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On Strict Higher Categories and their Application to Cobordism Theory

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Abstract

The main goal of the present thesis is an exposition of the Bökstedt-Madsen theorem ([BM]), which relates the classifying space of the embedded cobordism category to certain iterated loop spaces of the Thom space of universal vector bundles. To this end, we first give a brief exposition of higher category theory with emphasis on the k-fold categories and a general introduction to cobordism theory and moduli spaces of manifolds. In the final chapter, we explore the extension of the theory to accommodate manifolds with Baas-Sullivan singularities.

Sammendrag

Hovedmålet i denne oppgaven er en eksposisjon av Bökstedt-Madsen-teoremet ([BM]), som relaterer det klassifiserende rommet til den embeddede kobordismekategorien til visse itererte looprom til Thom-rommet assosiert til universelle vektorbunter. For dette formålet gir vi først en kort eksposisjon av høyere kategoriteori med fokus på k-foldige kategorier og en generell introduksjon til kobordismeteori og modulirom av mangfoldigheter. I det siste kapittelet utforsker vi en utvidelse av denne teorien som tar hånd om mangfoldigheter med Baas-Sullivan-singulariteter.

Preface

This thesis was written during my last year as a student for the degree of Master of Science in Mathematics at NTNU. It was written under supervision of Nils A. Baas in the field of algebraic topology.

The number of people to whom I owe thanks is too great to include in full. I am very grateful to Nils Baas for our helpful and enlightening discussions, offering me invaluable advice and for his patience and encouragement during the past two years.

I would also like to thank Richard Williamson for getting me interested in homotopy theory, and for countless helpful discussions which have helped to form my current perspective on mathematics as a whole, and Marius Thaule for helpful dicussions and advice.

I thank my family and friends for their encouragement and patience with me during the writing of this thesis.

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Chapter 1
Higher categories

Category theory has proven itself to be a ubiquitous presence in algebra, geometry and topology. However, in many situations arising naturally, imposing a category structure is either insufficient or forces us to make unnatural identifications. Let us look at an example of both. Let us look at the categories Top of topological spaces, sSet of simplicial sets , $\mathsf{Ch}(R)$ of chain complexes with values in R-modules for a commutative ring R and $\mathsf{QCoh}(X)$ of quasi-coherent sheaves of modules over a scheme X. In these categories it makes sense and it meaningful to talk about homotopies between morphisms. In fact, for many naturally arising functors between these, such as the singular complex functor $C_\bullet(-) : \mathsf{Top} \to \mathsf{Ch}(R)$, it is possible to extend the functor to take homotopies between morphisms in Top to homotopies between chain complexes in $\mathsf{Ch}(R)$, and so on for homotopies of homotopies, et cetera.

The central idea of higher category theory is that ordinary categories do not possess enough structure to describe the structures we are interested in. In particular, in situations where we may have many possible ways of composing a pair of morphisms, which are all equal up to some equivalence relation; we need to take equivalence classes to recover the associativity and unitality axioms of a category. The philosophy of higher category theory is to not quotient out this failure to satisfy the axioms, but treat it as extra structure. This extra structure takes the form of "morphisms between morphisms", or 2-morphisms. Iterating, we get 3-morphisms between 2-morphisms et cetera. We then weaken the notion of "associative" to mean "associative up to an invertible 2-morphism", where "invertible 2-morphism" in turn is weakened to "invertible up to an invertible 3-morphism" and so on. A higher category which has morphisms of dimensions up to some finite number n is called an n-category. There are several technical obstacles to rigorously defining n-categories. One of which is the problem of *coherence conditions.* If we stop adding higher morphisms after dimension n, we will need to impose some strict notion of associativity and unitality at that level. The equations encoding which n-morphisms are identified are called coherence conditions. The problem with coherence conditions is a combinatorial explosion which occurs as the dimension increases. We will examine this phenomenon shortly. First, we give some motivating examples of higher categories.

Example 1.1. The standard example is the homotopy groupoid of a topological space. Recall that given a topological space X, the homotopy groupoid is a category $\Pi_{\leq 1}(X)$ whose objects are the points of X, and a morphism $p : x \to y$ is a homotopy class of paths $p : [0,1] \to X$ such that $p(0) = x$ and $p(1) = y$. The need to take homotopy classes is immediate; in order to compose paths, we first need to reparameterize them. Let us fix a homeomorphism

$$[0,1] \times_{i_0,i_1} [0,1] \approx [0,2]$$

The composition map

$$\Pi_{\leq 1}(X) \times_{s,t} \Pi_{\leq 1}(X) \to \Pi_{\leq 1}(X)$$

extends through the canonical map of path spaces

$$\mathrm{Hom}_{\mathsf{Top}}([0,1], X) \times_{e_0,e_1} \mathrm{Hom}_{\mathsf{Top}}([0,1], X) \to \mathrm{Hom}_{\mathsf{Top}}([0,2], X)$$

