in the following PDE:

$$\frac{1}{2}\frac{\partial^2 F_1}{\partial \pi^2}\sigma_\pi^2 \pi^2 + \frac{1}{2}\frac{\partial^2 F_1}{\partial Q^2}\sigma_Q^2 Q^2 + \frac{\partial^2 F_1}{\partial \pi \partial Q}\sigma_\pi \sigma_Q \rho \pi Q + \frac{\partial F_1}{\partial \pi}\alpha_\pi \pi + \frac{\partial F_1}{\partial Q}\alpha_Q Q - rF_1 = 0. \tag{5.26}$$

A popular and powerful technique used to solve nonlinear PDEs such as Eq. (5.26) is similarity methods. These methods reduce n dimensional PDEs to $n-1$ dimensional PDEs resulting in less complexity. In our case, the PDE is two dimensional, hence it can be reduced to an ODE. This can be done as the transformation of variables leaves the underlying equation unchanged (see Pandey et al. (2009) and Budd and Piggott (2001)). By making the transformation $R = \pi Q$ (note that R is equivalent to the total annual profit T), implying that $F_1(\pi, Q) = F_1(R)$, and substituting we can reduce the original PDE to an ODE, without changing the original underlying equation.[5] This enables an analytical derivation of the decision boundary of F_1. We end up with the following ODE (a similar method is applied by Paxson and Pinto (2005)):

$$\frac{1}{2}F_1''(R)R^2(\sigma_\pi^2 + \sigma_Q^2 + 2\sigma_\pi\sigma_Q\rho) + F_1'(R)R(\alpha_\pi + \alpha_Q + \sigma_\pi\sigma_Q\rho) - rF_1(R) = 0, \tag{5.27}$$

which has a solution of the familiar form:

$$F_1 = A_1R^{\beta_1} + A_2R^{\beta_2}, \quad \text{for} \quad R \leq R_1^*,$$

where β_1 and β_2 are the roots of the following characteristic equation:

$$Q_1(\beta) = \frac{1}{2}\sigma_\pi^2\beta(\beta-1) + \frac{1}{2}\sigma_Q^2\beta(\beta-1) + \rho\sigma_\pi\sigma_Q\beta^2 + \alpha_\pi\beta + \alpha_Q\beta - r = 0,$$

where it can be shown that $\beta_1 > 1$ and $\beta_2 < 0$. Because $R = 0$ is an absorbing barrier, $F_1(0) = 0$, $A_2 = 0$. The value matching condition is (assuming that $R = R_1^*$ at time $t = 0$):

$$A_1R_1^{*\beta_1} = \mathcal{E}\int_{t=0}^{\infty} R_t(1+\tau)e^{-rt}dt - K = \int_0^{\infty} R_1^* e^{\mu_{\pi Q}t}(1+\tau)e^{-rt}dt - K = \frac{R_1^*(1+\tau)}{r-\mu_{\pi Q}} - K.$$

[5] see Appendix D for derivation

where the growth rate of R, i.e. the combined growth rate of π and Q, is denoted by $\mu_{\pi Q}$:[6]:

$$\mu_{\pi Q} = \alpha_\pi + \alpha_Q + \rho\sigma_\pi\sigma_Q. \tag{5.28}$$

The smooth pasting condition is:

$$\beta_1 A_1 R_1^{*(\beta_1 - 1)} = \frac{(1+\tau)}{r - \mu_{\pi Q}},$$

which by rearranging, and inserting into the value matching condition yields the following values for the constant A_1 and the threshold R_1^*:

$$A_1 = \frac{R_1^{*(1-\beta_1)}(1+\tau)}{\beta_1(r - \mu_{\pi Q})},$$

$$R_1^* = \frac{\beta_1}{\beta_1 - 1} K \frac{(r - \mu_{\pi Q})}{(1+\tau)}.$$

Now we look back at the first scenario, i.e. where $\delta = 0$. As long as the innovation is not introduced, the salmon farming company will hold the option denoted by F_0. However, note that the company will still get the benefit of the innovation when it arrives, regardless of investment timing.

The Bellman equation in this scenario becomes:

[6] See Appendix E for derivation of $\mu_{\pi Q}$

$$rF_0dt = \mathcal{E}(dF_0) + \lambda[F_1 - F_0]dt,$$

which says that the capital appreciation of holding the option over some infinitely small period of time dt must equal the sum of the expected drift in the option value and the expected gain from receiving the more valuable option F_1 in the next period of dt with probability λdt. Expanding the first term on the R.H.S. using multivariate Itô's lemma, and rearranging we get the following nonlinear PDE:

$$\frac{1}{2}\frac{\partial^2 F_0}{\partial \pi^2}\sigma_\pi^2\pi^2 + \frac{1}{2}\frac{\partial^2 F_0}{\partial Q^2}\sigma_Q^2 Q^2 + \frac{\partial^2 F_0}{\partial \pi \partial Q}\sigma_\pi\sigma_Q\rho\pi Q + \frac{\partial F_0}{\partial \pi}\alpha_\pi\pi + \frac{\partial F_0}{\partial Q}\alpha_Q Q - (r+\lambda)F_0 + \lambda F_1 = 0.$$

Like for Eq. (5.26), we solve Eq. (5.2) by using similarity methods. By making the transformation $R = \pi Q$, implying that $F_0(\pi, Q) = F_0(R)$ and substituting into the original PDE we leave the original equation unchanged, and get the following ODE:[7]

$$\frac{1}{2}F_0''(R)R^2(\sigma_\pi^2 + \sigma_Q^2 + 2\sigma_\pi\sigma_Q\rho) + F_0'(R)R(\sigma_\pi\sigma_Q\rho + \alpha_\pi + \alpha_Q) - (r+\lambda)F_0(R) + \lambda F_1(R) = 0. \tag{5.29}$$

This expression has a solution on the form (for $R < R_1^*$):

$$F_0 = C_1 R^{\theta_1} + C_2 R^{\theta_2} + A_1 R^{\beta_1}, \tag{5.30}$$

where the θ parameters are the roots of the following characteristic equation:[8]

[7] see Appendix D for derivation of Eq. (5.29)
[8] see Appendix F for derivation of Eq. (5.31)

$$Q_2(\theta) = \frac{1}{2}\sigma_\pi^2\theta(\theta-1) + \frac{1}{2}\sigma_Q^2\theta(\theta-1) + \rho\sigma_\pi\sigma_Q\theta^2 + \alpha_\pi\theta + \alpha_Q\theta - (r+\lambda) = 0, \tag{5.31}$$

where it can be shown that $\theta_1 > 1$ and $\theta_2 < 0$. Because $R = 0$ is an absorbing barrier, it follows that $F_0(0) = 0$, hence $C_2 = 0$.

Looking back at Eq. (5.24), we can define the value of the investment opportunity in three separate domains depending on the prevailing total profit R:

$$\mathrm{F} = \begin{cases} \big(C_1R^{\theta_1} + A_1R^{\beta_1}\big)\big(1-\delta\big) + A_1R^{\beta_1}\delta, & R < R_1^* \\ \left(\mathcal{E}_{R,Y}\left[\int_Y^\infty e^{-(r-\mu_{\pi Q})t}R(1+\tau)dt - e^{-rY}K\right] + D_1R^{\theta_1} + D_2R^{\theta_2}\right)\big(1-\delta\big) + \left(\frac{R(1+\tau)}{r-\mu_{\pi Q}} - K\right)\delta, & R_1^* \leq R < R_0^* \\ \mathcal{E}_{R,Y}\int_0^Y e^{-(r-\mu_{\pi Q})t}Rdt + \mathcal{E}_{R,Y}\int_Y^\infty e^{-(r-\mu_{\pi Q})t}R(1+\tau)dt - K, & R \geq R_0^* \end{cases} \tag{5.32}$$

where Y is an exponentially distributed variable indicating the time until the Poisson arrival (i.e. switch from $\delta = 0$ to $\delta = 1$). This is equal to:[9]

$$\mathrm{F} = \begin{cases} \big(C_1R^{\theta_1} + A_1R^{\beta_1}\big)\big(1-\delta\big) + A_1R^{\beta_1}\delta, & R < R_1^* \\ \left(\frac{\lambda R(1+\tau)}{(r-\mu_{\pi Q})(r-\mu_{\pi Q}+\lambda)} - \frac{\lambda K}{r+\lambda} + D_1R^{\theta_1} + D_2R^{\theta_2}\right)\big(1-\delta\big) + \left(\frac{R(1+\tau)}{r-\mu_{\pi Q}} - K\right)\delta, & R_1^* \leq R < R_0^* \\ \frac{R}{r-\mu_{\pi Q}}\left(1 + \frac{\lambda\tau}{r-\mu_{\pi Q}+\lambda}\right) - K, & R \geq R_0^* \end{cases} \tag{5.33}$$

[9] see Appendix G for the derivations

When $R < R_1^*$ and $\delta = 0$, the value of the investment opportunity equals the value of the option given that the innovation has arrived, $A_1 R^{\beta_1}$, adjusted by the term $C_1 R^{\theta_1}$, where $C_1 < 0$, which accounts for the fact that an innovation has not yet arrived. As λ increases, $C_1 R^{\theta_1}$ should converge towards zero, as the salmon farming company is closer to possessing the option to invest with an innovation.[10] At $\delta = 1$ the option value immediately switches to $F_1 = A_1 R^{\beta_1}$.

When $R_1^* \leq R < R_0^*$ and $\delta = 0$, the investment will take place immediately should the innovation arrive. Therefore the first two terms represent the expected value to invest given that the innovation has arrived at time Y, less the investment cost. The third term $D_1 R^{\theta_1}$ represents the value of the opportunity to invest at R_0^* and operate the facility until the arrival of the innovation. The fourth term $D_2 R^{\theta_2}$ represents the value of getting the investment opportunity in the region $R < R_1^*$, should the total annual profit fall beneath R_1^*. Therefore $D_1 R^{\theta_1}$ and $D_2 R^{\theta_2}$ are respectively increasing and decreasing functions in R.[11] As $\lambda \to 0$, the time the facility is likely to be operated without an innovation goes to zero, hence $D_1 R^{\theta_1}$ decreases.[12] The behaviour of $D_2 R^{\theta_2}$ for increasing λ values is more complex. When $\lambda \to 0$, $D_2 R^{\theta_2} \to 0$ as the value of the investment opportunity in the region $R < R_1^*$ has no value without the prospect of a future innovation. With increasing λ values, $D_2 R^{\theta_2}$ increases, as $C_1 R^{\theta_1} + A_1 R^{\beta_1}$ increases with higher λ values. However, when $\lambda \to 1$, the investment option is likely to be exercised in the region $R_1^* \leq R < R_0^*$ and $D_2 R^{\theta_2}$ decreases. Hence, the graph of $D_2 R^{\theta_2}$ plotted against λ form a bell curve.[13] At $\delta = 1$ the option value switches to $A_1 R^{\beta_1}$, which is exercised immediately in exchange for the termination value $\frac{R(1+\tau)}{r-\mu_{\pi Q}} - K$.

When $R \geq R_0^*$ the option is exercised regardless of whether an innovation has arrived or not. The termination value will be the sum of the two integrals in the respective domain

[10] see Table H.1 in Appendix H for numerical illustration
[11] see Table H.2 in Appendix H for numerical illustration
[12] see Table H.3 in Appendix H for numerical illustration
[13] see Table H.4 in Appendix H for numerical illustration

presented in (5.33). These represent the present value of the total annual profits without an innovation from the time of the investment until the innovation arrives, and the total annual profits after the innovation has arrived.

The endogenous constants C_1, D_1, D_2 and the optimal investment threshold when the innovation has not arrived, R_0^*, are obtained via value matching and smooth pasting conditions between the three branches of (5.33).

When $R = R_1^*$ and $\delta = 0$, we get the following value matching and smooth pasting condition (Eqs. (5.34) and (5.35) respectively):

$$C_1 R_1^{*\theta_1} + A_1 R_1^{*\beta_1} = \frac{\lambda R_1^*(1+\tau)}{(r-\mu_{\pi Q})(r-\mu_{\pi Q}+\lambda)} - \frac{\lambda K}{r+\lambda} + D_1 R_1^{*\theta_1} + D_2 R_1^{*\theta_2}, \tag{5.34}$$

$$\theta_1(C_1 - D_1) R_1^{*\theta_1 - 1} - \theta_2 D_2 R_1^{*\theta_2 - 1} = \frac{\lambda(1+\tau)}{(r-\mu_{\pi Q})(r-\mu_{\pi Q}+\lambda)} - \beta_1 A_1 R_1^{*\beta_1 - 1}. \tag{5.35}$$

When $R = R_0^*$, we get the following value matching and smooth pasting condition (Eqs. (5.36) and (5.37) respectively):

$$\frac{\lambda R_0^*(1+\tau)}{(r-\mu_{\pi Q})(r-\mu_{\pi Q}+\lambda)} - \frac{\lambda K}{r+\lambda} + D_1 R_0^{*\theta_1} + D_2 R_0^{*\theta_2} = \frac{R_0^*}{r-\mu_{\pi Q}}\left(1 + \frac{\lambda\tau}{r-\mu_{\pi Q}+\lambda}\right) - K \tag{5.36}$$

$$\theta_1 D_1 R_0^{*\theta_1 - 1} + \theta_2 D_2 R_0^{*\theta_2 - 1} = \frac{1}{r-\mu_{\pi Q}+\lambda} \tag{5.37}$$

Eqs. (5.34-5.37) gives the following system of nonlinear equations, which is solved in MATLAB using $fsolve$:

$$
\begin{aligned}
C_1 R_1^{*\theta_1} + A_1 R_1^{*\beta_1} - \tfrac{\lambda R_1^*(1+\tau)}{(r-\mu_{\pi Q})(r-\mu_{\pi Q}+\lambda)} + \tfrac{\lambda K}{r+\lambda} - D_1 R_1^{*\theta_1} - D_2 R_1^{*\theta_2} &= 0 \\
\theta_1 (C_1 - D_1) R_1^{*\theta_1 - 1} - \theta_2 D_2 R_1^{*\theta_2 - 1} - \tfrac{\lambda(1+\tau)}{(r-\mu_{\pi Q})(r-\mu_{\pi Q}+\lambda)} + \beta_1 A_1 R_1^{*\beta_1 - 1} &= 0 \\
\tfrac{\lambda R_0^*(1+\tau)}{(r-\mu_{\pi Q})(r-\mu_{\pi Q}+\lambda)} - \tfrac{\lambda K}{r+\lambda} + D_1 R_0^{*\theta_1} + D_2 R_0^{*\theta_2} - \tfrac{R_0^*}{r-\mu_{\pi Q}}\left(1 + \tfrac{\lambda\tau}{r+\mu_{\pi Q}+\lambda}\right) + K &= 0 \\
\theta_1 D_1 R_0^{*\theta_1 - 1} + \theta_2 D_2 R_0^{*\theta_2 - 1} - \tfrac{1}{r-\mu_{\pi Q}+\lambda} &= 0
\end{aligned}
\tag{5.38}
$$

Error of numerical method

In solving the set of nonlinear equations using $fsolve$, we incur an error which we denote ϵ_2. We use standard MATLAB settings with error toleration set to 10^{-6}, and therefore consider ϵ_2 to be an insignificant error.

5.3 Simulating expected optimal investment time

Determining optimal investment strategy and investment thresholds is the main focus of the majority of academic real options literature. The literature gives little attention to the expected time needed to reach the optimal investment thresholds, even though this is necessary information in order to express the present value of an investment opportunity. This point is emphasised by Himpler and Madlener (2014). In simpler models it is possible to derive the discount factor analytically. However, as our models are multi-factor, estimating the first passage time and expected stochastic discount factors is mathematically very complex. This is generally done numerically by simulation methods, such as Monte Carlo simulation. In the following we elaborate on the simulation procedures we constructed in order to estimate first passage time for Model 1 and 2.

Model 1: T_1^* denotes the expected time until the first passage of the investment threshold p^*. We simulate the development of the relationship $p = \frac{R}{K}$. The arrival of an innovation is simulated by drawing a random number from a uniform distribution and checking if it is less than the given λ value. In each discrete time step of one year we check if the value of p has exceeded the investment threshold p^*, and if so we save the number of time steps t_n in the array $T = [t_1, ..., t_n]$. To determine the expected first passage time T_1^* we run the procedure multiple times in order to achieve convergence in the simulations. We set T_1^*

equal to the median of the array T to filter out extreme values.

Model 2: T_2^* denotes the expected time until the first passage of one of the investment thresholds R_0^* or R_1^*. We simulate the development of the relationship $R = \pi Q$. The arrival of the innovation is simulated by drawing a random number from a uniform distribution and checking if it is less than the given λ value. Contrary to the Model 1 simulation, we need to check whether or not the innovation has arrived in order to determine if R_0^* or R_1^* is the optimal investment threshold. To arrive at the first passage time T_2^* we follow a similar procedure as for Model 1.

See Appendix I for MATLAB code of both simulation procedures.

Chapter 6

Model parametrisation

In this chapter we quantify the input variables used for the case study of an investment in a post-smolt production facility. A sensitivity analysis is conducted in Chapter 7. We quantify parameters related to investment cost, discount rate, production capacity, price, technology, growth rates, volatility and correlation.[1] The values have been chosen in close cooperation with industry experts.

Investment cost, K

The investment cost for the facility considered is set equal to MNOK 50. As stated in the previous, we perform a case study on a relatively small company, hence the investment cost reflects a small post-smolt production facility. The number is based on information received from the Norwegian aquaculture technology provider AKVAGroup and the Norwegian salmon farming company SalMar. SalMar has made several investments in land-based smolt production facilities based on the RAS technology. It is assumed that no equipment needs to be renewed during the time period considered. Thus, the investment cost is assumed to occur only at the time of investment. In Model 1, K is a stochastic variable with the initial value of MNOK 50.

Discount rate, r

The discount rate should reflect the risk embedded in the project. In theory it is calculated using e.g. the CAPM formula. However, to our knowledge it is not a common procedure in salmon farming companies to use theoretical models to derive the discount rate for individual

[1] The values are summarised in a table at the end of the chapter

projects. It is set by the company board and assumed equal among all projects. However, we believe it is an oversimplification to use the same discount rate regardless of project risk. Additionally, the discount rate is company sensitive information. Therefore we rely upon the advice of one of Norway's most acclaimed salmon analysts, Kolbjørn Giskeødegård at Nordea Markets. According to him the discount rate used in salmon farming companies normally range from 8 to 10 percent depending on the size of the company. However, given the risk inherent in the project considered, we follow Giskeødegård's advice and set the discount rate to 12 percent.

Production capacity, Q

In the case study we evaluate a facility with an initial annual production output of 500 metric tons of post-smolt. We set the individual post-smolt weight to 400 grams, which entails a production output of 1.25 million post-smolt. The specific fish weight is chosen based on advice from several of the largest salmon farming companies in Norway, including SalMar and Marine Harvest Group. They argue that the costs of land-based production increase fairly rapidly when surpassing a production weight of 400 grams.

In Model 2 we assume that there is growth and volatility in the production output. For this model, the growth rate is set to 1 percent annually. In Model 1 we wish to isolate the effects of uncertainty in profit development and investment cost. Therefore, the quantity of post-smolt produced is assumed constant and equal to 500 tons annually throughout the model.

We assume that the volatility in the production output stems from oscillating mortality rates. The average mortality rate for traditional salmon farming in Norway was 16 percent in 2013 (Terjesen and Handeland, 2014). The Norwegian salmon farming company Grieg Seafood has tested post-smolt production in a RAS facility, and according to Director of Biological Performance and Planning Frode Mathisen (2014), they have achieved a mortality rate of 2 percent in their most successful batches. However, as this is not an average and consistent rate, but merely the best case achieved in small-scale research facilities, this would be an optimistic number to apply as model input. Therefore, following advice from Associate Professor Torstein Kristensen at the Faculty of Biosciences and Aquaculture at the University of Nordland, we have set the mean annual survival rate to 90 percent, and the standard deviation of Q to 5 percent.

Unit profit, π

We need to determine three values related to unit profits: (1) initial unit profit, (2) volatility in unit profits, and (3) drift in unit profits. First, as most salmon farming companies considering post-smolt production aim to be self-sufficient, the usual method of calculating the price of the post-smolt is cost-based pricing with no mark-up. However, an investment in a facility can never be justified when the profit margin is constantly zero. The spot market for post-smolt exists mainly to serve companies' urgent demands for specific weight classes of post-smolt, caused by unexpected incidents of mass mortality. This creates market prices that are inelastic and highly volatile. Therefore we can not use spot market prices to determine initial unit profit. Based on advice from CEO Finn Christian Skjennum at Tjeldbergodden Settefisk, one of the few companies that specialises in commercialised post-smolt production, the post-smolt price should be set equal to the total production cost plus a margin. For the production cost we use a formula that is normally applied to calculate the price of smolt at a specific weight. This was recommended by Hatchery Manager in SalMar, Bjørn Hembre. The price is the sum of a fixed cost of NOK 4.5 per individual and NOK 0.05 per gram of fish. This equates to NOK 61.25 per kg.[2] The margin we use in the model is set to 20 percent on advice from Skjennum, which equates to a selling price of NOK 73.5 per kg. The unit profit will then be NOK 12.25 per kg in the first year.

Second, as already mentioned, we assume that the per unit profits of post-smolt production follow a GBM. In determining the volatility of this process we again rely upon the advice of Skjennum. He believes that the post-smolt price should be correlated with the salmon price in the long run. Therefore we use a volatility in the model in the same range as the salmon price volatility. From January 1998 to September 2014, annualised volatility of weekly salmon spot prices have ranged from about 16 to 35 percent (Skistad, 2014).

[2] $\frac{\text{NOK } 4.5 + \text{NOK } 0.05 \text{ per gram } \times 400 \text{ grams}}{0.4 \text{ kg}} = \text{NOK } 61.25$ per kg.

Based on this research, we set volatility of the per unit profits to 25 percent, but perform a sensitivity analysis to test how a higher (or lower) volatility affects the value of the investment opportunity. As we consider a perpetual option we cannot assume that the per unit profits can grow faster than the overall economy in eternity, which grows at around 2-3 percent annually (Koller et al., 2010). Additionally, as commodity prices often show a mean-reverting behavior in the long run and salmon closely resembles a commodity, the drift rate of the associated price should not be set too high. Therefore we have set the drift rate of the GBM representing per unit profits to 2 percent. This applies to both models presented.

Initial total annual profit, R_{DCF}

Given that the investment was made today, we assume an initial production volume $Q = 500$ tons and per unit profit $\pi =$ NOK 12.25, hence the total profit in the first year of operation, denoted by R_{DCF}, is equal to MNOK 6.125.

Technology, λ and ϕ

After consulting with Hatchery Manager in SalMar, Bjørn Hembre, we set the arrival rate of innovating technology in post-smolt production to 0.2. This would indicate an expected arrival every 5 years and applies to both models presented. An innovation is assumed to reduce investment cost by 5 percent in Model 1 and improve total annual profits by 10 percent in Model 2. The intuition behind using different innovation factors for the models is that only one innovation is allowed in Model 2, while there is no limit on the number of innovations (or Poisson jumps) that can occur in Model 1.

Correlation, ρ

For Model 2 we assume that the correlation between quantity and unit profits is zero due to the lack of data. The consequence of this assumption is that the growth rate of the model becomes the sum of the growth rates of quantity and unit profits, namely equal to 3 percent (See Eq. (5.28)). In Chapter 7 we test the sensitivity of the results to different values of ρ.

Parameter	**Symbol**	**Model 1**	**Model 2**
Investment cost	K	MNOK 50	MNOK 50
Discount rate	r	12%	12%
Innovation arrival rate	λ	0.2	0.2
Investment cost reduction factor	ϕ	0.95	-
Profit improvement factor	τ	-	0.1
Profit volatility	σ_R, σ_π	25%	25%
Profit drift	α_R, α_π	2%	2%
Quantity volatility	σ_Q	-	5%
Quantity drift	α_Q	-	1%
Initial total annual profit	R_{DCF}	MNOK 6.125	MNOK 6.125
Correlation	ρ	-	0
Combined growth rate	$\mu_{\pi Q}$	-	3%

Table 6.1: Input parameters for the models summarised

Chapter 7

Post-smolt case study

In this chapter we present a case study of Model 1 and Model 2 calibrated to an investment in a post-smolt facility. We present results using the parameter values from the previous chapter, and conduct sensitivity analyses. We treat the intrinsic value of the option as an approximate to the NPV given by a simplistic DCF analysis with no explicitly forecast cash flows, and perpetuity growth from year one. This enables us to make a comparison between the traditional DCF and the real options approach. The value of being able to wait for the optimal investment threshold is highlighted in two points; the initial annual profit R_{DCF} and the traditional DCF investment threshold $R_{NPV=0}$. The objective of the chapter is to find if real options valuation can uncover additional value compared to DCF analysis, and to give investment managers intuition on how technology and profit uncertainty affect an investment in a post-smolt production facility.

7.1 Multi-factor model with stochastic profit and investment cost

In this section we present the results of Model 1. Unless stated otherwise, we use the following values for the input parameters (see Chapter 6): $r = 0.12$, $\phi = 0.95$, $\sigma_R = 0.25$, $\alpha_R = 0.02$ and $\lambda = 0.2$. We will hereby refer to this as the base case for Model 1.

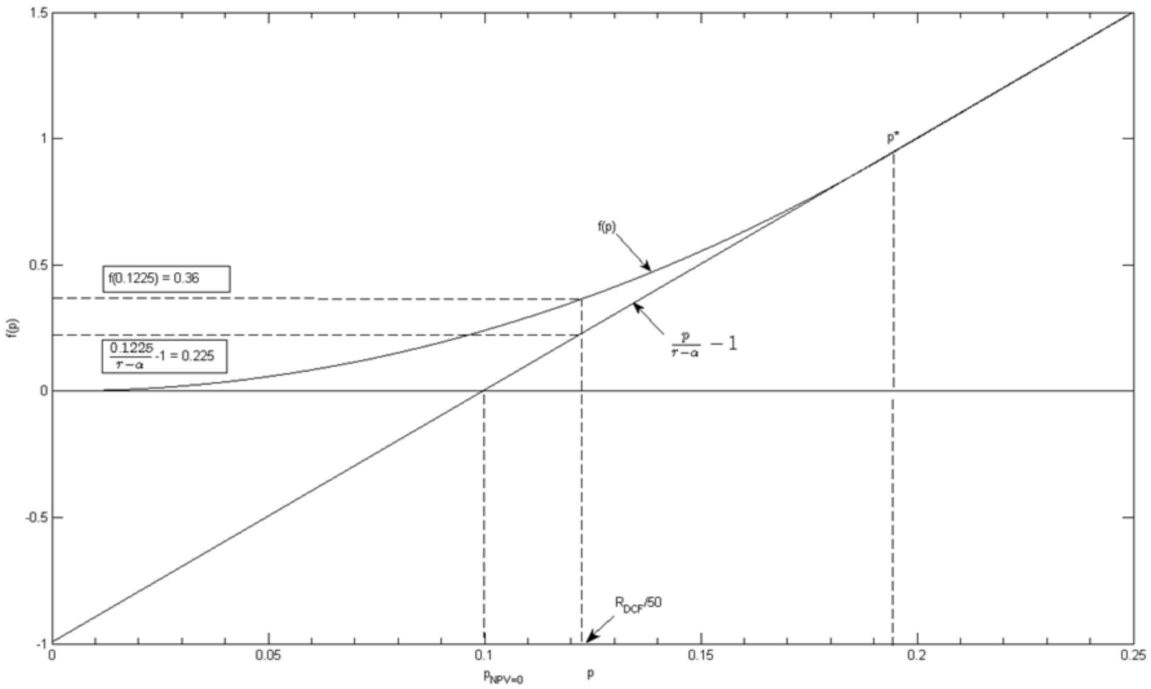


Figure 7.1: The ratios $f(p) = F(R, K)/K$, and the intrinsic value line $\frac{p}{r-\alpha} - 1$, under base case assumptions

Figure 7.1 displays the value of $f(p)$ and the intrinsic value line given by the function $\frac{p}{r-\alpha} - 1$. Referring to Section 5.1, $f(p) = F(R, K)/K$, i.e. it represents the ratio of the value of the investment opportunity to the investment cost. Similarly, the intrinsic value line given by $\frac{p}{r-\alpha} - 1 = \frac{V(R)-K}{K}$, i.e. it represents the net present value of the project relative to the investment cost. The intrinsic value line intersects with the horizontal axis at $p_{NPV=0} = R/K = 0.1$, indicating that the traditional DCF analysis would suggest investing given that first year total annual profit is at least 10 percent of the investment cost. The investment threshold $p^* = (R/K)^* = 0.1945$ is where the graph of $f(p)$ and the intrinsic value line meet tangentially. For $p < p^*$, $f(p)$ is larger than the intrinsic value, and the difference represents the value of the flexibility to wait for the optimal investment threshold p^*. This observation implies that modelling the investment problem by using a real options approach can uncover additional value. To quantify the excess value in a specific point, we evaluate $f(p)$ at $R_{DCF} = 6.125$ and $K =$ MNOK 50. In this point the value of the flexibility to delay an investment is approximately MNOK 7, an excess value of about 60 percent to the net present value.[1]

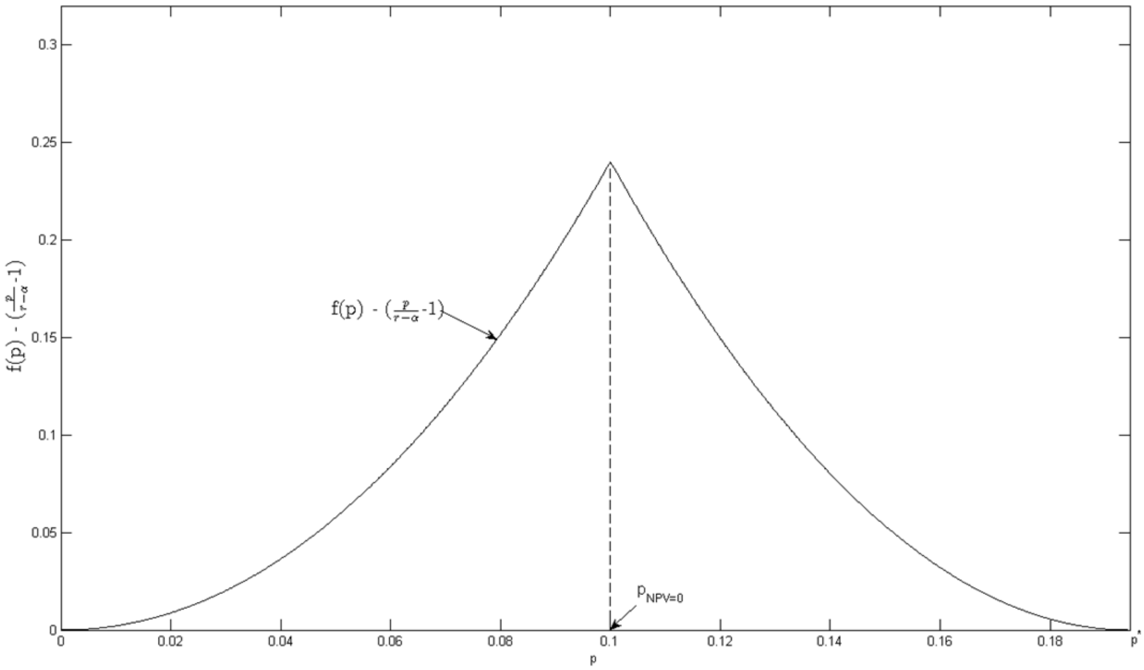


Figure 7.2: Value of the flexibility to delay investment, i.e. the difference between the option value and the intrinsic value, $f(p) - \left(\frac{p}{r-\alpha} - 1\right)$.

Figure 7.2 illustrates the value of the flexibility to delay investment. The value reaches a maximum of approximately 0.25 when p equals the break even point of the investment, i.e. where the traditional NPV equals zero. For an investment cost of MNOK 50, this corresponds to MNOK 12.5. The results shown in the graph is in line with our intuition - at low values of p, i.e. when K is large relative to R, the flexibility has little value as the option is unlikely to be exercised, and at high values of p the flexibility to delay investment also has low value as the option is likely to be exercised in the near future.

[1](0.36-0.225) × MNOK 50 ≈ MNOK 7

We will in the following conduct a sensitivity analysis aiming to determine how changes in the values of the input parameters affect $f(p)$ and the threshold p^*.

Sensitivity analysis

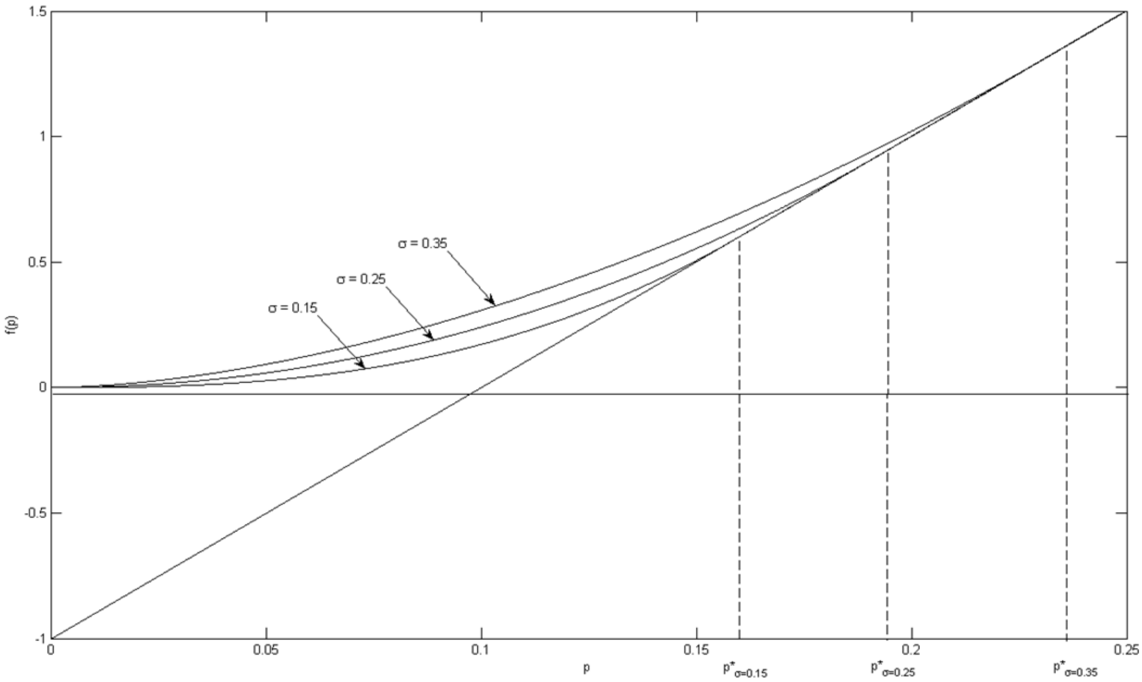


Figure 7.3: Value of investment opportunity, $f(p)$, for $\sigma = 0.15$, $\sigma = 0.25$ and $\sigma = 0.35$.

Figure 7.3 shows that a higher level of profit uncertainty, i.e. higher σ, increases the value of $f(p)$. A higher σ also leads to a higher p^*. This in line with classic options theory (see Dixit and Pindyck (1994)) as more uncertainty, all else held equal, should both increase the value of the investment opportunity but also make the salmon farming company more reluctant to invest in the project. The threshold development for increasing volatility is more directly shown in Figure 7.4 below.

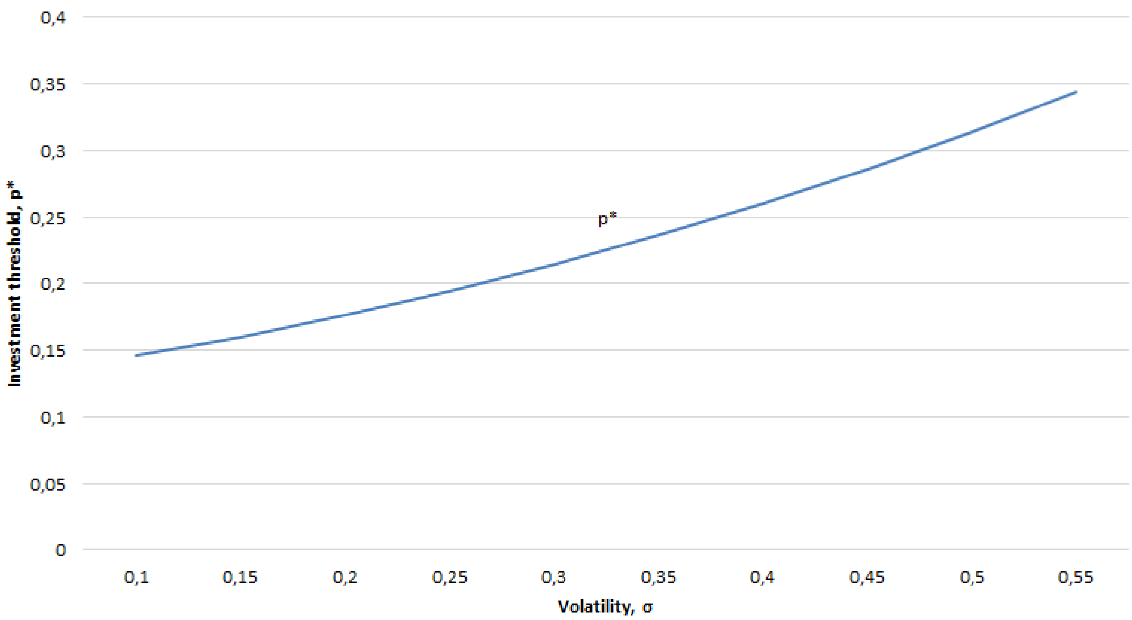


Figure 7.4: Investment threshold p^* as a function of σ

Figure 7.4 shows that the investment threshold, p^*, increases convexly for higher volatility. Thus, the salmon farming company is more reluctant to invest when volatility increases, despite the fact that $f(p)$ also increases.