followed by taking the precomposition with a map

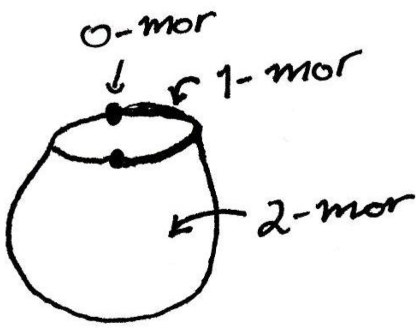
$$f : [0,1] \to [0,2]$$

for which $f(0) = 0$ and $f(1) = 2$. The problem is then that there is no canonical choice for such a map. While a linear map may seem natural, there is, for the purpose of composing paths, no reason to prefer it over, say, the quadratic function $x \mapsto 2x^2$. However, we are in luck, since the space of these functions f as above is contractible. Therefore we have a canonical composition, but only up to homotopy. Still, the need to take homotopy classes does not end at avoiding arbitrary choices. There is a real issue of associativity and unitality of the composition operation which necessitates it. Namely, if we fix any such f, we can build two canonical maps $[0,1] \to [0,3]$ from it. By utilizing our chosen homeomorphism $[0,1] \times_{i_0,i_1} [0,1] \approx [0,2]$, we get the two maps $(\mathrm{id} \times f) \circ f$ and $(f \times \mathrm{id}) \circ f$, and these are never equal. However, they are always homotopic. The same argument holds for unitality. Thus we must take homotopy classes in order to equip $\Pi_{\leq 1}(X)$ with the structure of a category.

Now, we might instead retain the structure of the homotopies witnessing the associativity and unitality. The set of these homotopies, along with rules for composing them, adds a second layer of structure. Now we are again faced with the problem of making this into an associative and unital structure, and the solution is again to take homotopy classes of homotopies. We may again iterate the procedure, an arbitrary number of times, either indefinitely or up to a certain finite number, say n layers of structure. In this case we obtain what is called the *fundamental n-groupoid* of X, denoted $\Pi_n(X)$. As the name suggests, this is one of the "tests" a definition of n-categories must pass - it must include $\Pi_n(X)$ as an example of an n-groupoid.

Example 1.2. Let M and N be n-manifolds with boundary, such that ∂M and ∂N have a common union of components P. Then we can glue M and N along P, obtaining the pushout $W = M \sqcup_P N$. W is uniquely determined up to isomorphism, but to construct it requires one to choose smooth collars around P in M and N.

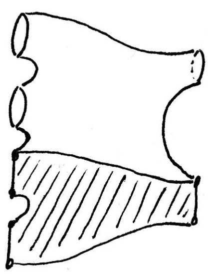
More generally, assume we are given an n-manifold V_n and a $0 \leq k \leq n$. Let V_{n-1} be an $(n-1)$-dimensional submanifold of ∂V_n. Continuing downward until we reach V_{n-k}, we reach a hierarchy of manifolds of increasing dimension which we want to realize as a geometric k-category.

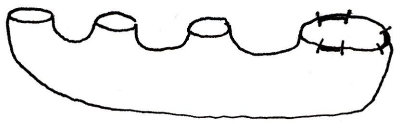


V_n n-manifold	k-morphism
$V_{n-1} \subset \partial V_n$	$(k-1)$-morphism
$V_{n-2} \subset \partial V_{n-1}$	$(k-1)$-morphism
$\vdots$	$\vdots$
$V_{n-k} \subset \partial V_{n-k+1}$	0-morphism

In this way a decomposition of the boundary of successively lower dimensional manifolds can give rise to a higher categorical structure. The nature of this structure will naturally depend on how we deal with this decomposition. In this thesis, we will focus on the *decomposed manifolds* of [Baa73], also called $\langle k \rangle$-manifolds in [Lau00], which naturally gives rise to a k-fold categorical structure.

Further structure imposed on the boundary decomposition can give rise to more exotic generalized manifolds, for example manifolds with Baas-Sullivan singularities, which we explore in Chapter 4.





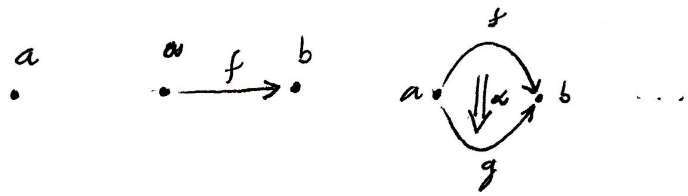
1.1 Strict n-categories

By far the easiest to handle flavours of higher categories are the so-called *strict* n-categories. The reason for this is that all the coherence relations are trivial.

Definition 1.1.1. We define a *strict n-category* inductively as follows.

- The category of strict 0-categories is just Set.
- For $n \geq 1$, the category $n\mathsf{StrCat}$ of strict n-categories is the category $(n-1)\mathsf{StrCat}\text{-}\mathsf{Cat}$ of categories enriched in strict $(n-1)$-categories.

Remark 1.1.2. Unravelling the definition, we see that a strict n-category has n layers of morphisms, where k-morphisms go between $(k-1)$-morphisms whose sources and targets agree. This is an example of a *globular* notion of higher categories. Furthermore, there is defined a notion of composition at each layer which is associative and unital. Morphisms in different dimensions are usually represented by extending the usual commutative diagrams into higher-dimensional cell complexes. I.e. a k-morphism for $k = 0, 1, 2, ...$ is represented as follows:



Example 1.3. The prototypical example of a strict $(n+1)$-category is the category of strict n-categories. Here the objects are given by the strict n-categories and the morphisms by enriched functors between these. Unravelling the definition of a functor $F : C \to D$ between strict n-categories, we see that it consists of an $(n+1)$-tuple of functions $F_k : C_k \to D_k$ taking k-morphisms to k-morphisms for $0 \leq k \leq n$, such that these are compatible with respect to all structure maps.

1.2 Coherence conditions

Let us illustrate how coherence conditions come into play, and how the combinatorial explosion occurs as the dimension increases. We will work out the 2-dimensional case explicitly, building upon the definition of strict 2-categories in the previous section.

Definition 1.2.1. A *weak* 2*-category* $\mathcal{C}$, also called a *bicategory*, is the data of a set of objects $\mathcal{C}_0$, and for each pair $a, b \in \mathcal{C}_0$, a category $\mathcal{C}(a,b)$, the objects of which are called 1-morphisms from a to b, and for two 1-morphisms $f, g : a \to b$, a morphism $\alpha : f \Rightarrow g$ is called a 2-morphism from f to g.

This data is equipped with a composition and unit structure maps. The former is given by, for each triple $a, b, c \in \mathcal{C}_0$, a specified functor

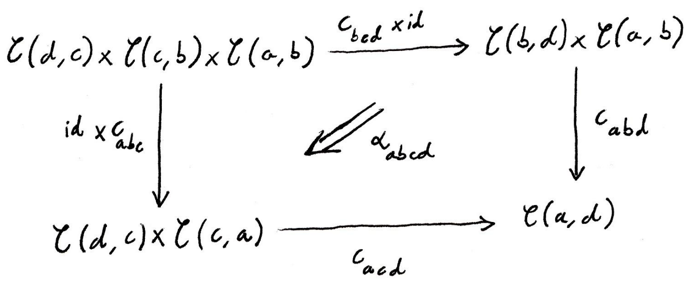
$$c_{abc} : \mathcal{C}(b,c) \times \mathcal{C}(a,b) \to \mathcal{C}(a,c)$$

and the latter by, for each $a \in \mathcal{C}_0$, a functor

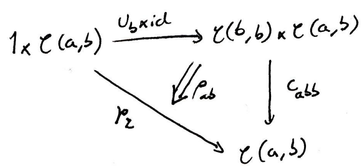
$$u_a : 1 \to \mathcal{C}(a,a)$$

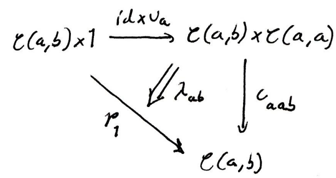
where 1 is the trivial one-object category.

This data is required to satisfy weak associativity and unitality. The former is given by, for each quadruple of objects $a, b, c, d \in \mathcal{C}_0$, an invertible 2-morphism

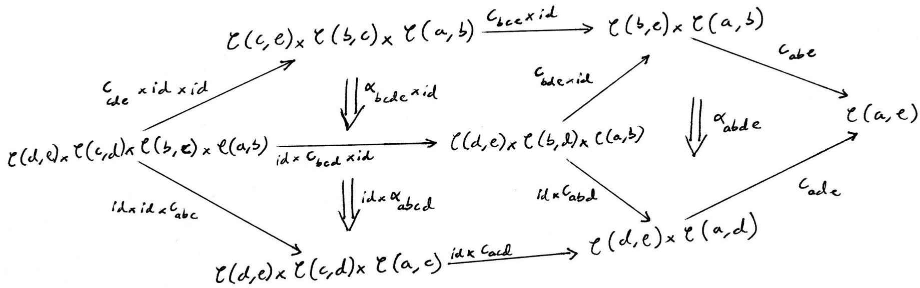


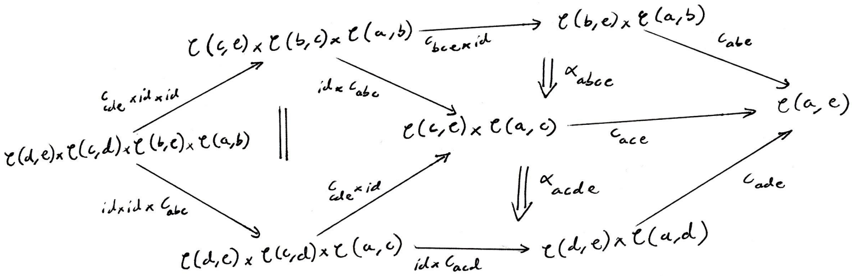
and the latter by, for each pair of objects $a, b \in \mathcal{C}_0$, a pair of invertible 2-morphisms



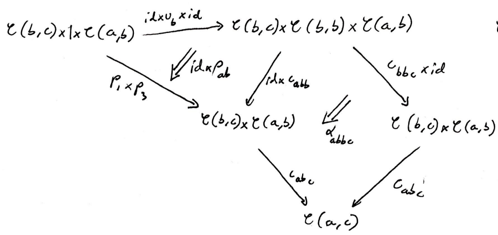


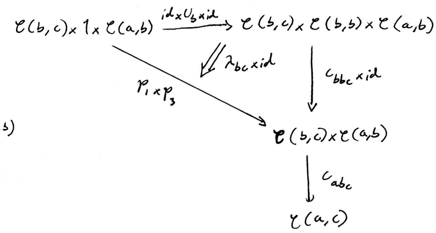
Finally, we require the 2-morphisms appearing in the axioms to satisfy some higher coherence conditions. Firstly, we have the following version of the pentagon axiom for monoidal categories: For any quintiplet of objects $a, b, c, d, e \in \mathcal{C}_0$, the following pastings of 2-morphisms are *equal*.