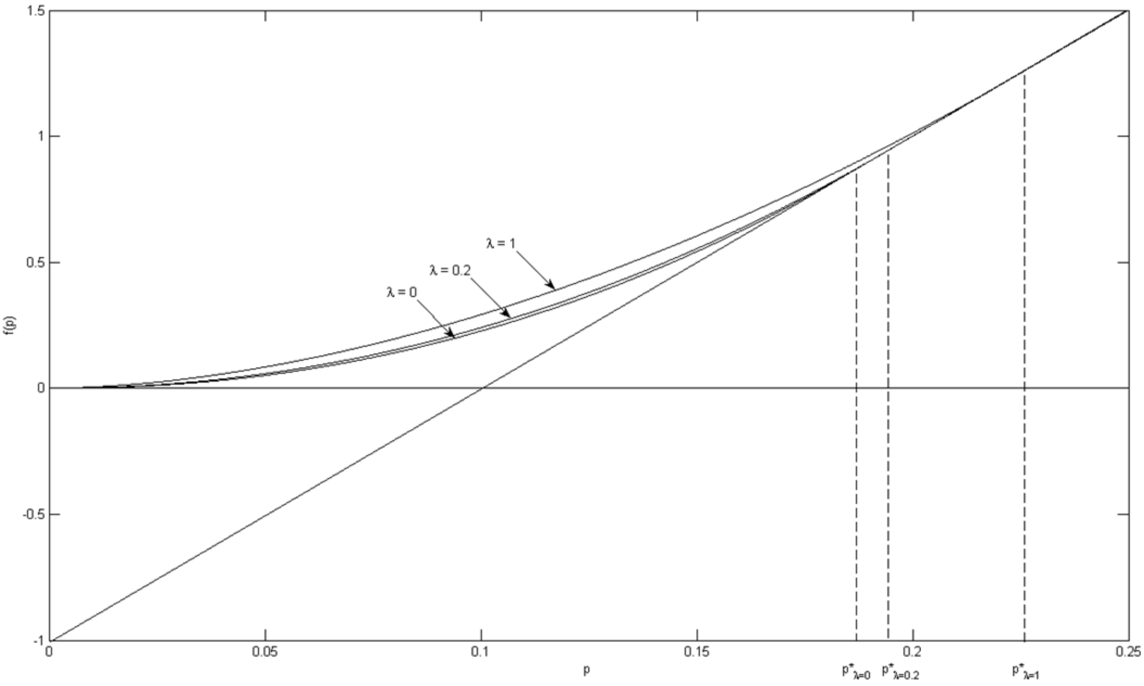


Figure 7.5: Value of investment opportunity, $f(p)$, for $\lambda = 0$, $\lambda = 0.2$ and $\lambda = 1$

Figure 7.5 shows that $f(p)$ increases for higher innovation arrival rates λ. A higher arrival rate of innovations reduces the expected investment cost, and thus increases the value of the investment opportunity. Additionally the investment threshold p^* increases with λ, i.e. the salmon farming company has stronger incentives to delay the investment. We consider two hypothetical cases to exemplify this: In a case where $\lambda = 0$, the innovations will never arrive, and there is no point in waiting for a lower investment cost for the salmon farming company. However, if $\lambda = 1$, the salmon farming company receives a reduced investment cost in the next time period, which gives stronger incentives to wait.

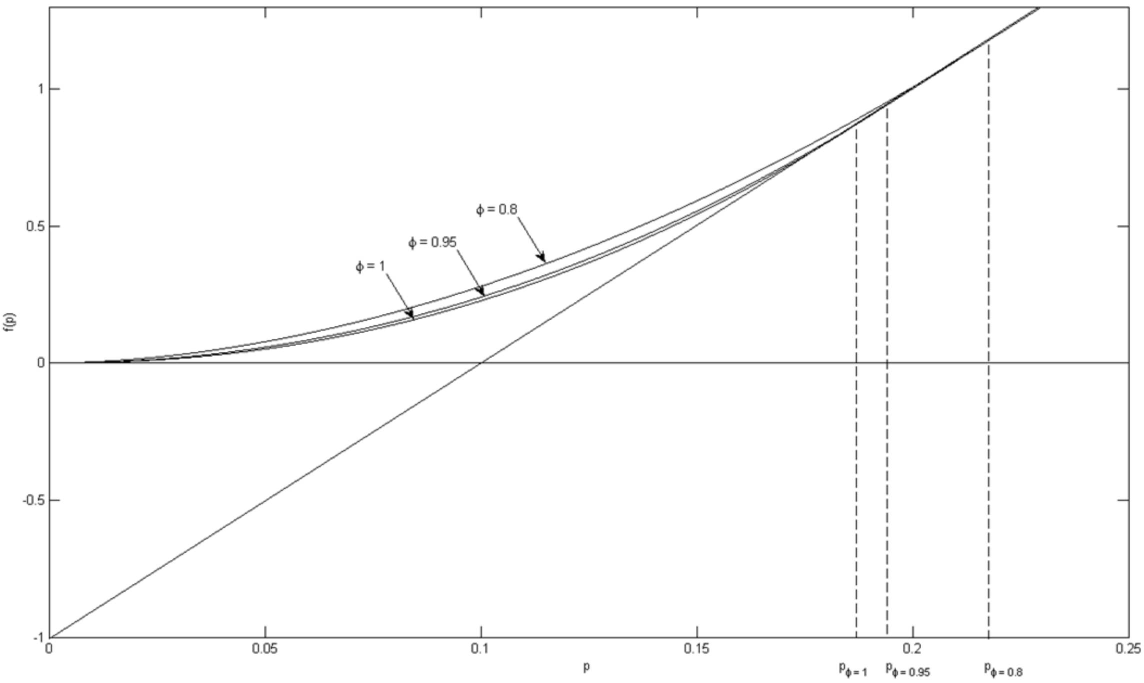
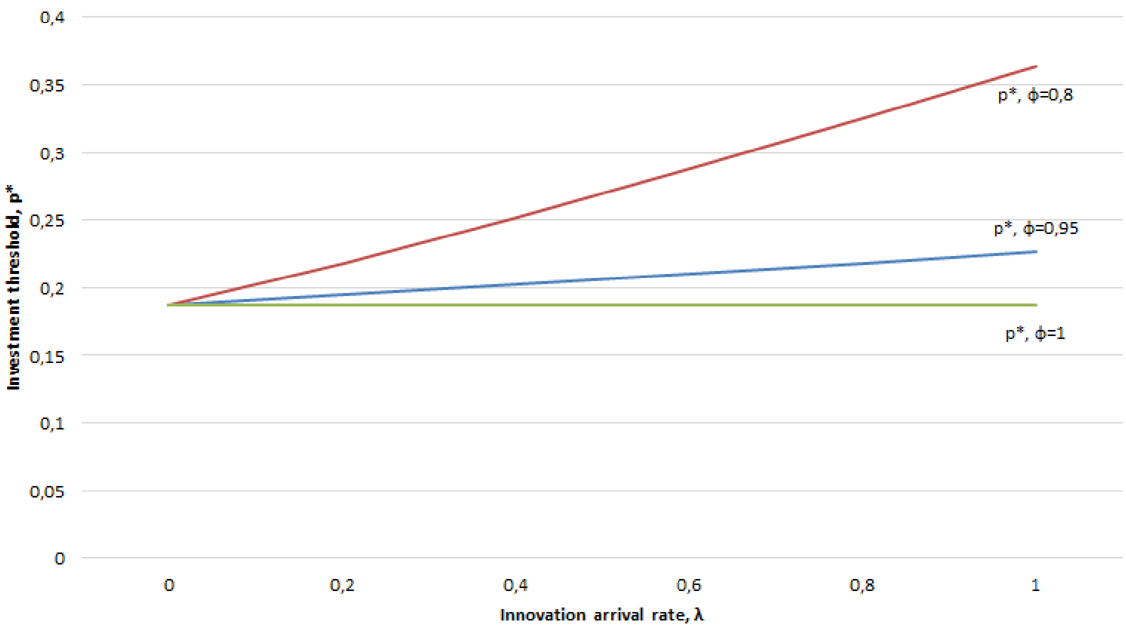


Figure 7.6: Value of investment opportunity, $f(p)$, for $\phi = 0.8$, $\phi = 0.95$ and $\phi = 1$

Figure 7.6 shows that a lower value of ϕ, i.e. a more significant cost reduction factor increases $f(p)$ and the investment threshold p^*. This result follows the same logic as Figure 7.5 - a more significant investment cost reduction factor lowers the expected investment cost, and hence the value of the investment opportunity increases. At the same time, the salmon farming company has stronger incentives to delay investment. In Figure 7.7 below we illustrate the combined effects of λ and ϕ on the investment threshold p^*.



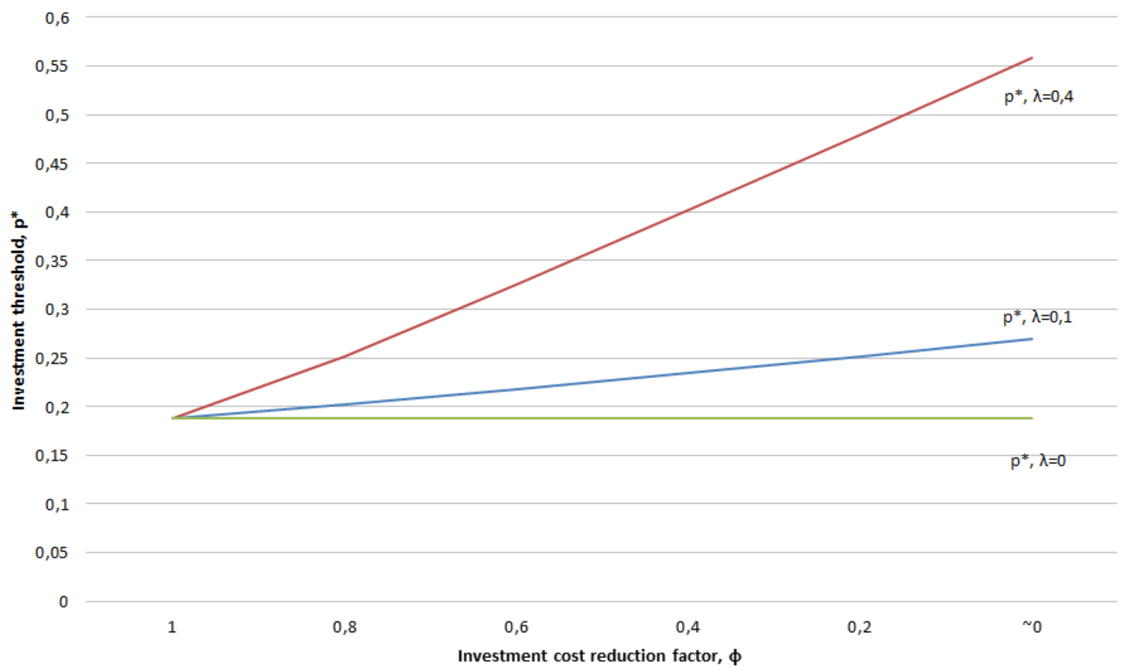


Figure 7.7: Investment threshold p^* as a function of λ with different ϕ values (top graph), and as a function of ϕ with different λ values (bottom graph)

In the top graph of Figure 7.7 p^* is plotted for $\lambda \in [0, 1]$ and for different values of the investment cost reduction factor ϕ. When $\phi = 1$, there is no investment cost reduction associated with the innovation, hence p^* is constant for all λ values. Conversely, when $\lambda = 0$, the innovation will never arrive and p^* is constant for all innovation sizes ϕ. We observe that the lower the value of ϕ, the steeper the graph is. This is in line with our intuition, as innovations with large impact on the investment cost and high arrival rate increase the incentives to wait. More importantly we see from Figures 7.5, 7.6 and Figure 7.7 that λ and ϕ have similar effects on $f(p)$ and p^*. In other words the expectation of frequent but small innovations (high λ and high ϕ), and few but large innovations (low λ and low ϕ), will have a similar effect on both option value and investment threshold. Thus the combination of λ and ϕ can be interpreted as the total amount of technological uncertainty inherent in the investment opportunity.

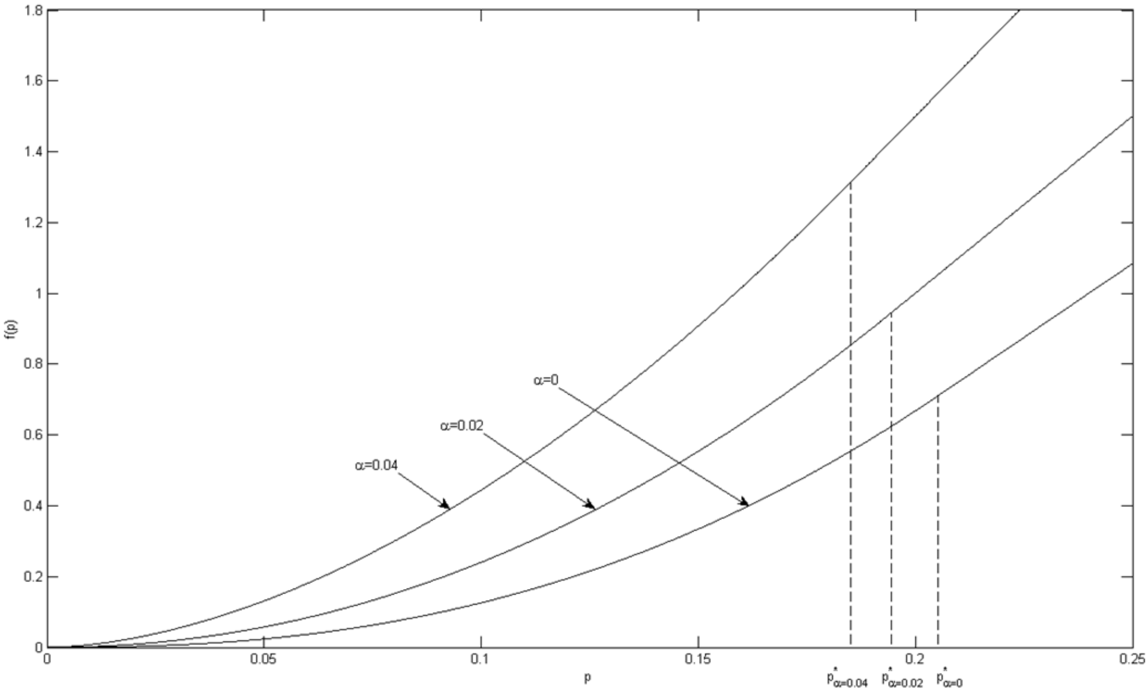


Figure 7.8: Value of investment opportunity, $f(p)$, for $\alpha = 0$, $\alpha = 0.02$ and $\alpha = 0.04$

Figure 7.8 shows that the value of $f(p)$ increases with higher profit growth rates, while the investment threshold, p^* decreases (see Figure 7.8). A higher growth rate increases the expected value of the project, and incentivises investment. For further elaboration we refer to classic real options theory such as the work of Dixit and Pindyck (1994).

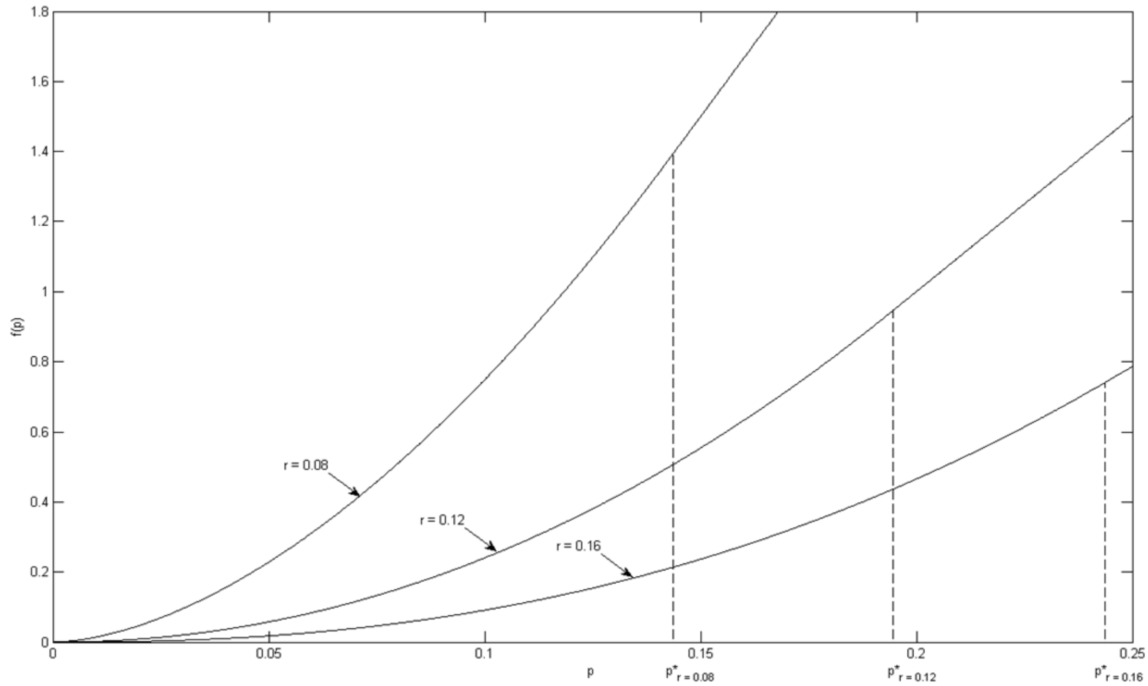


Figure 7.9: Value of investment opportunity, $f(p)$, for $r = 0.08$, $r = 0.12$ and $r = 0.16$

A higher discount rate reduces the expected value of the project. This effect is shown in Figure 7.9, where we see lower values of $f(p)$ and higher thresholds p^* for higher discount rates r. This observation is also in line with classic option theory and again we refer to Dixit and Pindyck (1994) for elaboration. Figures 7.8 and 7.9 illustrate that both $f(p)$ and p^* are highly sensitive to α and r. Therefore it is important for the salmon farming company to be precise when quantifying these parameters.

In the following we will perform a similar case study on Model 2.

7.2 Multi-factor model with stochastic profit and quantity, and sudden arrival of an innovation

In this section we present the results of Model 2. Unless stated otherwise, we use the following values for the input parameters (see Chapter 6): $r = 0.12$, $\tau = 0.1$, $K =$ MNOK 50, $\sigma_\pi = 0.25$, $\sigma_Q = 0.05$, $\alpha_\pi = 0.02$, $\alpha_Q = 0.01$, $\rho = 0$ and $\lambda = 0.2$. In the following we will refer to this parametrisation as the base case for Model 2.

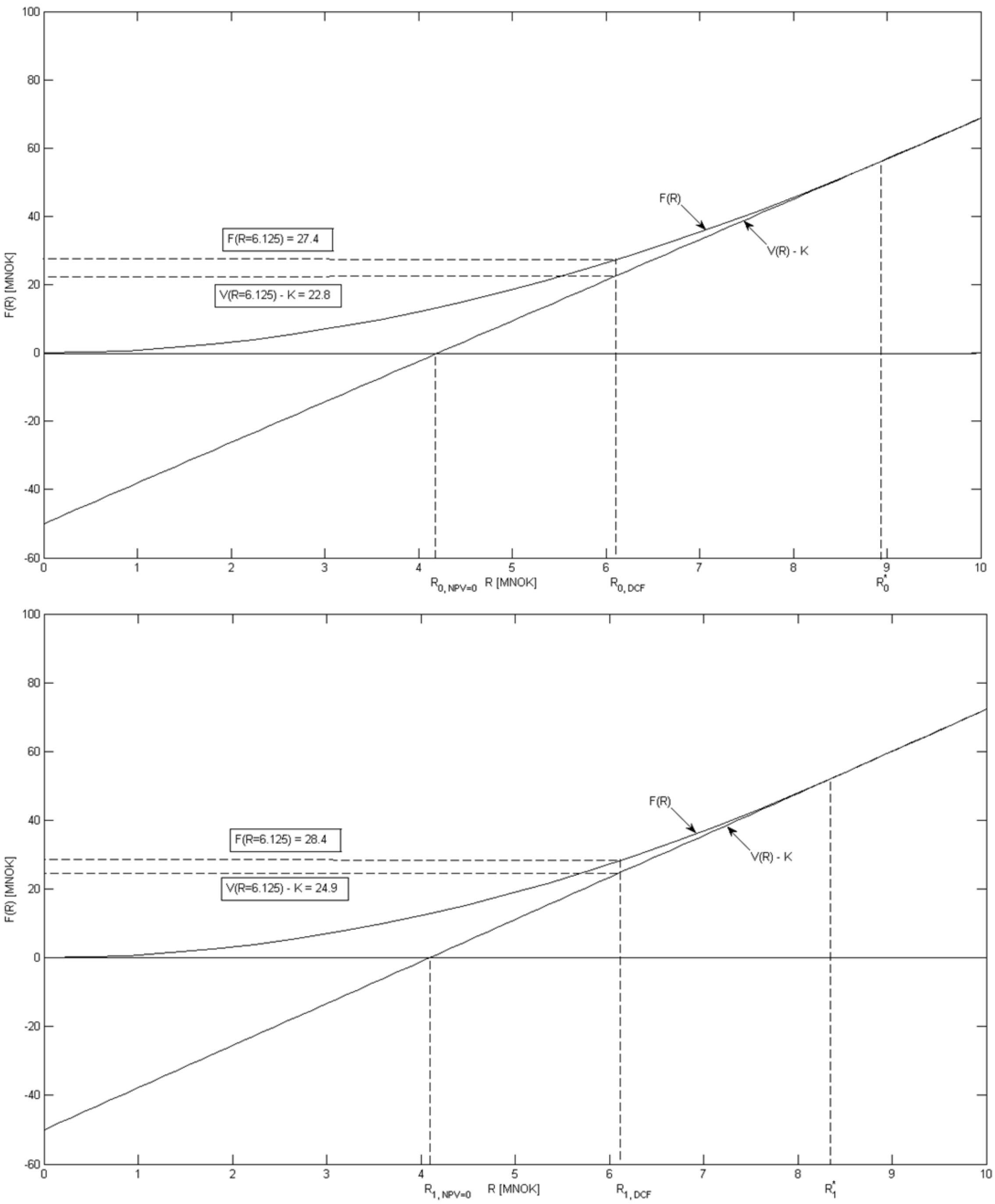


Figure 7.10: Value of investment opportunity, $F(R)$, and intrinsic value of the option, $V(R) - K$ as a function of R, for Model 2 base case assumptions. The top graph represents the value when the innovation has not arrived ($\delta = 0$). The bottom graph represents the value when the innovation has arrived ($\delta = 1$). R is the prevailing total annual profit in MNOK

Figure 7.10 displays the value of the option to invest, $F(R)$, and the intrinsic value of the option, $V(R)-K$ as functions of the prevailing total annual profit R. The top graph represents the value when the innovation has not arrived ($\delta = 0$). The bottom graph represents the value when the innovation has arrived ($\delta = 1$). Both $F(R)$ and $V(R) - K$ are higher when the innovation has arrived, as R is increased by $(1+\tau)$.[2] Initially the salmon farming company should evaluate the optimal investment strategy based on the graph for $\delta = 0$. If the innovation arrives, the company must instead consider the graph for $\delta = 1$.