Secondly, we have the following relation on the unitors. For each triplet of objects $a, b, c \in \mathcal{C}_0$, the following pastings of 2-morphisms are *equal*.





Example 1.4. **[B$\acute{6}$7]** Let C be a category with pushouts. We define its bicategory of cospans $\mathrm{Cosp}(C)$ as follows. The objects of $\mathrm{Cosp}(C)$ are simply the objects of C. A 1-morphism of $\mathrm{Cosp}(C)$ is a cospan in C, that is a diagram

$$P_0 \xrightarrow[f_0]{} S \xleftarrow[f_1]{} P_1$$

which we denote by (f_0, f_1) A 2-morphism is a morphism of cospans, that is a commutative diagram

$$\begin{array}{ccccc} & & S_0 & & \\ & \nearrow & \big\downarrow & \nwarrow & \\ P_0 & & & & P_1 \\ & \searrow & & \swarrow & \\ & & S_1 & & \end{array}$$

For each pair of cospans

$$P_0 \xrightarrow[f_0]{} S \xleftarrow[f_1]{} P_1$$

$$P_1 \xrightarrow[g_0]{} S' \xleftarrow[g_1]{} P_2$$

we choose a pushout of the diagram

$$S \xleftarrow{f_1} P_1 \xrightarrow{g_0} S'$$

and this pushout is defined as the composition of the two 1-morphisms. The associator and unitors now follow from the fact that the pushout is a universal construction.

Definition 1.2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. A *psuedofunctor* $P : \mathcal{C} \to \mathcal{D}$ is the data of:

- a function $P_0 : \mathcal{C}_0 \to \mathcal{D}_0$,
- for each pair $x, y \in \mathcal{C}_0$, a functor $P_{xy} : \mathcal{C}(x, y) \to \mathcal{D}(P_0(x), P_0(y))$,
- for each object $x \in C_0$, an invertible 2-morphism $P_{\mathrm{id}_x} : \mathrm{id}_{P_0(x)} \to P_{xx}(\mathrm{id}_x)$,

– for each triple $x, y, z \in C_0$, an isomorphism, natural in $f : x \to y$ and $g : y \to z$,

$$P_{xyz}(f, g) : P_{yz}(g) \circ P_{xy}(f) \to P_{xz}(g \circ f)$$

these data are required to satisfy the following axioms. For each hom-category $C(x, y)$ and each $f \in C(x, y)$, the following diagrams commute.

$$\begin{array}{ccc}
P_{xy}(f) \circ \mathrm{id}_{P_0(x)} & \xrightarrow{\lambda_{P_{xy}(f)}} & P_{xy}(f) \\
{\scriptstyle \mathrm{id}_{P_{xy}(f)} \circ P_{\mathrm{id}_x}} \downarrow & & \uparrow {\scriptstyle P_{xy}(\lambda_f)} \\
P_{xy}(f) \circ P_{xx}(\mathrm{id}_x) & \xrightarrow[P_{xxy}(f \circ \mathrm{id}_x)]{} & P_{xy}(f \circ \mathrm{id}_x)
\end{array}$$

$$\begin{array}{ccc}
\mathrm{id}_{P_0(y)} \circ P_{xy}(f) & \xrightarrow{\rho_{P_{xy}(f)}} & P_{xy}(f) \\
{\scriptstyle P_{\mathrm{id}_y} \circ \mathrm{id}_{P_{xy}(f)}} \downarrow & & \uparrow {\scriptstyle P_{xy}(\rho_f)} \\
P_{yy}(\mathrm{id}_y) \circ P_{xy}(f) & \xrightarrow[P_{xyy}(\mathrm{id}_y \circ f)]{} & P_{xy}(\mathrm{id}_y \circ f)
\end{array}$$

Furthermore, for each quadruple $w, x, y, z \in C_0$ and 1-morphisms $f \in C(w, x)$, $g \in C(x, y)$ and $h \in C(y, z)$, the following diagram commutes:

$$\begin{array}{ccccc}
P_{yz}(h) \circ (P_{xy}(g) \circ P_{wx}(f)) & \xrightarrow{\alpha_{P_{wx}(f), P_{xy}(g), P_{yz}(h)}} & (P_{yz}(h) \circ P_{xy}(g)) \circ P_{wx}(f) & \xrightarrow{P_{xyz}(g,h) \circ \mathrm{id}_{P_{wx}(f)}} & P_{xz}(g \circ h) \circ P_{wx}(f) \\
{\scriptstyle P_{wxy}(f,g) \circ \mathrm{id}_{P_{yz}(h)}} \downarrow & & & & \downarrow {\scriptstyle P_{wxz}(f, h \circ g)} \\
P_{yz}(h) \circ P_{wy}(g \circ f) & \xrightarrow{P_{wyz}(g \circ f, h)} & P_{wz}(h \circ (g \circ f)) & \xrightarrow{P_{wz}(\alpha_{f,g,h})} & P_{wz}((h \circ g) \circ f)
\end{array}$$