The two vertical lines marked by R_1^* and R_0^*, represent the investment thresholds respectively with and without the presence of the innovation. As described by the value matching and smooth pasting conditions given by Eqs. (5.34-5.37), $F(R)$ and $V(R)-K$ meet tangentially at the investment thresholds. R_0^* is higher than R_1^*, as the salmon farming company needs a higher total annual profit to be willing to invest without the presence of an innovation. $R_{1,NPV=0}$ and $R_{0,NPV=0}$ represent the traditional NPV thresholds, i.e. where the NPV equals zero. At the initial total annual profit R_{DCF} = MNOK 6.125, the difference between the net present value and the real options value is MNOK 4.6 or approximately 20 percent of the net present value when $\delta = 0$. For $\delta = 1$ the corresponding numbers are MNOK 3.5 and 14 percent. This shows that the real options model can uncover additional value compared to the traditional DCF method.

For the base case we included a separate graph for $\delta = 0$ and $\delta = 1$ to show that their behaviour coincides and to highlight the difference in investment value and optimal thresholds. For the remainder of the section we focus on the case where the innovation has not arrived ($\delta = 0$), as this is the most relevant and interesting scenario to study.

[2]To account for the expected future arrival of the innovation, $V(R)-K$ for $\delta = 0$ is given by the expression $\frac{R}{r-\mu_{\pi Q}}\left(1+\frac{\lambda\tau}{r-\mu_{\pi Q}+\lambda}\right) - K$, see Eq. (5.33)

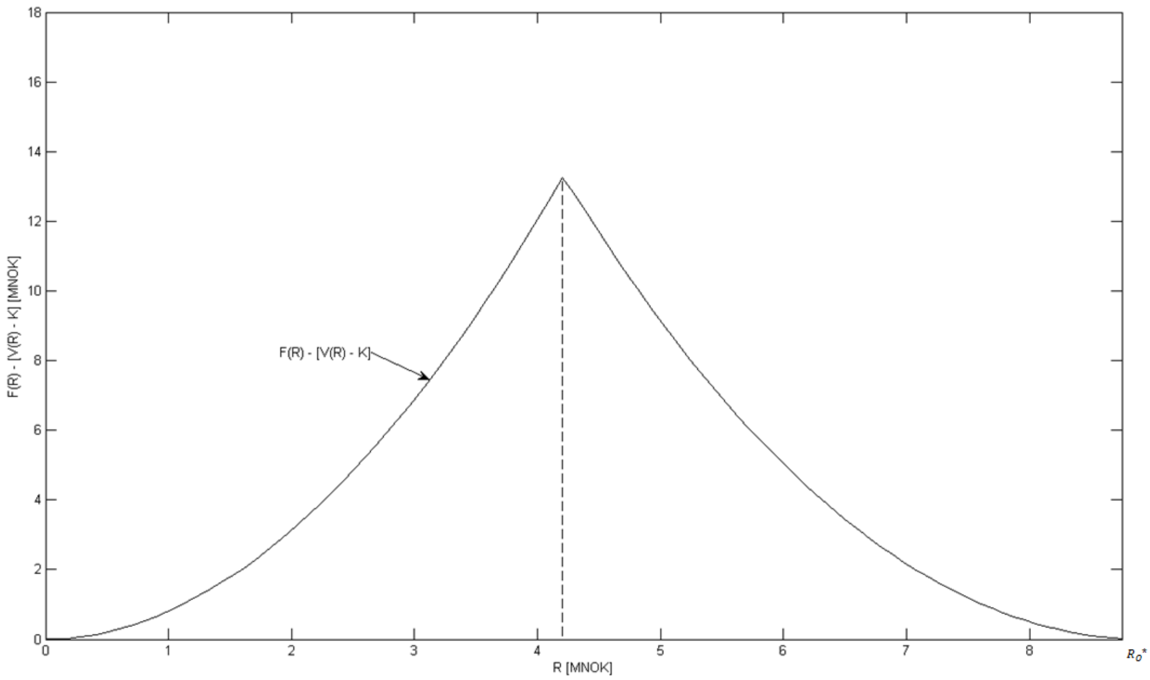


Figure 7.11: Value of the flexibility to delay investment, i.e. the difference between the option value and the intrinsic value, $F(R) - [V(R) - K]$.

In Figure 7.11 we have displayed the value of the flexibility to delay the investment. We see from the graph that the value reaches a maximum of approximately MNOK 14 when $R =$ MNOK 4.2, i.e. where the traditional NPV equals zero. This is in line with our intuition - at low values of R the flexibility has little value as the option is unlikely to be exercised, and at high values of R the flexibility also has low value as the option is likely to be exercised in the near future.

Sensitivity analysis

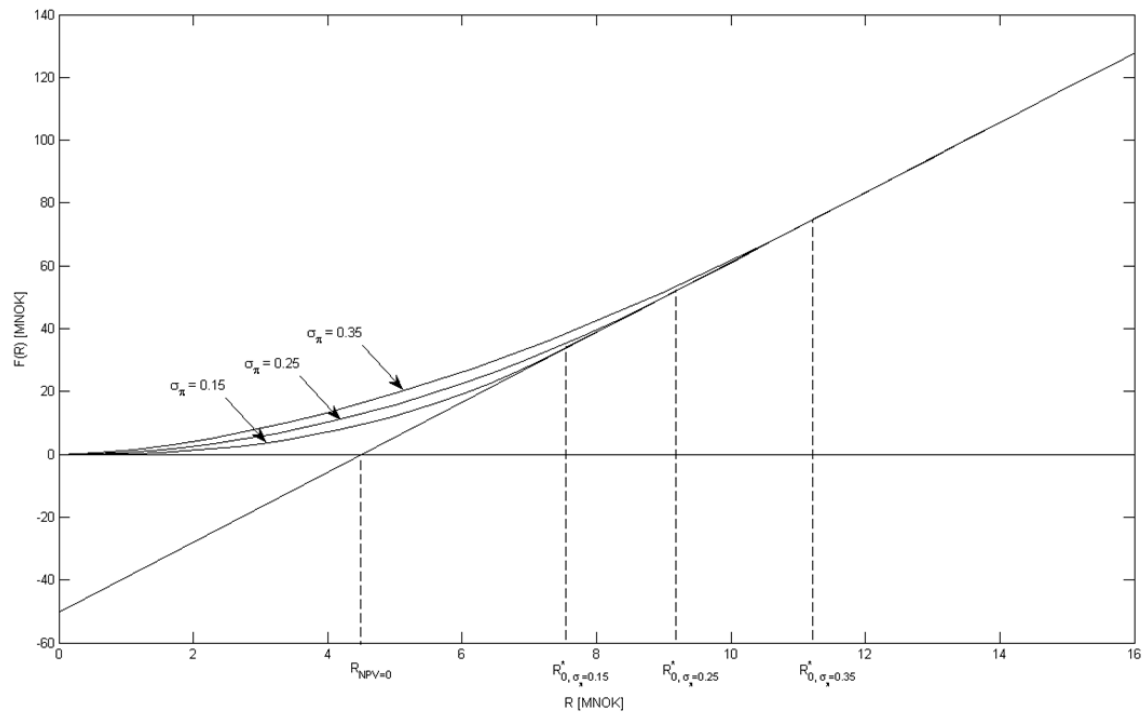


Figure 7.12: Value of investment opportunity, $F(R)$, for $\sigma_\pi = 0.15$, $\sigma_\pi = 0.25$ and $\sigma_\pi = 0.35$

In Figure 7.12 we show the value of the investment opportunity, $F(R)$, for different values of the unit profit volatility σ_π. The intrinsic value of the real option $V(R) - K$ is also indicated in the figure. As expected, $V(R) - K$ and all $F(R, K)$ meet tangentially in the R_0^*-thresholds. As changes in both σ_π and σ_Q affect the option value in the same way, we have not included a separate graph for σ_Q. The results show that a higher level of uncertainty leads to higher option value. However, higher uncertainty also increases the associated investment thresholds as the salmon farming company is more reluctant to invest in the project when the uncertainty is higher. As long as $\rho = 0$ the intrinsic value line $V(R) - K = \frac{R}{r-\mu_{\pi Q}}\left(1 + \frac{\lambda\tau}{r-\mu_{\pi Q}+\lambda}\right) - K$ does not vary with different values of σ_π. This is because $\mu_{\pi Q}$ is dependent on ρ. However, each different value of the parameters $\lambda, \tau, \rho, \alpha, r$ or K, has its unique intrinsic line. Therefore, to preserve simplicity in the graphs intrinsic value lines will not be included for the remainder of the sensitivity analysis.

The threshold development for increasing volatility is more directly shown in Figure 7.13 below. It shows that R_0^* and R_1^* are increasing convexly for higher per unit profit volatility.

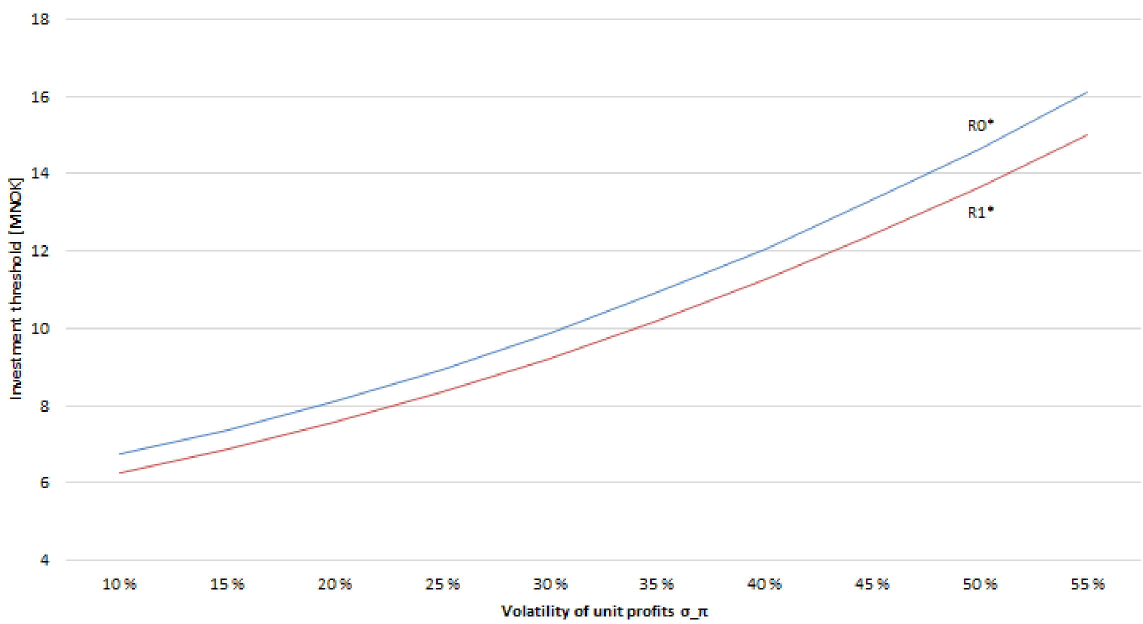


Figure 7.13: Investment thresholds, R_1^* and R_0^*, as a function of σ_π

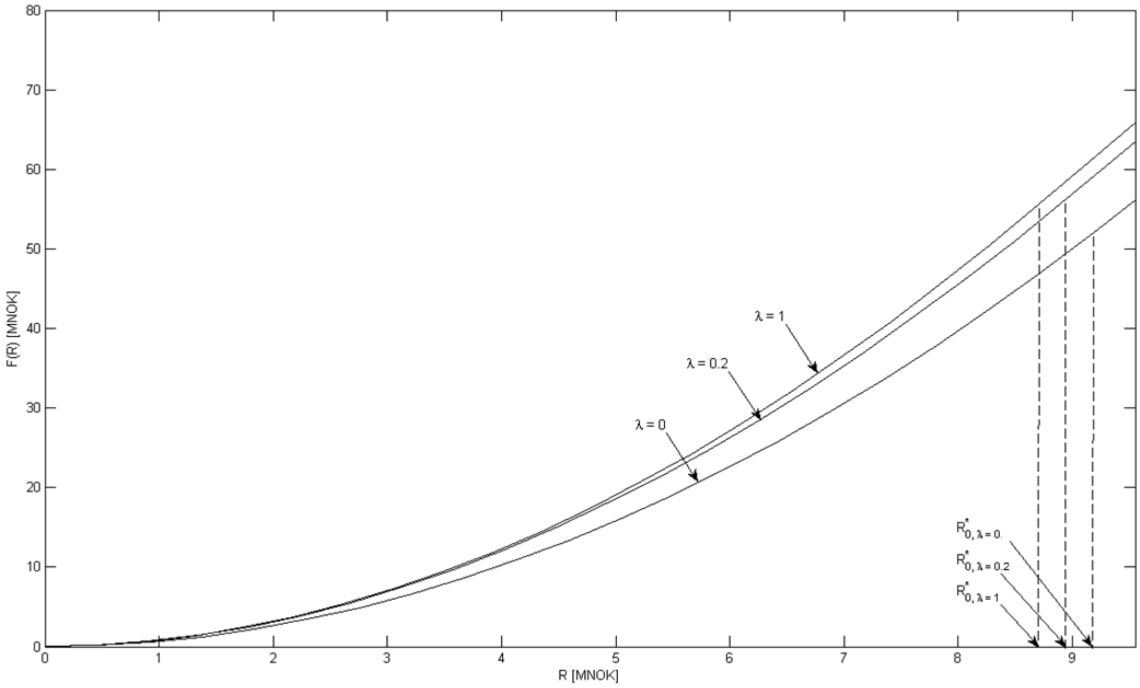


Figure 7.14: Value of investment opportunity, $F(R)$, for $\lambda = 0$, $\lambda = 0.2$ and $\lambda = 1$

Figure 7.14 displays the sensitivity of the option value to different λ values, i.e. the arrival rate of the innovation. The option value is an increasing function in λ. A higher λ decreases the expected time to the arrival of the innovation, increasing the expected value of the investment opportunity. The option value is more sensitive in the λ-range $[0, 0.2]$ than in $[0.2, 1]$. Figure 7.15 below shows directly how λ and τ affect the investment thresholds.

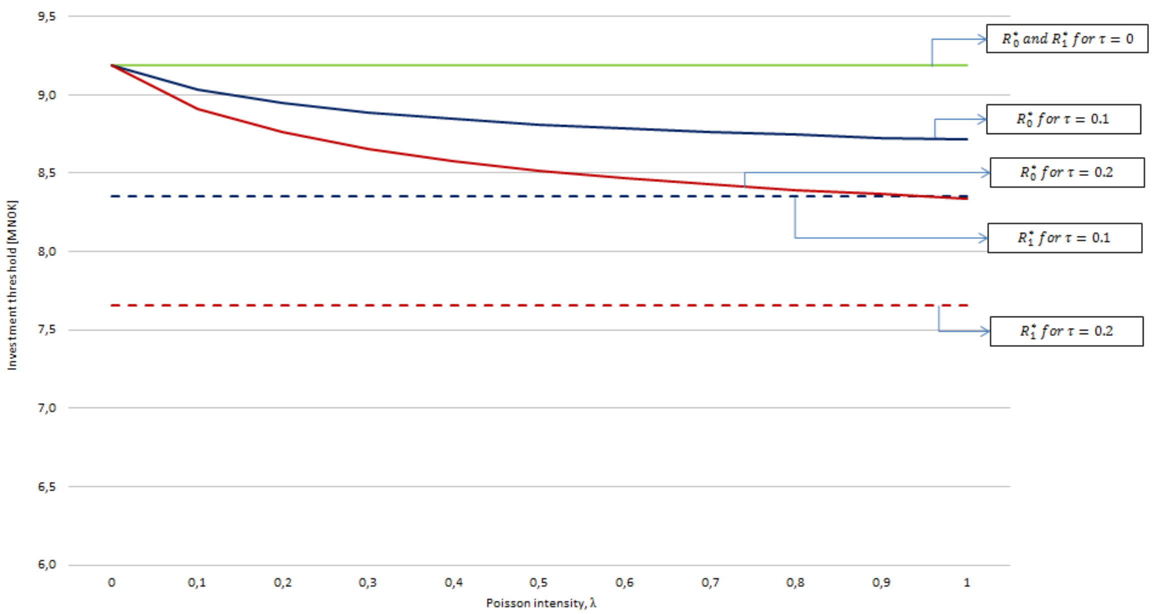


Figure 7.15: Investment thresholds, R_1^* and R_0^*, as a function of λ for $\tau = 0$, $\tau = 0.1$ and $\tau = 0.2$.

Figure 7.15 shows that R_0^* is a decreasing function in λ for $\tau > 0$. The decreasing convex shape of R_0^*, is a result of the expected time to arrival of the innovation, $\mathcal{E}_\lambda[Y] = \frac{1}{\lambda}$, which is a convex function. When $\lambda = 0$, the innovation will never arrive and the investment threshold R_0^* is constant for all τ values. Conversely, when $\tau = 0$, the innovation rate λ is insignificant as there is no real benefit associated with the arrival of the innovation, and therefore $R_0^* = R_1^*$. For the boundary value $\lambda = 1$, the innovation will arrive in the next time period. Hence the investment threshold R_0^*, approaches the investment threshold R_1^*. To understand why the threshold R_0^* approaches but does not meet R_1^* perfectly at $\lambda = 1$ ($\tau > 0$), it is necessary to consider the integral given by Eq. (7.1) describing the value of the investment opportunity when $R \geq R_0^*$ and $\delta = 0$

$$\mathcal{E}_{R,Y} \int_0^Y e^{-(r-\mu_\pi Q)t} R dt + \mathcal{E}_{R,Y} \int_Y^\infty e^{-(r-\mu_\pi Q)t} R(1+\tau) dt - K \tag{7.1}$$

Y is an exponentially distributed variable describing the time until the Poisson event, indicating the arrival of a technological innovation. The expected value of Y, $\mathcal{E}[Y] = 1$ for $\lambda = 1$, and therefore the integral is equal to:

$$\mathcal{E}_R \int_0^1 e^{-(r-\mu_{\pi Q})t} R dt + \mathcal{E}_R \int_1^\infty e^{-(r-\mu_{\pi Q})t} R(1+\tau) dt - K \tag{7.2}$$

The value of the investment is therefore close, but not equal to the expected value of the investment when $R = R_1^*$ and $\delta = 1$, expressed as:

$$\mathcal{E}_R \int_0^\infty e^{-(r-\mu_{\pi Q})t} R(1+\tau) dt - K = \frac{R_1^*(1+\tau)}{r - \mu_{\pi Q}} - K \tag{7.3}$$

By definition, R_0^* describes the investment threshold for investing before the arrival of the technological innovation. Even if $\lambda = 1$, the post-smolt facility still has to be operated for one time period without the extra profits associated with the innovation. As $\Delta t \to 0$ the facility will be operated without the innovation for an infinitesimally small period, and R_0^* will converge towards R_1^* as $\lambda \to 1$.[3]

Eq. (7.1) is also key to understanding the threshold behavior in between the two boundary cases, $\lambda = 0$ and $\lambda = 1$. As the salmon farming company gets the benefits of the innovation at no cost, regardless of investment timing, R_0^* must be a strictly decreasing function in λ bounded by R_0^* at $\tau = 0$ and R_1^*. The only change incurred by an increasing λ, is the proportion of time operated with and without the increased profits, as illustrated in figure 7.16. This result deviates from the result of Chronopoulos et al. (2015) who model a subsidy which is not received if introduced after investment is made.

[3] Solving the value matching and smooth pasting equation set when replacing Eq. (7.1) with Eq. (7.3) for the domain $R \geq R_0^*$ and $\delta = 0$, gives $R_1^* = R_0^*$.

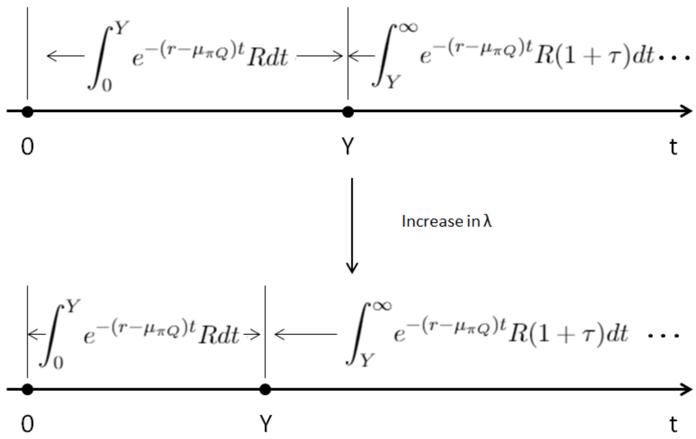


Figure 7.16: An increase in λ decreases the expected time until the Poisson event, $\mathcal{E}_\lambda[Y]$. Hence, the proportion of time the facility is operated with increased profits is larger.

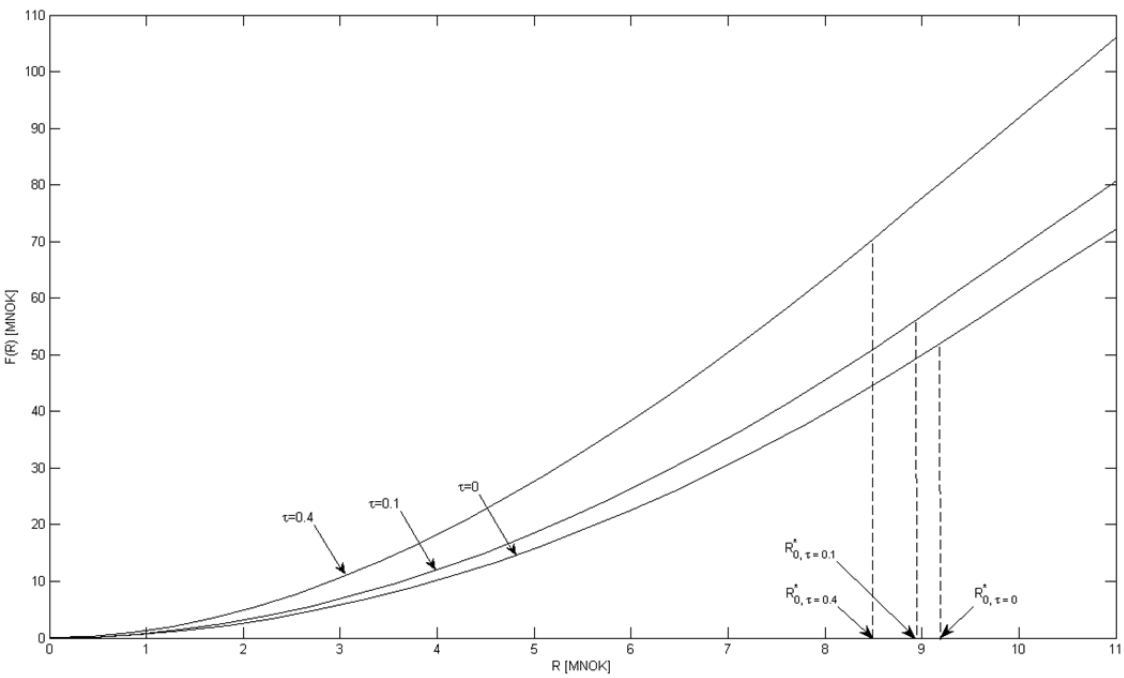


Figure 7.17: Value of investment opportunity, $F(R)$, for $\tau = 0$, $\tau = 0.1$ and $\tau = 0.4$

As illustrated in Figure 7.17, the value of the investment opportunity, $F(R)$, increases and the investment threshold decreases with the size of the profit improvement factor τ. This is in line with our intuition, as a higher τ yields a higher expected value of the project, making the salmon farming company willing to make the investment at a lower total annual profit.

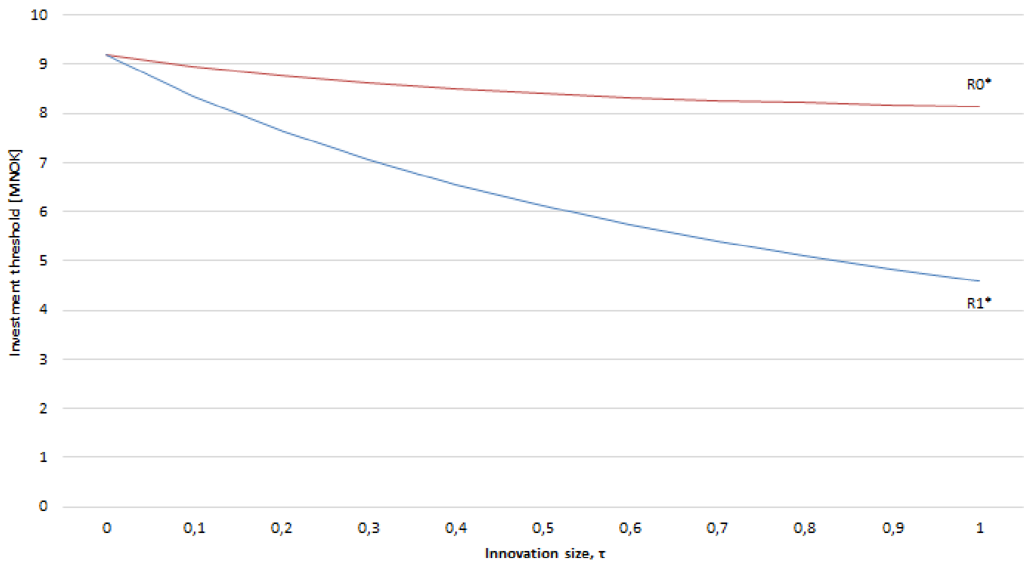


Figure 7.18: Investment thresholds, R_1^* and R_0^*, as a function of τ

Figure 7.18 shows that an increase in the innovation size τ lowers both investment thresholds. The decrease is steeper for R_1^* than for R_0^*, as a salmon farming company that invests at R_1^* enjoys the benefit of the innovation for the whole lifetime of the project, while an investment made at R_0^* implies that the benefit from the innovation is received further out in time.

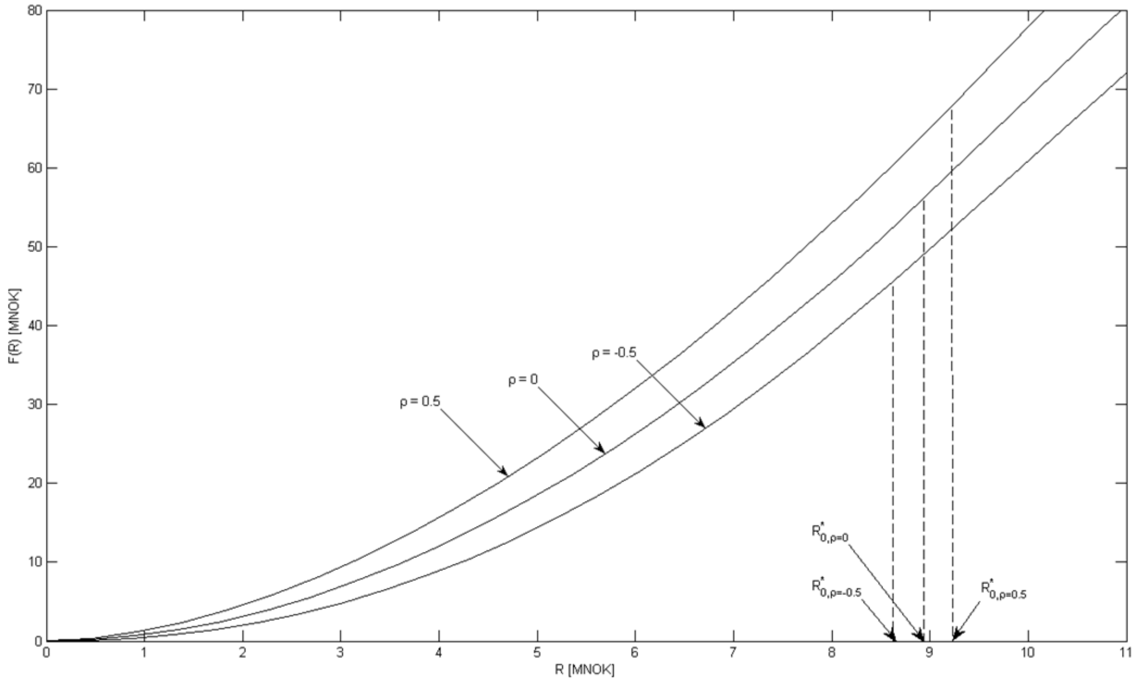


Figure 7.19: Value of investment opportunity, $F(R)$, for $\rho = -0.5$, $\rho = 0$ and $\rho = 0.5$

ρ denotes the correlation between unit profit π, and annual quantity produced Q. The response of $F(R)$ to changes in ρ is similar to the response to changes in σ. A positive ρ gives both a higher option value and higher investment thresholds, while a negative ρ gives lower option value and lower investment thresholds. The intuition is that positive correlation increases the uncertainty in the model, as a change in one parameter is likely to be amplified by a change in the other parameter. For instance, an increase in unit profit would likely mean an increase in quantity, which would impact R more than if the correlation was zero. In the case of negative correlation, the impact of a change in unit profits on R is likely to be dampened by an opposite change in quantity.

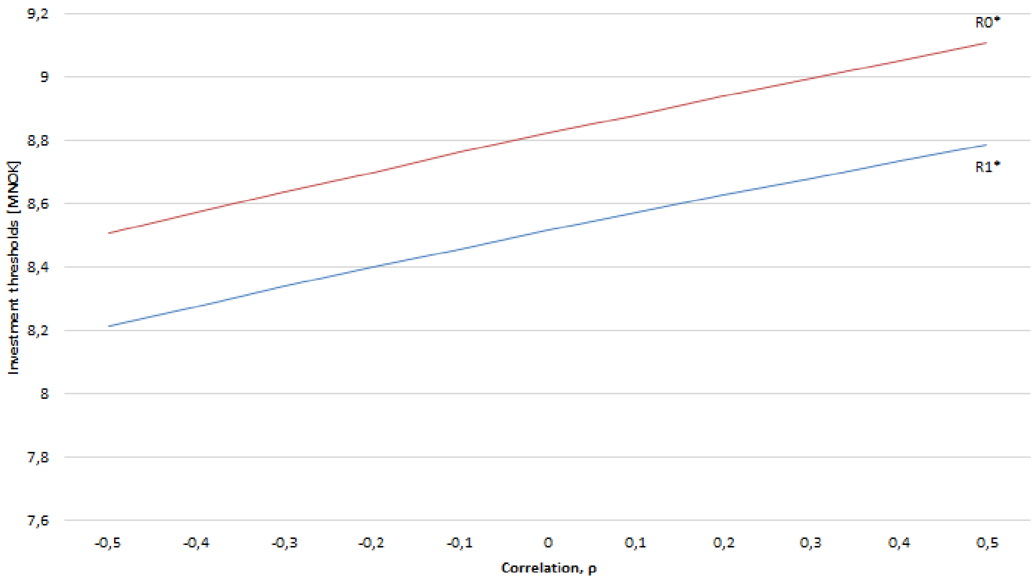


Figure 7.20: Investment thresholds, R_1^* and R_0^*, as a function of ρ

Figure 7.20 shows that an increase in the correlation of the two underlying stochastic processes gives a increase in the investment threshold both with and without the presence of an innovation. This is in line with our intuition, as an increase in ρ increases the uncertainty in the model.

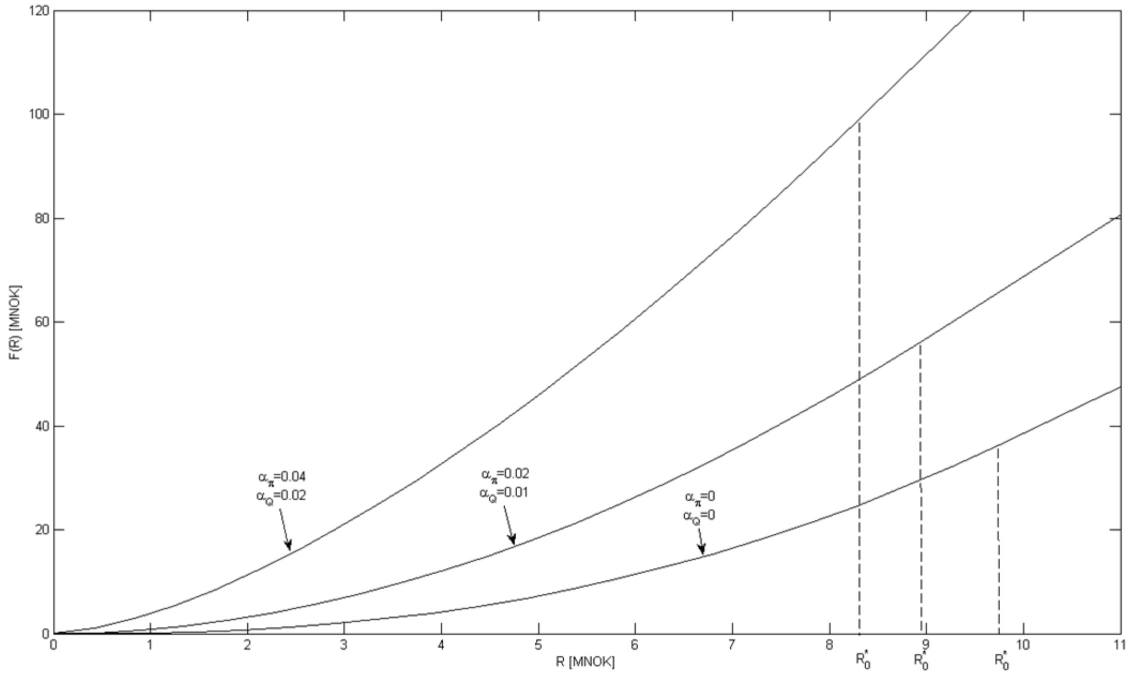


Figure 7.21: Value of investment opportunity, $F(R)$, for $\alpha_\pi = 0$ and $\alpha_Q = 0$, $\alpha_\pi = 0.02$ and $\alpha_Q = 0.01$ and $\alpha_\pi = 0.04$ and $\alpha_Q = 0.02$

Figure 7.21 shows that a higher growth rate increases the value of the investment opportunity and decrease the investment thresholds. The threshold development for increasing growth rates is more directly shown in Figure 7.22. It shows that R_0^* and R_1^* are decreasing convexly in both higher per unit profit and quantity growth rates.

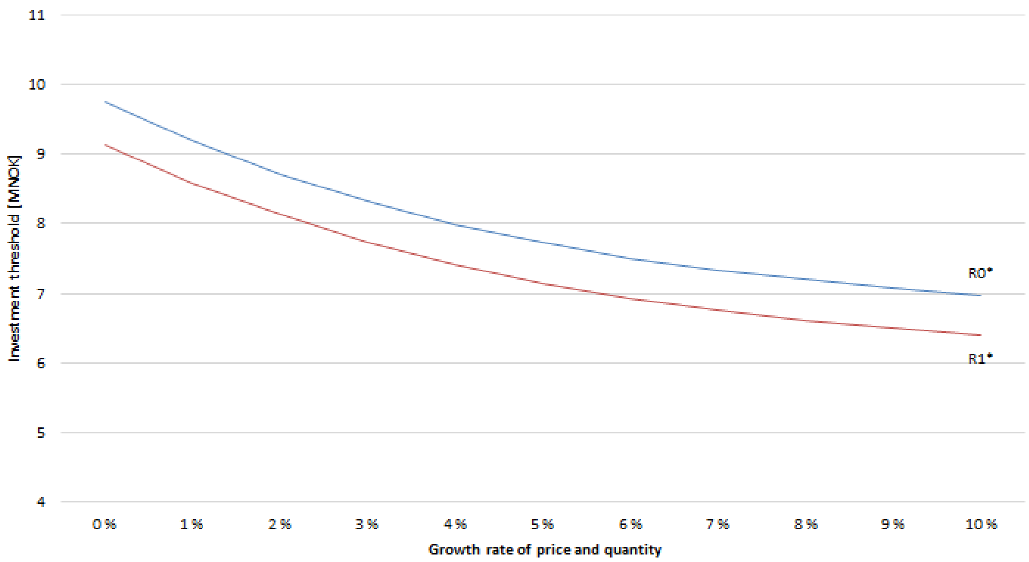


Figure 7.22: Investment thresholds, R_1^* and R_0^*, as a function of α

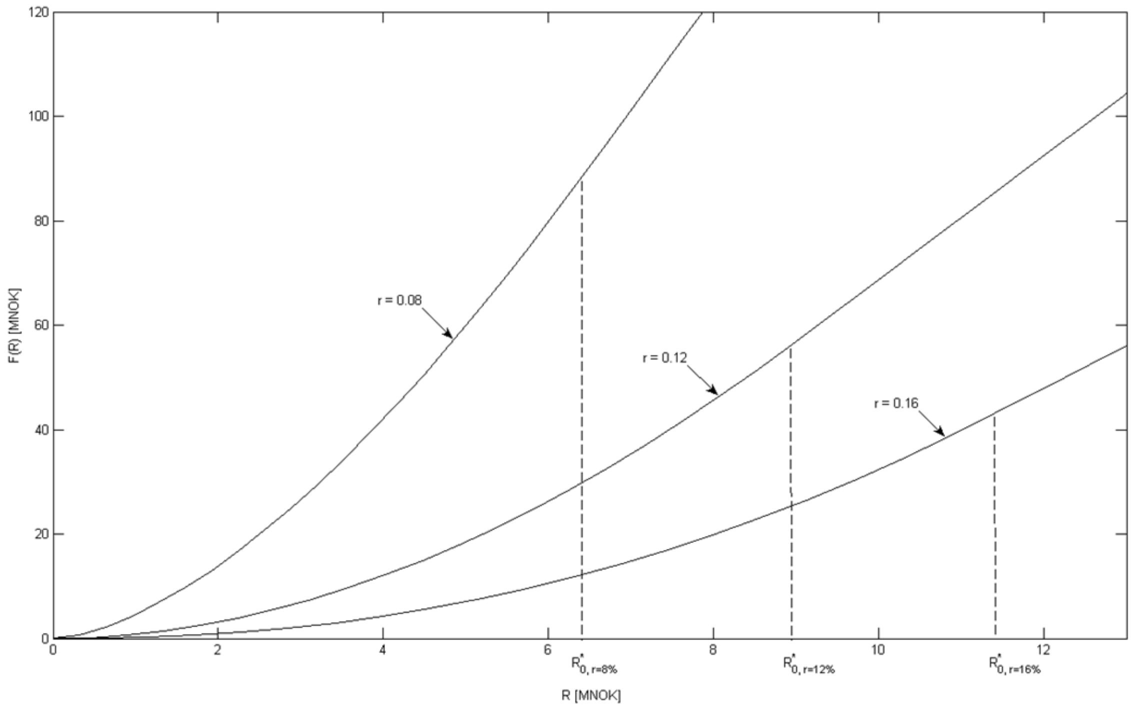


Figure 7.23: Value of investment opportunity, $F(R)$, for $r = 0.08$, $r = 0.12$ and $r = 0.16$

Figure 7.23 shows how $F(R)$ and the threshold R_0^* vary for different values of the discount rate r. An increase in the discount rate leads to a reduced value of the investment opportunity and higher investment thresholds. As neoclassical investment theory suggests, an increase in r reflects a more risky investment and leads to higher investment thresholds as the salmon farming company demands a higher return on its investment. The threshold development for increasing discount rates is more directly shown in Figure 7.24. It shows that R_0^* and R_1^* are increasing in higher discount rates. As for Model 1, we see that F(R) and the thresholds are highly sensitive to changes in growth and discount rate.