1.3 k-tuple categories

Idea 1.3.1. Just as a strict k-category was defined as a category enriched in strict $(k-1)$-categories, a k-tuple category is defined as a category *internal* to the category of $(k-1)$-tuple categories. k-tuple categories first appeared in [Ehr63] in the case $k = 2$. The notion of a category defined *internally* to another category is an essential concept for this idea, so we will spend some time developing the theory of internal categories. Readers who are familiar with this notion may safely skip to the next section.

1.3.1 The 2-category $\mathsf{Cat}(\mathcal{A})$

Remark 1.3.2. In this section, we will develop some central aspects of internal categories. We choose to do this in the most general setting possible, and to work explicitly within the base category. Although what follows is an independent development, the ideas and constructions are certainly not new. Examples of existing treatments can be found in for instance [Bor94, Chapter 8]. Throughout this section, let us fix a category $\mathcal{A}$, which we for convenience assume to admit pullbacks, although the below discussion makes sense as long as $\mathcal{A}$ has the pullbacks appearing in definitions.

Definition 1.3.3. A *category internal to* $\mathcal{A}$, also called an $\mathcal{A}$-category, is the data of

- an object C_0 of $\mathcal{A}$, called an *object of objects*,
- an object C_1 of $\mathcal{A}$, called an *object of arrows*,
- a pair of arrows

$$C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

  of $\mathcal{A}$, called *source* and *target*, respectively,
- an arrow

$$C_0 \xrightarrow{\mathrm{Id}} C_1$$

of $\mathcal{A}$, called *identity*,

– a pullback square

$$\begin{array}{ccc} C_1 \times_{s,t} C_1 & \xrightarrow{u_t} & C_1 \\ {\scriptstyle u_s}\downarrow & & \downarrow{\scriptstyle t} \\ C_1 & \xrightarrow[s]{} & C_0 \end{array}$$

and an arrow

$$C_1 \times_{s,t} C_1 \xrightarrow{c} C_1$$

of $\mathcal{A}$, called *composition*.

This data is required to satisfy the following axioms. Namely, the following diagrams commute in $\mathcal{A}$.

– (Source and target of composition)

$$\begin{array}{ccc} C_1 \times_{s,t} C_1 & \xrightarrow{c} & C_1 \\ {\scriptstyle u_t}\downarrow & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow[s]{} & C_0 \end{array}$$

$$\begin{array}{ccc} C_1 \times_{s,t} C_1 & \xrightarrow{c} & C_1 \\ {\scriptstyle u_s}\downarrow & & \downarrow{\scriptstyle t} \\ C_1 & \xrightarrow[t]{} & C_0 \end{array}$$

– (Associativity)

$$\begin{CD}
C_1 \times_{s,t} C_1 \times_{s,t} C_1 @>{c \times_{s,t} \mathrm{id}}>> C_1 \times_{s,t} C_1 \\
@V{\mathrm{id} \times_{s,t} c}VV @VV{c}V \\
C_1 \times_{s,t} C_1 @>>{c}> C_1
\end{CD}$$

– (Source and target of identity arrows)

$$\begin{array}{ccccc}
 & & C_0 & & \\
 & {}^{\mathrm{id}}\swarrow & \downarrow{\scriptstyle \mathrm{Id}} & \searrow^{\mathrm{id}} & \\
C_0 & \xleftarrow[s]{} & C_1 & \xrightarrow[t]{} & C_0
\end{array}$$

– (Compositions with identity arrows)

$$\begin{array}{ccc}
C_0 \times_{\mathrm{id},t} C_1 & \xrightarrow{\mathrm{Id} \times_{\mathrm{id},t} \mathrm{id}} & C_1 \times_{s,t} C_1 \\
 & {}_{u_t}\searrow & \downarrow{\scriptstyle c} \\
 & & C_1
\end{array}
\qquad
\begin{array}{ccc}
C_1 \times_{s,\mathrm{id}} C_0 & \xrightarrow{\mathrm{id} \times_{s,\mathrm{id}} \mathrm{Id}} & C_1 \times_{s,t} C_1 \\
 & {}_{u_s}\searrow & \downarrow{\scriptstyle c} \\
 & & C_1
\end{array}$$

Remark 1.3.4. There are a few immediate observations from Definition 1.3.3:

(i) The source and target morphisms s, t and the composition morphism c are necessarily epimorphisms.

(ii) The identity morphism Id is necessarily a monomorphism.

Lemma 1.3.5. Let $C = (C_0, C_1, s, t, \mathrm{Id}, c)$ be an $\mathcal{A}$-category. Then Id is the unique morphism in $\mathcal{A}$ satisfying its axioms.

Proof. Assume we have two identity morphisms

$$C_0 \underset{\mathrm{Id}'}{\overset{\mathrm{Id}}{\rightrightarrows}} C_1$$

and consider the morphism

$$C_0 \xrightarrow{\mathrm{Id}' \times \mathrm{Id}} C_0 \times_{\mathrm{id},t} C_1$$

By the "compositions with identity arrows" axioms, we have commutative diagrams

$$\begin{array}{ccc} C_0 & \xrightarrow{\mathrm{Id}' \times \mathrm{Id}} & C_1 \times_{s,t} C_1 \\ & \searrow^{\mathrm{Id}} & \downarrow c \\ & & C_1 \end{array} \qquad \begin{array}{ccc} C_0 & \xrightarrow{\mathrm{Id}' \times \mathrm{Id}} & C_1 \times_{s,t} C_1 \\ & \searrow^{\mathrm{Id}'} & \downarrow c \\ & & C_1 \end{array}$$

such that $\mathrm{Id} = \mathrm{Id}'$. □

Definition 1.3.6. Let $C = (C_0, C_1, s, t, \mathrm{Id}, c)$ and $D = (D_0, D_1, s', t', \mathrm{Id}', c')$ be $\mathcal{A}$-categories. A *functor*

$$C \xrightarrow{F} D$$

is the data of

- an arrow

$$C_0 \xrightarrow{F_0} D_0$$

- an arrow

$$C_1 \xrightarrow{F_1} D_1$$

such that the following diagrams commute.

– (Source and target)

$$\begin{array}{ccc} C_1 & \xrightarrow{F_1} & D_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ C_0 & \xrightarrow[F_0]{} & D_0 \end{array} \qquad \begin{array}{ccc} C_1 & \xrightarrow{F_1} & D_1 \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t'} \\ C_0 & \xrightarrow[F_0]{} & D_0 \end{array}$$

– (Composition)

$$\begin{array}{ccc} C_1 \times_{s,t} C_1 & \xrightarrow{c} & C_1 \\ {\scriptstyle F_1 \times F_1}\downarrow & & \downarrow{\scriptstyle F_1} \\ D_1 \times_{s',t'} D_1 & \xrightarrow[c']{} & D_1 \end{array}$$

– (Identity)

$$\begin{array}{ccc} C_0 & \xrightarrow{F_0} & D_0 \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}'} \\ C_1 & \xrightarrow[F_1]{} & D_1 \end{array}$$

Definition 1.3.7. Let $C = (C_0, C_1, s, t, \mathrm{Id}, c)$ and $D = (D_0, D_1, s', t', \mathrm{Id}', c')$ be $\mathcal{A}$-categories, and let

$$C \underset{G}{\overset{F}{\rightrightarrows}} D$$

be a pair of functors from C to D. A *natural transformation*

$$F \xrightarrow{\eta} G$$

is the data of an arrow

$$C_0 \xrightarrow{\eta} D_1$$

such that the following diagrams commute.

– (Source and target)

$$\begin{array}{ccc} C_0 & \xrightarrow{\eta} & D_1 \\ & {\scriptstyle F_0}\searrow & \downarrow{\scriptstyle s'} \\ & & D_0 \end{array} \qquad \begin{array}{ccc} C_0 & \xrightarrow{\eta} & D_1 \\ & {\scriptstyle G_0}\searrow & \downarrow{\scriptstyle t'} \\ & & D_0 \end{array}$$

– (Naturality)

$$\begin{array}{ccccc} C_1 & \xrightarrow{\mathrm{id} \times s} & C_1 \times_{s,\mathrm{id}} C_0 & \xrightarrow{G_1 \times \eta} & D_1 \times_{s',t'} D_1 \\ {\scriptstyle t \times \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle c'} \\ C_0 \times_{id,t} C_1 & \xrightarrow{\eta \times F_1} & D_1 \times_{s',t'} D_1 & \xrightarrow{c'} & D_1 \end{array}$$

Definition 1.3.8. – Given three functors F, G, H from C to D and natural transformations

$$F \xrightarrow{\eta} G$$

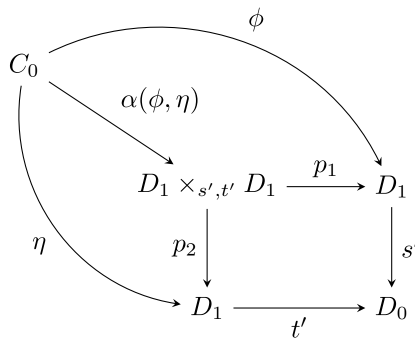
and

$$G \xrightarrow{\phi} H$$

we have commutativity of

$$\begin{CD} C_0 @>{\phi}>> D_1 \\ @V{\eta}VV @VV{s'}V \\ D_1 @>>{t'}> D_0 \end{CD}$$

so there is a unique factorization



Define the *vertical composition*

$$F \xrightarrow{\phi \circ \eta} H$$

to be the composition

$$C_0 \xrightarrow{\alpha(\phi, \eta)} D_1 \times_{s', t'} D_1 \xrightarrow{\quad \circ \quad} D_1$$