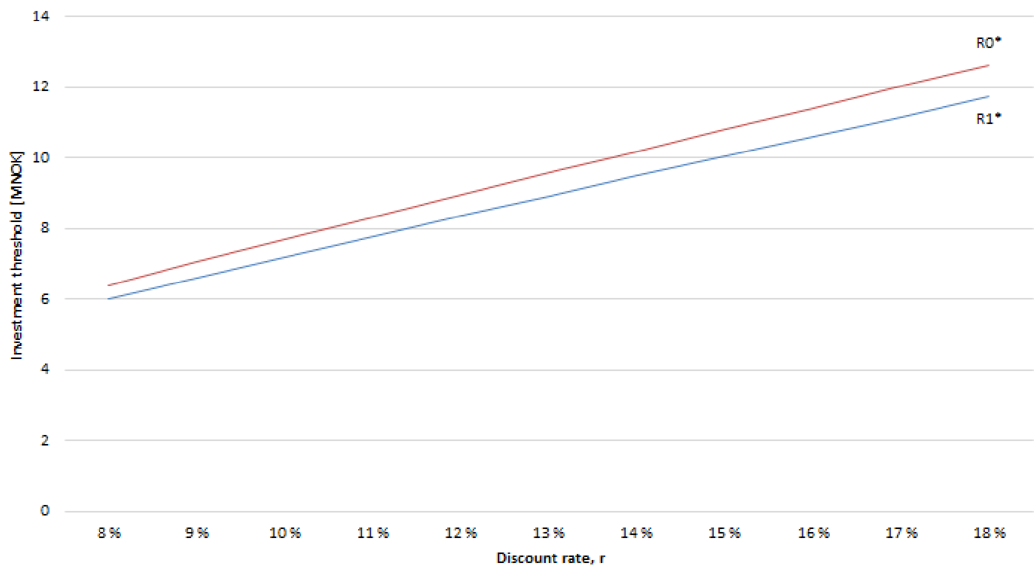


Figure 7.24: Investment thresholds, R_1^* and R_0^*, as a function of r

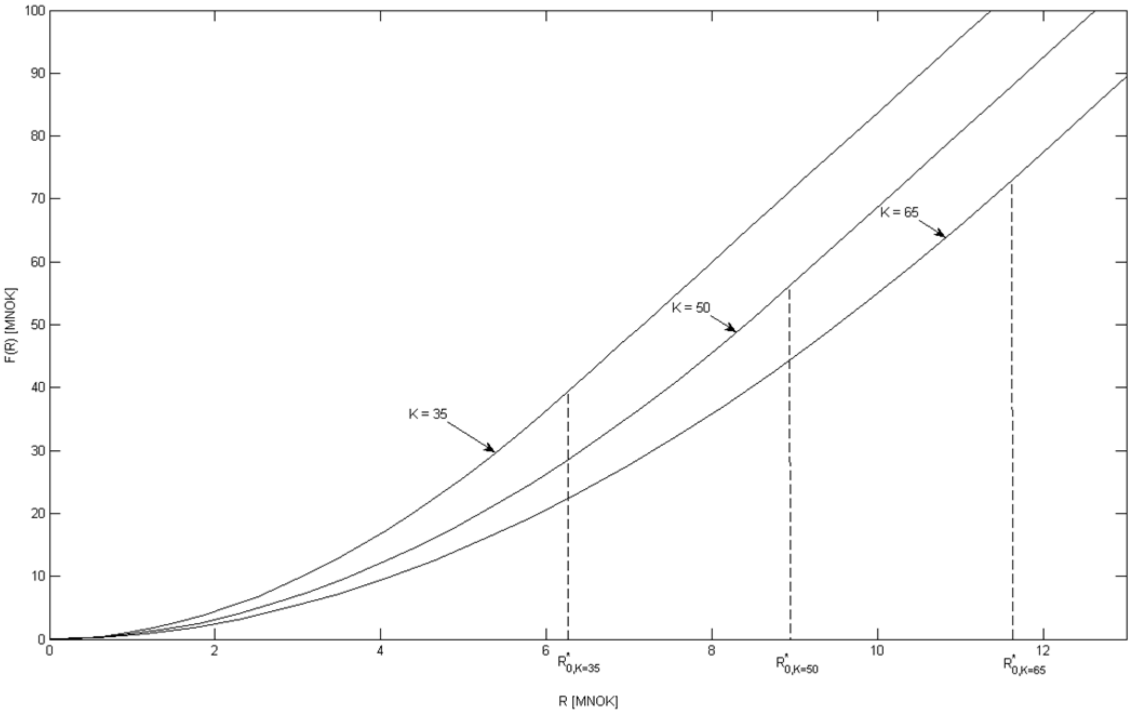


Figure 7.25: Value of investment opportunity, $F(R)$, for $K = 35$, $K = 50$ and $K = 65$

Figure 7.25 shows that an increase in the investment cost K, gives a lower value of the investment opportunity $F(R)$, and higher thresholds. At a higher K, the salmon farming company demands a higher profit level to compensate for the larger investment cost. The increase of the investment thresholds in K is shown in Figure 7.26.

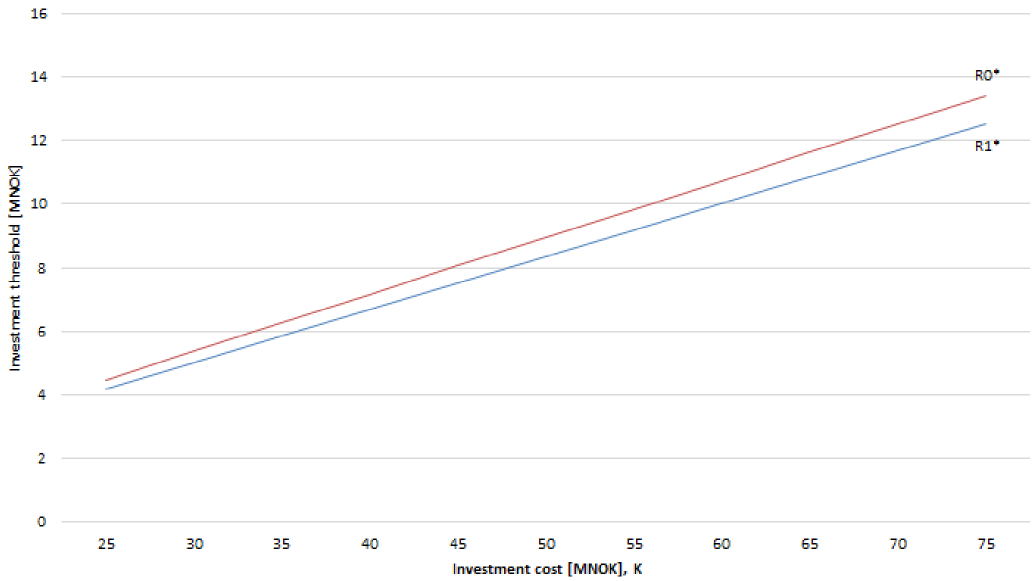


Figure 7.26: Investment thresholds, R_1^* and R_0^*, as a function of K

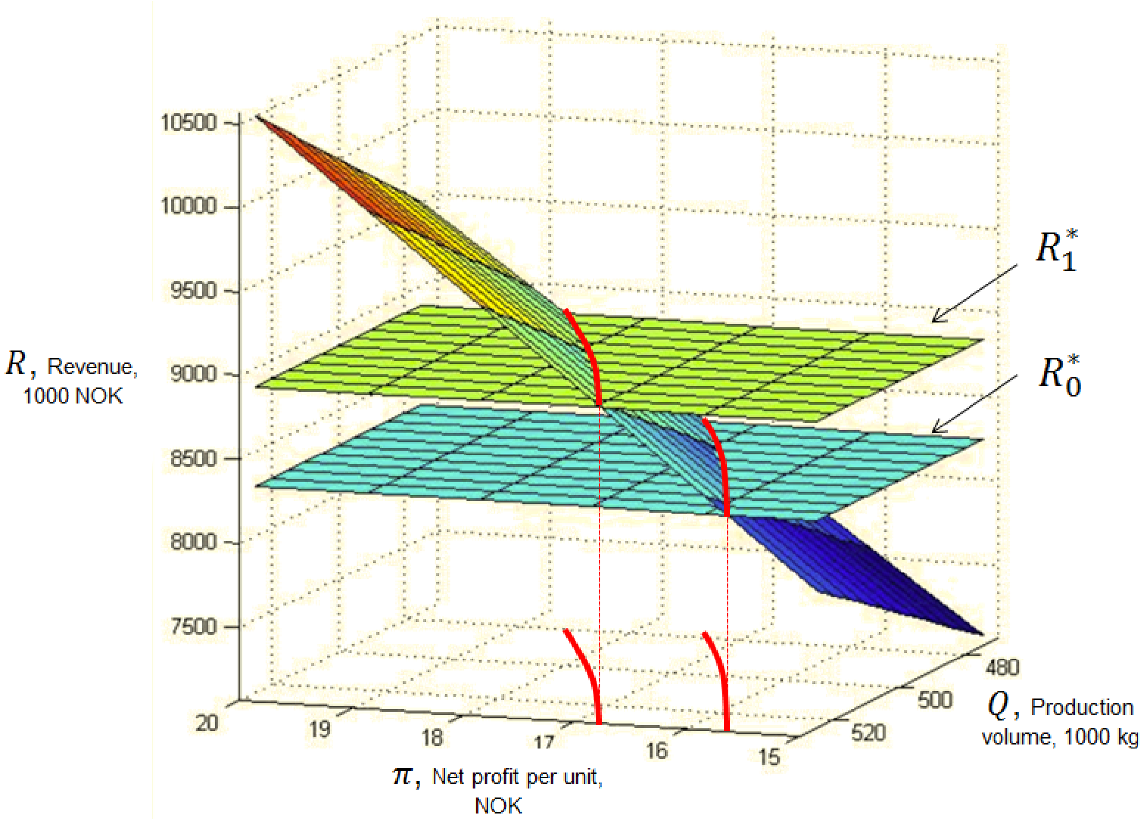


Figure 7.27: Investment thresholds, R_1^* and R_0^*, as a function of π and Q

So far, we have given the investment thresholds as a function of the total annual profit R. As stated in Section 5.2, R is equal to the product of the per unit profit π and the annual quantity produced Q, i.e. $R = \pi Q$. In Figure 7.27 we show how our results coincide with the original two dimensional problem. The red lines highlight the intersection between the total profit function and the horizontal threshold planes R_0^* and R_1^*. The curved shape indicates that the thresholds can be reached either by a high Q and low π, low Q and high π or a combination of both. By mirroring the intersection lines of the profit surface and the threshold surfaces to the (π, Q)-plane, we can read that at a production quantity Q of 520 tons per year, for $\delta = 0$, the salmon farming company would demand a per unit profit π of approximately NOK 17 in order to make the investment, while for $Q = 500$, $\delta = 0$, it would demand $\pi \approx$ NOK 17.7.

7.3 Summary of findings

Our results show that evaluating the specific investment case using real options valuation uncovers significant excess value compared to a traditional discounted cash flow analysis. In the traditional DCF investment threshold, i.e. when NPV is zero, the value of being able to delay the investment, given an investment cost of MNOK 50, exceeds MNOK 12 in both real options models (see Figures 7.2 and 7.11). More specifically, when the DCF analysis suggests investment, the real options models find that the value of being flexible to wait for optimal investment condition is almost 30 percent of the initial investment cost. Given initial conditions (R_{DCF} = MNOK 6.125), the intrinsic values in both Model 1 and 2 are positive, meaning that a traditional DCF analysis would suggest immediate investment. However, both real options analyses suggest waiting for a more beneficial investment timing. From running the simulation procedures presented in Section 5.3 under base case assumptions, we find the expected first passage time of the investment thresholds to be 11 and 5 years for Model 1 and Model 2 respectively (see Appendix I for MATLAB code).

In Tables 7.1 and 7.2 below we summarise the main findings of the sensitivity analysis. The direction of the arrows in the tables indicate the effect of an increase in the specific input parameter on the value of investment opportunity and the investment threshold. The tables show that the two real options models generally react similarly to changes in the input parameters. The exceptions are changes in innovation arrival rate, λ. For higher values of the innovation arrival rate λ, the investment threshold in Model 1 rises, while it decreases in Model 2. By modelling technological uncertainty in two different ways we prove that the investment strategy is greatly influenced by whether or not the benefit of technological progress is gained after the investment is made - the salmon farming company has weaker incentives to delay investment in the former than in the latter case. The total amount of technological uncertainty is governed by the relationship between innovation arrival rate and innovation impact. The expectation of frequent but small innovations, and few but large innovations, have a similar effect on both option value and investment threshold in each of the models. On a final note we emphasise that the behaviour of our models coincide with those of classic real options theory (see Dixit and Pindyck (1994)) adding credibility to the study. In the next chapter we discuss our findings w.r.t related literature, as well as the modelling assumptions and practical applicability of the case study.

Parameter	**Symbol**	$f(p)$	p^*
Investment cost	K	$\downarrow$	$\uparrow$
Discount rate	r	$\downarrow$	$\uparrow$
Innovation arrival rate	λ	$\uparrow$	$\uparrow$
Investment cost reduction factor	ϕ	$\downarrow$	$\downarrow$
Profit volatility	σ_R	$\uparrow$	$\uparrow$
Profit drift	α_R	$\uparrow$	$\downarrow$

Table 7.1: Effect of increase in input parameters on the value of $f(p)$ and on the investment threshold p^* in Model 1

Parameter	**Symbol**	$F(R)$	R_0^*, R_1^*
Investment cost	K	$\downarrow$	$\uparrow$
Discount rate	r	$\downarrow$	$\uparrow$
Innovation arrival rate	λ	$\uparrow$	$\downarrow$
Profit improvement factor	τ	$\uparrow$	$\downarrow$
Profit volatility	σ_π	$\uparrow$	$\uparrow$
Profit drift	α_π	$\uparrow$	$\downarrow$
Quantity volatility	σ_Q	$\uparrow$	$\uparrow$
Quantity drift	α_Q	$\uparrow$	$\downarrow$

Table 7.2: Effect of increase in input parameters on the value of the investment opportunity $F(R)$ and on the investment thresholds R_0^* and R_1^* in Model 2

Chapter 8

Discussion

Through forefront multi-factor real options modelling we have managed to uncover additional value compared to the traditional DCF method. In this chapter we discuss our main contributions w.r.t. related literature. We also discuss some of the challenges associated with implementing a real options approach and why it is not currently used by the salmon farming industry.

Main contributions w.r.t. related literature

In order to consider both the impact of uncertainty in technology and profits, we have applied forefront multi-factor real options modelling. This allowed us to gain insight in how technology and profit uncertainty affect the optimal investment strategy, and to derive the optimal investment timing. Model input parameters have been quantified in close cooperation with industry experts.

We adapt the model for investment under technology and revenue related uncertainty proposed by Murto (2007) to the case of post-smolt investment. To derive analytical solutions, Murto (2007) only considers three special cases: (1) Deterministic price process ($\sigma = 0$), (2) deterministic technological progress ($\lambda \to \infty$ and $\phi \to 1$) and (3) full collapse of the investment cost ($\phi = 0$). Doing so he is able to give intuition on how revenue and technology related uncertainty affects the investment problem, when the relationship between the two factors affecting technological uncertainty, arrival rate λ and investment cost reduction factor ϕ, is fixed. However, he is not able to assess the effect of independent changes in λ and ϕ, nor is he able to determine the option value $f(p)$ or the investment threshold p^* for arbitrary input parameters. Hence, solving the model throughout the entire solution space adds significant value. Therefore we have extended his work by developing a numerical solutions approach that allows us to solve the model. We are thereby able to give additional and more nuanced insight as to how profit and technology uncertainty affect the investment problem.

Additionally, an important contribution to the existing literature is the application of real options techniques originally proposed in papers considering investment in power generation plants under policy uncertainty, to an investment under technological uncertainty. Adkins and Paxson (2013) and Chronopoulos et al. (2015) consider an investment in a power generation plant with the possible sudden provision or retraction of a government subsidy. The subsidy is in the form of a cash payment proportional to the revenues of the firm and can only be gained if it is introduced before the investment is made. Adkins and Paxson (2013) consider uncertainty in both price and quantity, whereas Chronopoulos et al. (2015) only consider price uncertainty while finding optimal quantity. We find that the sudden provision of a subsidy resembles the arrival of a technological innovation, enabling us to adapt and apply the frameworks of Adkins and Paxson (2013) and Chronopoulos et al. (2015) to an investment under technological uncertainty. One important difference between a subsidy and a technological innovation is that the subsidy cannot be gained after the investment is made. We have therefore extended their models so that they fit our case, where the benefit of a technological innovation is gained also after the investment is made. Our analyses show that the salmon farming company has stronger incentives to invest early if the benefit of the technological innovation is gained regardless of investment timing. This insight can be related back to the investment problem considered by Adkins and Paxson (2013) and Chronopoulos et al. (2015): As the main purpose of a subsidy scheme is to strengthen incentives to invest sooner rather than later, our analysis indicates that subsidy schemes should be adapted so that a subsidy can be gained regardless of investment timing.