– Given a functor

$$C \xrightarrow{F} D$$

define the *identity natural transformation* at F to be the arrow

$$C_0 \xrightarrow{\mathrm{id}_F} D_1$$

such that the following diagram commutes:

$$\begin{array}{ccc} C_0 & & \\ {\scriptstyle F_0}\downarrow & \searrow^{\mathrm{id}_F} & \\ D_0 & \xrightarrow[\mathrm{Id}']{} & D_1 \end{array}$$

Then clearly the diagram

$$\begin{array}{ccccc} C_1 & \xrightarrow{\mathrm{id}\times s} & C_1 \times_{s,\mathrm{id}} C_0 & \xrightarrow{F_1 \times \mathrm{Id}' F_0} & D_1 \times_{s',t'} D_1 \\ {\scriptstyle t\times \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle c'} \\ C_0 \times_{id,t} C_1 & \xrightarrow{\mathrm{Id}' F_0 \times F_1} & D_1 \times_{s',t'} D_1 & \xrightarrow{c'} & D_1 \end{array}$$

commutes, so id_F is a natural transformation.

Lemma 1.3.9. The vertical composition defined in Definition 1.3.8 is an internal natural transformation $F \to H$.

Proof. We have to show that the diagram

$$\begin{array}{ccccc} C_1 & \xrightarrow{\mathrm{id}\times s} & C_1 \times_{s,\mathrm{id}} C_0 & \xrightarrow{H_1 \times (\phi\circ\eta)} & D_1 \times_{s',t'} D_1 \\ {\scriptstyle t\times \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle c'} \\ C_0 \times_{id,t} C_1 & \xrightarrow{(\phi\circ\eta)} & D_1 \times_{s',t'} D_1 & \xrightarrow{c'} & D_1 \end{array}$$

commutes. We can fill it in with commutative diagrams as follows.

$$\begin{tikzcd}
 & C_1 \times_{s,\mathrm{id}} C_0 \arrow[r, "H_1 \times \phi \times \eta"] & D_1 \times_{s',t'} D_1 \times_{s',t'} D_1 \arrow[r, "c' \times \mathrm{id}"] & D_1 \times_{s',t'} D_1 \\
C_1 \arrow[ru, "\mathrm{id} \times s"] \arrow[rr, "t \times \mathrm{id} \times s"] \arrow[rd, "t \times \mathrm{id}"'] & & C_0 \times_{\mathrm{id},t} C_1 \times_{s,\mathrm{id}} C_0 \arrow[r, "\phi \times G_1 \times \eta"] & D_1 \times_{s',t'} D_1 \times_{s',t'} D_1 \arrow[u, "c' \times \mathrm{id}"'] \arrow[d, "\mathrm{id} \times c'"] \\
 & C_0 \times_{\mathrm{id},t} C_1 \arrow[r, "\phi \times \eta \times F_1"] & D_1 \times_{s',t'} D_1 \times_{s',t'} D_1 \arrow[r, "\mathrm{id} \times c'"] & D_1 \times_{s',t'} D_1
\end{tikzcd}$$

And then gluing the following diagram along the arrows between the D-nodes.

$$\begin{tikzcd}
D_1 \times_{s',t'} D_1 \times_{s',t'} D_1 \arrow[rr, "\mathrm{id} \times c'"] \arrow[rd, "c' \times \mathrm{id}"] & & D_1 \times_{s',t'} D_1 \arrow[dd, "c'"] \\
 & D_1 \times_{s',t'} D_1 \arrow[rd, "c'"] & \\
D_1 \times_{s',t'} D_1 \times_{s',t'} D_1 \arrow[ru, "c' \times \mathrm{id}"] \arrow[rd, "\mathrm{id} \times c'"] & & D_1 \\
 & D_1 \times_{s',t'} D_1 \arrow[ru, "c'"] & \\
D_1 \times_{s',t'} D_1 \times_{s',t'} D_1 \arrow[rr, "c' \times \mathrm{id}"] \arrow[ru, "\mathrm{id} \times c'"] & & D_1 \times_{s',t'} D_1 \arrow[uu, "c'"']
\end{tikzcd}$$

□

Terminology 1.3.10. If there is a risk of confusion, we will refer to functors between $\mathcal{A}$-categories as $\mathcal{A}$-functors, and natural transformations between $\mathcal{A}$-functors as $\mathcal{A}$-natural transformations.

Lemma 1.3.11. Let A, B, C be $\mathcal{A}$-categories.

– Assume we have an $\mathcal{A}$-functor

$$A \xrightarrow{F} B$$

and a pair of $\mathcal{A}$-functors

$$B \underset{H}{\overset{G}{\rightrightarrows}} C$$

such that there is a natural transformation

$$G \xrightarrow{\alpha} H$$

Then there is a natural transformation

$$GF \xrightarrow{\alpha \cdot F} HF$$

called the *whiskering* of α and F, given by the composition

$$A_0 \xrightarrow{F_0} B_0 \xrightarrow{\alpha} C_1$$

– Assume we have a pair of $\mathcal{A}$-functors

$$A \underset{G}{\overset{F}{\rightrightarrows}} B$$

and an $\mathcal{A}$-functor

$$B \xrightarrow{H} C$$

such that there is a natural transformation

$$F \xrightarrow{\alpha} G$$

Then there is a natural transformation

$$HF \xrightarrow{H \cdot \alpha} HG$$

called the *whiskering* of H and α, given by the composition

$$A_0 \xrightarrow{\alpha} B_1 \xrightarrow{H_1} C_1$$

Proof. In the first case, the source and target axioms are clearly satisfied. As for the commutativity axiom, we have