Modelling assumptions and practical applicability

In the following we discuss how our modelling assumptions limit the practical applicability of the case study. The discussion is based on Lander and Pinches (1998)'s three proposed reasons why real options models are not applied more widely in practice:

1. The required modeling assumptions are often violated in a practical real options application,

2. The necessary additional assumptions required for mathematical tractability limit the scope of applicability,

3. The types of models currently used are not well known or understood by corporate managers and practitioners.

First, we have assumed that the state variables in the models follow well-defined stochastic processes. However, it is not clear what is the correct stochastic processes, and our models may be highly sensitive to how we have modelled the behaviour of the state variables. To enable a realistic comparison of the traditional net present value and the real options value we needed to consider variable costs. We did this by assuming that per unit profits follow a GBM. This is not a common assumption in theoretic real options literature, however a similar assumption is made by Himpler and Madlener (2014), who also extend Murto (2007). A more common assumption in real options literature is that there are no variable costs. This entails that unit price equals unit profit. Making this assumption would however reduce the realism of our results, as the payback period would be less than one and a half years given initial parameter values (see Chapter 6).[1] Consequently, our results would give a false impression of how favorable an investment in post-smolt production is.

Second, we have made additional assumptions to enable mathematical tractability. The investment problem is assumed to only contain one option: invest now, or otherwise wait. In reality however, there would be several more options embedded in the investment problem, such as abandonment, mothballing, stepwise investment and stepwise R&D (see Dixit and Pindyck (1994); Chronopoulos et al. (2015); Bowe and Lee (2004)). By adding flexibility to the investment problem, the real options value would increase and our results would be more nuanced. However, including these embedded options would increase the mathematical complexity of the problem significantly. Furthermore, we consider a small salmon farming company that does not have the same financial flexibility as the industry majors. Hence, it is not relevant to model embedded options such as stepwise R&D, or mothballing the facility in the likely scenario of an economic downturn. By extension, one

[1] Payback period $= \frac{\text{Investment cost}}{\text{Volume per year} \times \text{unit profit}} = \frac{\text{MNOK 50}}{\text{500,000kg per year} \times \text{NOK 73.5per kg}} = 1.36$ years

can argue that it currently makes more sense for the majors to invest in post-smolt. They have more options embedded, increasing their flexibility and in turn the real options value of the investment. For the smaller salmon farming companies it is generally better to let the majors drive the technological innovation, and invest when the technology is sufficiently mature.

To further allow for mathematical tractability, the investment opportunity is assumed to be perpetual in both models. This is not necessarily a realistic assumption, as most investment opportunities have a finite time horizon. Due to the high discount rate of 12 percent (See Chapter 6), the present value of cash flows far into the future is close to zero, and has negligible impact on the investment decision. Thus in practice, our models consider a time-bound investment opportunity.

Similarly to e.g. Bernanke (1980) and Cukierman (1980), we have assumed in both models that profit generation starts instantaneously after the investment. Implicitly we assume it takes no time to set up the facility, which we acknowledge as an unrealistic assumption. However, the time lag affects both the DCF and the real options value similarly, and is therefore not expected to affect the comparison. In addition, we do not distinguish implementation from *successful* implementation. Stenbacka and Tombak (1994) however argue that the new technology will improve the equilibrium profits of the adopting firm only if it is successfully implemented. They model uncertainty w.r.t. the time between adoption of the new technology and successful implementation. The probability of success in a given time period increases as the firm gains experience using the new technology. Therefore, Stenbacka and Tombak (1994) present the process of technology adoption as a "time consuming activity". However, as we focused on the uncertainty related to the arrival of technological innovations, we chose not to consider uncertainty related to implementation.

In Model 1 we have not accounted for a possible correlation between profits and the innovation arrival rate. There are however arguments supporting both a positive and a negative correlation. If profits in post-smolt production were to rise, it is likely that the market would become more attractive and as a result grow larger. Consequently the technology suppliers would likely increase their R&D efforts to gain market share, leading to an increase in the innovation arrival rate. Conversely, in an economic downturn companies would demand more cost effective technology, which could also increase innovation arrival rate. Due to the lack of available empirical data, we are not able to draw conclusions about correlation, and it is therefore omitted.

Our models do not consider the effect of competition between salmon farming companies. In reality a company's choice of technology migration strategy would be conditioned on its expectations about its rivals migration strategies. This could have been modelled using a game theoretic real options approach similar to Huisman and Kort (2004). However, incorporating game theory in the models would entail a significant increase in the complexity and might deprive focus from the main objective of the thesis. Furthermore, we have previously argued that it makes more sense for a small salmon farming company to be a second-mover compared to the industry majors in terms of a post-smolt investment. Hence modelling the game between first- and second-movers is not relevant for our investment case.

In Chapter 7 we show that a higher level of uncertainty increases the real options value of an investment opportunity. To exploit the additional value, the salmon farming company must be able to respond to new information about the market conditions. Koller et al. (2010) illustrate this by Figure 8.1 below:

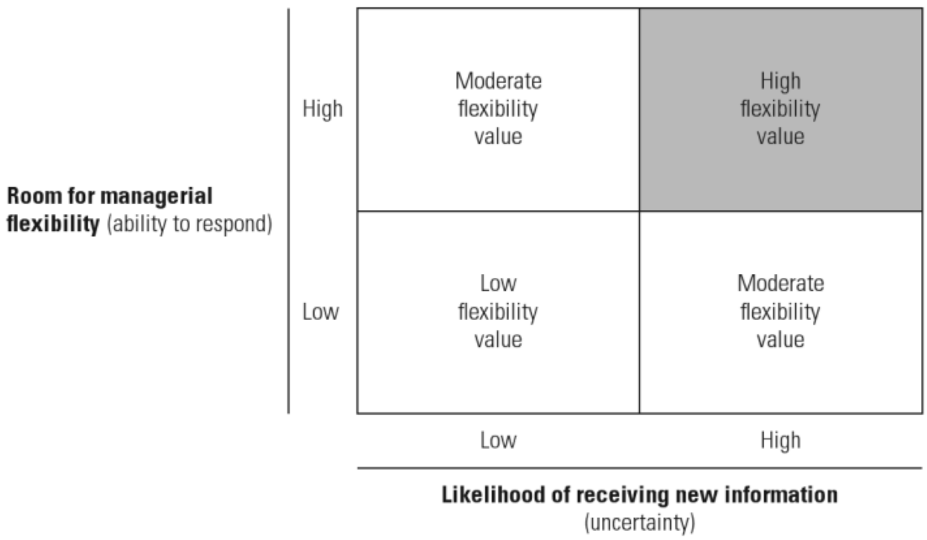


Figure 8.1: The value of a real option is high when uncertainty is high, and the management have the ability to respond to new information.

In large companies an investment decision needs to pass several layers of hierarchy, limiting their flexibility. This places them in the bottom right corner of Figure 8.1, as they are only moderately able to exploit the additional value given by a real options analysis. The small company we consider is however assumed to have a flat structure, which enables a quick investment decision. This places the company in the top right corner of Figure 8.1, as they have a high ability to exploit the additional value given by a real options analysis. This supports our choice of applying a real options approach to the specific investment problem.

Third, considering the mathematical complexity of the real options models, we acknowledge the fact that they can be difficult to grasp without a solid understanding of real options theory. Naturally, the lack of understanding might cause reluctance among investment managers to rely on the case study results. This poses a serious challenge in terms of making salmon farming companies confident on the business potential real options analysis can uncover. Interviewing executives in the salmon farming industry we have found that they are generally quite satisfied with the traditional capital budgeting methods, and do not see the need to value uncertainty. Bodén and Åhlén (2007) draw similar conclusions. A common response among executives when confronted with the possibility of applying

ROV was: "What's in it for me?". Thus, we conclude that challenging the incumbent DCF method is going to be a tough and timely task.

Chapter 9

Conclusion

The objective of this thesis has been to solve the following problem:

Can real options analysis uncover additional value compared to traditional capital budgeting methods when evaluating an investment in a post-smolt facility under technology and profit uncertainty?

Our main findings show that evaluating the specific investment case by real options valuation uncovers significant excess value compared to a traditional discounted cash flow method. In both real options models, the value of being able to delay the investment in the traditional DCF investment threshold, i.e. when NPV is zero, is close to 30 percent of the investment cost (see Figures 7.2 and 7.11). By modelling technological uncertainty in two different ways we prove that the investment strategy is greatly influenced by whether or not the benefit of technological progress is gained after the investment is made - the salmon farming company has weaker incentives to delay investment in the former than in the latter case. The total amount of technological uncertainty is governed by the relationship between innovation arrival rate and innovation impact. The expectation of frequent but small innovations, and few but large innovations, have a similar effect on both option value and investment threshold in each of the models. In practice our findings implicate that by relying on traditional capital budgeting methods, the salmon farming companies can end up rejecting sensible business opportunities within post-smolt production. More specifically, given the current characteristics of post-smolt production, traditional capital budgeting methods underestimate its potential, possibly preventing the industry from taking the next step towards meeting the growing global demand.

In order to consider the impact of uncertainty in technology and profits simultaneously, we have applied forefront multi-factor real options modelling. We have also shown how technology and profit uncertainty affect the optimal investment strategy deriving the optimal investment timing. Model input parameters have been quantified in close cooperation with industry experts. In terms of investment strategy, given initial conditions, the intrinsic values in both Model 1 and 2 are positive, meaning that a traditional DCF analysis would suggest immediate investment. However, both real options analyses suggest waiting for more beneficial investment conditions.

Our main contribution has been threefold: 1) We have examined the potential of real options valuation in salmon farming; 2) We have extended the real options theory within the area of theoretical multi-factor models; 3) We have applied real options valuation to a specific post-smolt investment case, thereby adding to the ongoing discussion of the economic viability of post-smolt production.

First, to the best of our knowledge we are among the first to examine the potential of real options valuation within salmon farming. The majority of academic literature focuses on the biology rather than the economics of salmon farming. The few available economic papers are based either on traditional capital budgeting methods or optimisation of operations. We have contributed to the salmon farming industry by providing a study giving intuition on how uncertainty in technology and profits affects an investment in post-smolt production. The study can improve decision-making under uncertainty and potentially help the industry to take the next step towards meeting the growing global demand.

Second, multi-factor real options models are a recent development in theoretical real options literature. Hence, developing these models further is an important contribution. By developing two multi-factor real options models, we have captured different aspects of investing under technology and profit uncertainty. We have adapted the model for investment under technology and revenue related uncertainty proposed by Murto (2007) to the case of post-smolt investment, and extended the work by developing a numerical procedure that allows us to solve the model. Additionally, we have applied real options techniques originally proposed in papers considering investment in power generation plants under policy uncertainty, to an investment under technological uncertainty. Adkins and Paxson (2013) and Chronopoulos et al. (2015) consider an investment in a power generation plant with the possible sudden provision or retraction of a government subsidy. The subsidy is in the form of a cash payment proportional to the revenues of the firm and can only be gained if it is introduced before the investment is made. Adkins and Paxson (2013) consider uncertainty in both price and quantity, whereas Chronopoulos et al. (2015) only consider price uncertainty while finding optimal quantity. We have found that the sudden provision of a subsidy resembles the arrival of a technological innovation, which has enabled us to adapt and apply the frameworks of Adkins and Paxson (2013) and Chronopoulos et al. (2015) to investments under technological uncertainty. One important difference between a subsidy and a technological innovation is that the subsidy cannot be gained after the investment is made. We have therefore extended their models so that they fit our case, where the benefit of a technological innovation is gained also after the investment is made. Our analyses show that the salmon farming company has stronger incentives to invest early if the benefit of the technological innovation is gained regardless of investment timing. This insight can be related back to the investment problem considered by Adkins and Paxson (2013) and Chronopoulos et al. (2015): As the main purpose of a subsidy scheme is to strengthen incentives to invest sooner rather than later, our analysis indicates that subsidy schemes should be adapted so that a subsidy can be gained regardless of investment timing.

Third, due to the high degree of uncertainty related to post-smolt production technology, the industry is reluctant to undertake investments. Currently the industry relies on traditional capital budgeting methods that only treats the downside potential of uncertainty. We have contributed by proposing multi-factor real options models that also capture the upside potential of the uncertainty embedded in post-smolt production. As there is little available data related to post-smolt production and market prices, the models' underlying assumptions and parameters have been chosen in close collaboration with both biological researchers and representatives from the industry majors. Thus, our results represent a serious contribution to further discussions of the economical viability of post-smolt production.

However, regardless of the compelling results, we emphasise the application value of our models as a study giving intuition on how uncertainty in technology and profits affects the investment problem. The models are based on several assumptions that limit the applicability of the absolute values presented. As Luehrman (1998) and Van Putten and MacMillan (2004) we conclude that real options valuation should be used as a complement to, and not a substitute for traditional DCF analysis. We also emphasise that the two real options models presented are complements, and not substitutes for each other, as they give insight on two different forms of technology and profit uncertainty.

9.1 Further research

This study has offered an economic perspective on post-smolt production, one of the most promising developments within salmon farming. The scope of the thesis has been to examine the effects of technology and profit uncertainty on project value and optimal investment strategy. In the following we suggest five exciting extensions for potential future research:

First, we assumed profit development to follow a GBM. As salmon has many of the characteristics of a commodity, the post-smolt price can be modelled as a mean-reverting process. Alternatively, more complex price developments such as mean reversion with jumps could be implemented, to also allow for the post-smolt price to jump as a result of salmon farming companies having encountered mass mortality and are willing to pay a high price for larger smolt to recover production. Second, as a post-smolt case study on a larger company would include more options than just the option to defer investment, our models could be extended by including embedded options such as stepwise investment, and expansion or abandonment options. Third, there is uncertainty tied to the future policies set for post-smolt production. Currently, salmon produced in closed production facilities is not included in the MAB, which represents a great advantage for post-smolt production. A possible alteration of the current policy poses a risk for salmon farming companies, as they might be forced to include the post-smolt in their MAB. Thus, including policy uncertainty represents another possible extension of the models presented. Fourth, it could be interesting to view the investment problem from a game theoretic perspective of a large company. Given the current lack of commercial suppliers in the post-smolt market, first movers can achieve premium prices. At the same time the first mover would risk losing terrain to second-movers who have awaited superior technology. Fifth, our models could be adapted to specific investment cases with similar properties as post-smolt production. One example is an investment in the development of an oil field where the technology is in rapid development, and where there is uncertainty related to the oil recovery rates and prices.

To conclude, we have proven that taking a real options approach to an investment under profit and technology uncertainty uncovers excess value compared to a traditional discounted cash flow method. The models developed extend the existing real options literature on multi-factor and technology uncertainty modelling. The result is a study giving salmon farming companies intuition on how uncertainty about technology and potential future profits affects an investment in post-smolt production. The insight gained from our thesis can support post-smolt production in the debate of its economic viability. This can potentially revolutionise the salmon farming industry, enabling it to meet growing global demand.

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Appendix A

Derivation details of Model 1 PDE

In this appendix we derive Eq. (5.4):

$$rF(R,K)dt = \mathcal{E}[dF(R,K)] + \lambda[F(R,\phi K) - F(R,K)]dt \tag{A.1}$$

$$\mathcal{E}[dF(R,K)] = \mathcal{E}\left[\frac{\partial F}{\partial R}dR + \frac{\partial F}{\partial K}dK + \frac{1}{2}\left(\frac{\partial^2 F}{\partial R^2}(dR)^2 + 2\frac{\partial^2 F}{\partial R \partial K}dRdK + \frac{\partial^2 F}{\partial K^2}(dK)^2\right)\right]$$

$$= \mathcal{E}\left[F_R\alpha_R Rdt + F_R\sigma_R RdZ_R + F_K Kdq_1 + \frac{1}{2}\left(F_{RR}\alpha_R^2 R^2 dt^2 + F_{RR}\sigma_R^2 R^2 dZ_R^2\right.\right.$$

$$\left.\left. + 2\alpha\sigma RKdtdZ_R + 2F_{RK}(\alpha_R RKdtdq_1 + \sigma_R RKdZ_R dq_1) + F_{KK}(Kdq_1)^2\right)\right]$$

$$= F_R\alpha_R Rdt + F_K K\lambda dt + \left(F_{RR}\alpha_R^2 R^2 dt^2 + F_{RR}\sigma_R^2 R^2 dt + 2F_{RK}\alpha_R RK\lambda dt^2\right.$$

$$\left. + F_{KK}K^2\mathcal{E}(dq_1^2)\right)$$

$$= F_R\alpha_R Rdt + F_K K\lambda dt + \frac{1}{2}F_{RR}\sigma_R^2 R^2 dt$$

Substituting into Eq. (A.1):

$$rF(R,K)dt = F_R\alpha_R Rdt + F_K K\lambda dt + \frac{1}{2}F_{RR}\sigma_R^2 R^2 dt + \lambda[F(R,\phi K) - F(R,K)]dt$$

Reorganising and divinding by dt, we end get the following PDE:

$$\frac{1}{2}\sigma_R^2 R^2 F_{RR}(R,K) + \alpha_R R F_R(R,K) - rF(R,K) + \lambda[F(R,\phi K) - F(R,K)] = 0$$

Appendix B

Explicit Euler scheme to solve Model 1

We rewrite the second order ODE (5.8), into a system of two first degree ODEs, in order to be able to apply Euler's method:

$$y_1 = y$$

$$y_2 = y^{'}$$

$$y_2^{'} = y^{''} = \frac{2[-\alpha_R p f'(p) + (r+\lambda)f(p) - \lambda\phi f(\frac{p}{\phi})]}{\sigma_R^2 p^2}$$

This is implemented in MATLAB by the code displayed in Figure B.1 below:

```
function [ y,z,x ] = case1new( h,p )

x = 0:h:p;%grid
y = zeros(1,length(x));%y1
z = zeros(1,length(x));%y2=y1'

mu = 0.03; %growth rate
r = 0.12; %discount rate
phi = 0.95;%innovation size
lambda = 0.2;% arrival rate
sigma = 0.25; %volatility

y(length(x)) = (p/(r-mu)) -1;    % Boundary condition
z(length(x)) = 1/(r-mu);         % Boundary condition


k1y = z(length(x))*h; %first iteration
k1z = h*2*(-mu*x(length(x))*z(length(x)) +(r + lambda)*y(length(x)) ...
    - lambda*phi*((p/phi*(r-mu))-1))/((sigma)^2*(x(length(x)))^2);
%first iteration,%can use the fact that we know f(p/phi) for f>p*

y(length(x)-1) = y(length(x)) - k1y; %first iteration
z(length(x)-1) = z(length(x)) - k1z; %first iteration

for i=(length(x)-1):-1:2   % calculation loop

    k1y = z(i)*h;
    k1z = h*2*(-mu*x(i)*z(i) +(r + lambda)*y(i)...
        - lambda*phi*(y(i)+((x(i)/phi)-x(i))*z(i)))/((sigma)^2*(x(i))^2);

    y(i-1) = y(i) - k1y;
    z(i-1) = z(i) - k1z;
end
figure
plot(x,y);

end
```

Figure B.1: Explicit Euler scheme implemented in MATLAB.

Appendix C

Implicit finite difference scheme to solve Model 1

```
function [test] = FiniteDifference5_1( h,stop,p )
%Solves model 5.1 using an implicit finite difference scheme

mu = 0; %growth rate
r = 0.12; %discount rate
phi = 0.95;%innovation size
lambda = 0.2;% arrival rate
sigma = 0.25; %volatility

boundary = round((p/h)+1);% converts double to integer

% set up matrices
a = zeros(boundary,boundary);
x = ones(boundary,1);
b = zeros(boundary,1);
grid = 0:h:p;
% Fill in boundary conditions
a(1,1) = 1;
a(boundary,boundary) = 1;
b(boundary,1) = (p/(r-mu)) -1;

c = 2; %counter

% Fills in the matrix "a" along the diagonal
for i= 2:1:length(x)-1

 a(i,c-1)= 0.5*(sigma^2)*grid(c)^2; %f(i-1)
 a(i,c)   = -(sigma^2)*grid(c)^2 - mu*grid(c)*h - (h^2)*(r+lambda) ...
           + lambda*phi*(h^2) - h*lambda*phi*((grid(c)/phi)-grid(c)); %f(i)
 a(i,c+1)= 0.5*(sigma^2)*grid(c)^2 + mu*grid(c)*h...
           + h*lambda*phi*((grid(c)/phi)-grid(c)); %f(i+1)
 c = c+1; %counter ++
end
x = a\b; %solves the matrix equation

condition = 1/(r-mu); % smooth pasting test condition

deriv = (x(boundary)- x(boundary-1))/h;% finds f_n'(p*)

test = condition-deriv; % Checks if we have found the correct p*
```

```
% Code to plot f(p) for p > p*
gridextra = p:h:stop; % extra grid from p* to stop
temp = (stop-p)/h +1; % number of grid points in gridextra
boundary2 = round(temp); %converts double to integer
extra= zeros(boundary2,1);%creates matrix
e = p; % counter

% for loop filling the array extra
for i = 1:1:length(extra)
    extra(i) = (e/(r-mu)) -1;
    e = e + h; %counter++
end

newx = [grid gridextra]; %concatinates the old and new x-axis
newy = [x; extra]; % concatinates the old and new y-axis

hold on;
plot(newx,newy,'Color','k');
xlabel('p');
ylabel('f(p)');

pstarx = [p p];
pstary = [0 x(length(x))];
hold on;
plot(pstarx,pstary,'Color','k');

end
```

Appendix D

Derivation details of Model 2 PDE

Let Eqs. (D.1) and (D.2) represent the value function from Eqs. (5.26) and (5.2) respectively:

$$\frac{1}{2}\frac{\partial^2 F_1}{\partial \pi^2}\sigma_\pi^2\pi^2 + \frac{1}{2}\frac{\partial^2 F_1}{\partial Q^2}\sigma_Q^2 Q^2 + \frac{\partial^2 F_1}{\partial \pi \partial Q}\sigma_\pi \sigma_Q \rho \pi Q + \frac{\partial F_1}{\partial \pi}\alpha_\pi \pi + \frac{\partial F_1}{\partial Q}\alpha_Q Q - rF_1 = 0. \tag{D.1}$$

$$\frac{1}{2}\frac{\partial^2 F_0}{\partial \pi^2}\sigma_\pi^2\pi^2 + \frac{1}{2}\frac{\partial^2 F_0}{\partial Q^2}\sigma_Q^2 Q^2 + \frac{\partial^2 F_0}{\partial \pi \partial Q}\sigma_\pi \sigma_Q \rho \pi Q + \frac{\partial F_0}{\partial \pi}\alpha_\pi \pi + \frac{\partial F_0}{\partial Q}\alpha_Q Q - (r+\lambda)F_0 + \lambda F_1 = 0. \tag{D.2}$$

By using similarity methods, letting $R = \pi Q$ and using the chain rule we get the following:

$$F(R) = F(\pi, Q)$$

$$\frac{\partial F(\pi, Q)}{\partial Q} = \frac{\partial F(R)}{\partial R}\pi$$

$$\frac{\partial F(\pi, Q)}{\partial \pi} = \frac{\partial F(R)}{\partial R}Q$$

$$\frac{\partial^2 F(\pi, Q)}{\partial^2 Q} = \frac{\partial^2 F(R)}{\partial R^2}\pi^2$$

$$\frac{\partial^2 F(\pi, Q)}{\partial \pi^2} = \frac{\partial^2 F(R)}{\partial R^2}Q^2$$

$$\frac{\partial F(\pi,Q)}{\partial\pi\partial Q} = \frac{\partial^2 F(R)}{\partial R^2}\pi Q + \frac{\partial F(R)}{\partial R}$$

Substituting into Eq. (D.1) we get the following:

$$\frac{1}{2}\frac{\partial^2 F_1(R)}{\partial R^2}Q^2\sigma_\pi^2\pi^2 + \frac{1}{2}\frac{\partial^2 F_1(R)}{\partial R^2}\pi^2\sigma_Q^2 Q^2 + \left(\frac{\partial^2 F_1(R)}{\partial R^2}\pi Q + \frac{\partial F_1(R)}{\partial R}\right)\sigma_\pi\sigma_Q\rho\pi Q$$
$$+ \frac{\partial F_1(R)}{\partial R}Q\alpha_\pi\pi + \frac{\partial F_1(R)}{\partial R}\pi\alpha_Q Q - rF_1(R) = 0.$$

$$\Downarrow$$

$$\frac{1}{2}F_1''(R)R^2(\sigma_\pi^2 + \sigma_Q^2 + 2\sigma_\pi\sigma_Q\rho) + F_1'(R)R(\alpha_\pi + \alpha_Q + \sigma_\pi\sigma_Q\rho) - rF_1(R) = 0,$$

Substituting into Eq. (D.1) we get the following:

$$\frac{1}{2}\frac{\partial^2 F_0(R)}{\partial R^2}Q^2\sigma_\pi^2\pi^2 + \frac{1}{2}\frac{\partial^2 F_0(R)}{\partial R^2}\pi^2\sigma_Q^2 Q^2 + \left(\frac{\partial^2 F_0(R)}{\partial R^2}\pi Q + \frac{\partial F_0(R)}{\partial R}\right)\sigma_\pi\sigma_Q\rho\pi Q$$
$$+ \frac{\partial F_0(R)}{\partial R}Q\alpha_\pi\pi + \frac{\partial F_0(R)}{\partial R}\pi\alpha_Q Q - (r+\lambda)F_0(R) + \lambda F_1(R) = 0.$$

$$\Downarrow$$

$$\frac{1}{2}F_0''(R)R^2(\sigma_\pi^2 + \sigma_Q^2 + 2\sigma_\pi\sigma_Q\rho) + F_0'(R)R(\sigma_\pi\sigma_Q\rho + \alpha_\pi + \alpha_Q) - (r+\lambda)F_0(R) + \lambda F_1(R) = 0.$$

Appendix E

Derivation of combined growth $\mu_{\pi Q}$

The combined growth of π and Q is denoted by $\mu_{\pi Q} = \alpha_\pi + \alpha_Q + \rho\sigma_\pi\sigma_Q$, is derived by applying Ito's product rule (Joshi, 2003, p. 100) to $R = \pi Q$:

$$
\begin{aligned}
\mathcal{E}[d(\pi Q)] &= \mathcal{E}[\pi dQ + Qd\pi + d\pi dQ] \\
&= \pi Q\alpha_Q dt + \pi Q\alpha_\pi dt + \pi Q\sigma_\pi\sigma_Q\mathcal{E}(dZ_Q dZ_\pi) \\
&= \pi Q(\alpha_Q + \alpha_\pi + \rho\sigma_\pi\sigma_Q)dt
\end{aligned}
$$

Appendix F

Derivation of characteristic equation

Given the following ODE:

$$\frac{1}{2}F_0''(R)R^2(\sigma_\pi^2+\sigma_Q^2+2\sigma_\pi\sigma_Q\rho)+F_0'(R)R(\sigma_\pi\sigma_Q\rho+\alpha_\pi+\alpha_Q)-(r+\lambda)F_0(R)+\lambda F_1(R)=0. \tag{F.1}$$

With a solution of the form:

$$F_0 = C_1R^{\theta_1}+C_2R^{\theta_2}+A_1R^{\beta_1},$$

Letting CR^θ denote a linear combination of $C_1R^{\theta_1}+C_2R^{\theta_2}$,it can be shown that $C_2=0$, and that the values of the θ parameters are the roots of the characteristic equation derived by implying:

$$F_1 = A_1R^{\beta_1}$$

$$F_0 = CR^\theta + A_1R^{\beta_1}$$

$$F_0' = \theta CR^{\theta-1}+\beta_1A_1R^{\beta_1-1}$$

$$F_0'' = \theta(\theta-1)CR^{\theta-2}+\beta_1(\beta_1-1)A_1R^{\beta_1-2}$$

Substituting into Eq. (F.1) yields:

$$
\begin{aligned}
&\frac{1}{2}(\theta(\theta-1)CR^{\theta}+\beta_1(\beta_1-1)A_1R^{\beta_1})(\sigma_\pi^2+\sigma_Q^2+2\sigma_\pi\sigma_Q\rho)\\
&+(\theta CR^{\theta}+\beta_1A_1R^{\beta_1-1})(\sigma_\pi\sigma_Q\rho+\alpha_\pi+\alpha_Q)-(r+\lambda)(CR^{\theta}+A_1R^{\beta_1})+\lambda A_1R^{\beta_1}=0
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\frac{1}{2}\theta(\theta-1)CR^{\theta}(\sigma_\pi^2+\sigma_Q^2+2\sigma_\pi\sigma_Q\rho)+\theta CR^{\theta}(\sigma_\pi\sigma_Q\rho+\alpha_\pi+\alpha_Q)-CR^{\theta}(r+\lambda)\\
&+\frac{1}{2}\beta_1(\beta_1-1)A_1R^{\beta_1}(\sigma_\pi^2+\sigma_Q^2+2\sigma_\pi\sigma_Q\rho)+\beta_1A_1R^{\beta_1}(\sigma_\pi\sigma_Q\rho+\alpha_\pi+\alpha_Q)-A_1R^{\beta_1}r=0
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&CR^{\theta}\left(\frac{1}{2}\theta(\theta-1)(\sigma_\pi^2+\sigma_Q^2+2\sigma_\pi\sigma_Q\rho)+\theta(\sigma_\pi\sigma_Q\rho+\alpha_\pi+\alpha_Q)-(r+\lambda)\right)\\
&+A_1R^{\beta_1}\left(\frac{1}{2}\beta_1(\beta_1-1)(\sigma_\pi^2+\sigma_Q^2+2\sigma_\pi\sigma_Q\rho)+\beta_1(\sigma_\pi\sigma_Q\rho+\alpha_\pi+\alpha_Q)-r\right)=0
\end{aligned}
$$

Note that the last part of the expression is the same as the characteristic equation Q_1 (Eq. (5.2)). Since β_1 is a root of Q_1, β_1 has a value making $Q_1=0$, hence we can eliminate this part. Dividing by CR^{θ} on both sides we end up with the following characteristic equation:

$$\frac{1}{2}\theta(\theta-1)(\sigma_\pi^2+\sigma_Q^2+2\sigma_\pi\sigma_Q\rho)+\theta(\sigma_\pi\sigma_Q\rho+\alpha_\pi+\alpha_Q)-(r+\lambda)=0 \tag{F.2}$$

Appendix G

Derivation of integrals

In this appendix we will derive one the simplification done to arrive at the range $R_1^* \leq R < R_0^*$ in domain (5.33).

$$\mathcal{E}_{R,Y}\left[\int_Y^\infty e^{-(r-\mu_{\pi Q})t}R(1+\tau)dt - e^{-rY}K\right]$$

$$= \mathcal{E}_{R,Y}\left[\left[-\frac{e^{-(r-\mu_{\pi Q})t}R(1+\tau)}{r-\mu_{\pi Q}}\right]_Y^\infty - e^{-rY}K\right]$$

$$= \mathcal{E}_{R,Y}\left[\frac{e^{-(r-\mu_{\pi Q})Y}R(1+\tau)}{r-\mu_{\pi Q}} - e^{-rY}K\right]$$

$$= \int_0^\infty e^{-(r-\mu_{\pi Q})Y}R(1+\tau)\lambda e^{-\lambda Y} - e^{-rY}K\lambda e^{-\lambda Y}dY$$

$$= \left[-\frac{\lambda e^{-(r-\mu_{\pi Q}+\lambda)Y}R(1+\tau)}{r-\mu_{\pi Q}+\lambda} + \frac{\lambda e^{-(r+\lambda)Y}K}{r+\lambda}\right]_0^\infty$$

$$= \frac{\lambda R(1+\tau)}{r-\mu_{\pi Q}+\lambda} + \frac{\lambda K}{r+\lambda}$$

Here we used the fact that $\mathcal{E}[g(X)] = \int_{-\infty}^\infty g(x)f(x)dx$ where $f(x)$ is the probability density function of $g(x)$. Because time cannot have negative value we integrate from 0 to ∞. Note that Y is exponentially distributed, and hence its density function is $\lambda e^{-\lambda x}$. It should from this derivation also be clear to the reader how the last integrals of the domain (5.33) were simplified.

Appendix H

Numerical illustrations of Model 2 option value components

λ	C_1	θ_1	R	$C_1R^{\theta_1}$
0	-0.13840	2.00	8	-8.05
0.2	-0.00316	3.24	8	-2.35
0.4	-0.000300	4.12	8	-1.31
0.6	-0.00004	4.84	8	-0.92
0.8	-0.000018	5.47	8	-0.69
1	0	6.03	8	-0.55

Table H.1: Table displaying the increase in $C_1R^{\theta_1}$ as λ increases

R	D_1	θ_1	D_2	θ_2	$D_1R^{\theta 1}$	$D_2R^{\theta_2}$
8.35	0.01	3.18	1075.3	-3.1	8.75	1.49
8.52	0.01	3.18	1075.3	-3.1	9.15	1.43
8.68	0.01	3.18	1075.3	-3.1	9.57	1.37
8.85	0.01	3.18	1075.3	-3.1	10.00	1.31
9.02	0.01	3.18	1075.3	-3.1	10.43	1.26
9.19	0.01	3.18	1075.3	-3.1	10.89	1.21

Table H.2: Table displaying the change in $D_1R^{\theta_1}$ and $D_2R^{\theta_2}$ as R increases

λ	D_1	θ_1	R	$D_1 R^{\theta_1}$
0	0.673693	1.96	8.6	45.76
0.2	0.010331	3.18	8.6	9.61
0.4	0.000823	4.04	8.6	4.89
0.6	0.000117	4.75	8.6	3.19
0.8	0.000023	5.35	8.6	2.34
1	0.000006	5.91	8.6	1.84

Table H.3: Table displaying the decrease in $D_1 R^{\theta_1}$ as λ increases

λ	D_1	θ_1	R	$D_1 R^{\theta_1}$
0	0.00002	-1.88	8.6	0
0.2	1075.311	-3.10	8.6	1.36
0.4	5807.22	-3.96	8.6	1.15
0.6	22016.26	-4.67	8.6	0.96
0.8	69722.75	-5.28	8.6	0.81
1	196415.84	-5.83	8.6	0.70

Table H.4: Table displaying the change in $D_2 R^{\theta_2}$ as λ increases. The change follows a bell shaped curve

Appendix I

Simulation procedures

```
function [] = M1FirstPassage(mu, sigma, sim, prob, phi)

% Code used to simulate the expeted first passage time of Model 1.
% Simulates GBM, P(profit/investment cost) and a Poisson process(lambda)
% Starts at initial values given in chapter 6

n=sim; % Sets number of simulations
dt = 1; % Length of time steps

% Initialise arrays
dW = zeros(1,n);
dX = zeros(1,n);
X = zeros(1,n);
a = zeros(1,n);
steps = zeros(1,n); % array to hold first passage times

dq = (1/phi) - 1; % Size of innovations / Poisson jump
X(1) = 0.1225; % Initial value of p


i = 1; % counter
k = 1; % counter

while k < (n+1) % While loop that controls the simulations

    dW(i) = sqrt(dt)*randn; % wiener increment of p
    a(i) = rand(); % Uniformly distributed variable to simulate innovation

    if a(i) <= prob % Checks if innovation has arrived
        dX(i) = mu*X(i)*dt + sigma*X(i)*dW(i) + X(i)*dq;
        else
        dX(i) = mu*X(i)*dt + sigma*X(i)*dW(i);
    end

    X(i+1) = X(i) + dX(i); % Sets new p value

    if (X(i+1) >= 0.18) % Cheks if investment threshold p^* is hit
        steps(k) = i; % First passage time = i
        plot(1:1:i+1, X(1:i+1));
        hold on
        X = zeros(1,n); % for efficiency
        i = 0;
        X(1) = 0.1;
        k = k + 1;
    end
    i = i + 1;
end
disp(mean(steps)); %Display median first passage time to filter extremas
hold off;
end
```

Figure I.1: Simulation procedure programmed in MATLAB to simulate the expected first hitting time for Model 1.

```matlab
function [] = M2FirstPassage(alphaPi,alphaQ,sigmaPi,sigmaQ,sim,prob,tau)

% Code used to simulate the expeted first passage time of Model 2.
% Simulates two GBMs, Pi(Profit) and Q(Annual quantity) and
% a single innovation a (lambda)
% Starts at initial values given in chapter 6

n =sim; % Number of time simulations
dt = 1; % Length of time steps

% Initialise arrays
dzPi = zeros(1,n);
dzQ = zeros(1,n);
dPi = zeros(1,n);
dQ = zeros(1,n);
Pi = zeros(1,n);
Q = zeros(1,n);
R = zeros(1,n);
a = zeros(1,n);

Pi(1) = 12.25; % Initial per unit profit
Q(1) = 500000; % Initial annual quantity produced
R(1)=Pi(1)*Q(1); % Initial total profit


steps = zeros(1,n); % array to hold the first passage times

% Counting variables
i = 1;
k = 1;
jcount = 0;

% While loop that controls the simulation

while k < (n+1)
    dzPi(i) = sqrt(dt)*randn; % wiener increment pi
    dzQ(i) = sqrt(dt)*randn; % wiener increment Q
    dPi(i) = alphaPi*Pi(i)*dt + sigmaPi*Pi(i)*dzPi(i); % change in Pi
    dQ(i) = alphaQ*Q(i)*dt + sigmaQ*Q(i)*dzQ(i); % change in Q
    a(i) = rand(); % Randon uniformly distributed variable

        if (a(i) <= prob && jcount==0) % checks if innovation has arrived
            Pi(i+1) = Pi(i) + dPi(i);
            Q(i+1) = Q(i) + dQ(i);
            R(i+1) = (Pi(i+1)*Q(i+1))*(1+tau);
            jcount = 1;
        elseif jcount==1 % Checks if innovation allready has arrived
            Pi(i+1) = Pi(i) + dPi(i);
            Q(i+1) = Q(i) + dQ(i);
            R(i+1) = (Pi(i+1)*Q(i+1))*(1+tau);
        else
            Pi(i+1) = Pi(i) + dPi(i);
            Q(i+1) = Q(i) + dQ(i);
            R(i+1) = Pi(i+1)*Q(i+1);
        end
```

```
        if (R(i+1) >= 8950000 && jcount==0)% Checks if R0^* is hit
            steps(k) = i; % First passage time = i
            plot(1:1:i+1, R(1:i+1));
            hold on
            Pi = zeros(1,n);
            Q = zeros(1,n);
            R = zeros(1,n);
            i = 0; % Reset Counter
            jcount = 0;
            Pi(1) = 12.25;
            Q(1) = 500000;
            R(1)=Pi(1)*Q(1);
            k = k + 1;

        elseif (R(i+1) >= 8350000 && jcount==1)% Checks if R1^* is hit
            steps(k) = i; % First passage time = i
            plot(1:1:i+1, R(1:i+1));
            hold on
            Pi = zeros(1,n);
            Q = zeros(1,n);
            R = zeros(1,n);
            i = 0; % Reset Counter
            jcount = 0;
            Pi(1) = 12.25;
            Q(1) = 500000;
            R(1)=Pi(1)*Q(1);
            k = k + 1;

        elseif (R(i+1) <= 0 ) % If GBM =0, then restart simulation
            Pi = zeros(1,n);
            Q = zeros(1,n);
            R = zeros(1,n);
            i = 0; % Reset Counter
            jcount = 0;
            Pi(1) = 12.25;
            Q(1) = 500000;
            R(1)=Pi(1)*Q(1);

        end
    i = i+1;
end
disp(median(steps)); % Display median first passage time to filter extremas
hold off;
end
```

Figure I.2: Simulation procedure programmed in MATLAB to simulate the expected first hitting time for Model 2.