$$c_C \circ (\alpha F_0 \times G_1 F_1) \circ (t_A \times \mathrm{id}) = c_C \circ (\alpha F_0 t_A \times G_1 F_1)$$
$$= c_C \circ (\alpha t_B F_1 \times G_1 F_1) = c_A(\alpha \times G_1) \circ (t_B \times \mathrm{id}) \circ (F_1 \times F_1)$$
$$= c_C \circ (H_1 \times \alpha) \circ (\mathrm{id} \times s_B) \circ (F_1 \times F_1) = c_C \circ (H_1 F_1 \times \alpha F_0) \circ (\mathrm{id} \times s_A)$$

The second case is similar. □

Lemma 1.3.12. Let A, B, C be $\mathcal{A}$-categories. Assume we have a pair of $\mathcal{A}$-functors

$$A \underset{F'}{\overset{F}{\rightrightarrows}} B$$

and a pair of $\mathcal{A}$-functors

$$B \underset{G'}{\overset{G}{\rightrightarrows}} C$$

such that there are $\mathcal{A}$-natural transformations

$$F \xrightarrow{\phi} F'$$

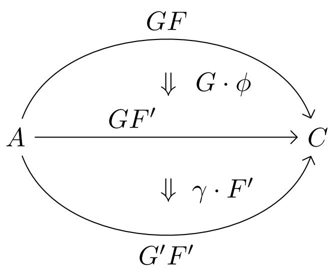
and

$$G \xrightarrow{\gamma} G'$$

The following defines $\mathcal{A}$-natural transformations

$$GF \longrightarrow G'F'$$

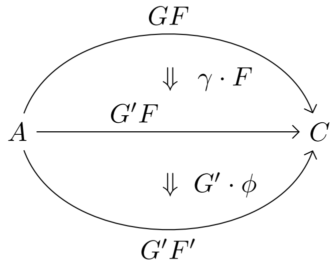
First whiskering ϕ and G and then vertically composing with the whiskering of F' and γ:



Denote the resulting $\mathcal{A}$-natural transformation by

$$GF \xrightarrow{\alpha} G'F'$$

We can also whisker γ with F first and then vertically compose with the whiskering of G' and ϕ:

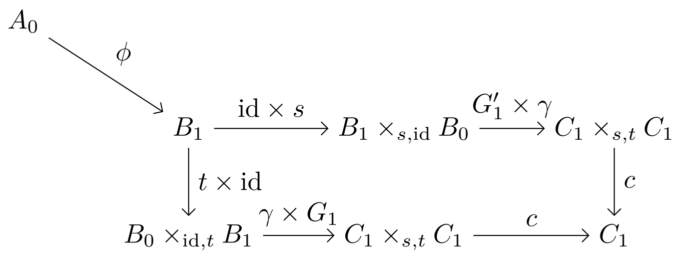


Denote the resulting $\mathcal{A}$-natural transformation by

$$GF \xrightarrow{\beta} G'F'$$

Then $\alpha = \beta$.

Proof. By precomposing the naturality hexagon of γ with ϕ, be obtain the commutative diagram



where we see that

$$\begin{aligned} c(G_1' \times \gamma)(s \times \mathrm{id})\phi &= c(G_1' \times \gamma)(F_0 \times \phi) \\ &= c\left[(G_1'\phi) \times (\gamma F_0)\right] = \alpha \end{aligned}$$

and

$$\begin{aligned} c(\gamma \times G_1)(t \times \mathrm{id})\phi &= c(\gamma \times G_1)(F_0' \times \phi) \\ &= c\left[(\gamma F_0) \times (G_1\phi)\right] = \beta \end{aligned}$$

and we have that $\alpha = \beta$. □

Definition 1.3.13. The resulting $\mathcal{A}$-natural transformation in Lemma 1.3.12 is denoted $\gamma \cdot \phi$ and is called the *horizontal composition* of γ and ϕ.

Lemma 1.3.14. Assume we have $\mathcal{A}$-categories A, B, C, $\mathcal{A}$-functors $F, F', F'' : A \to B$ and $G, G', G'' : B \to C$. Assume we have $\mathcal{A}$-natural transformations

$$F \xrightarrow{\eta} F' \xrightarrow{\eta'} F''$$

and

$$G \xrightarrow{\gamma} G' \xrightarrow{\gamma'} G''$$

Then

$$(\gamma' \cdot F) \circ (\gamma \cdot F) = (\gamma' \circ \gamma) \cdot F$$
$$(G \cdot \eta') \circ (G \cdot \eta) = G \cdot (\eta' \circ \eta)$$

Proof. The first equality is immediate by precomposing the definition of $\gamma' \circ \gamma$ with F_0. For the second relation, we have the following commutative diagram:

$$\begin{array}{ccccc}
A_0 & \xrightarrow{\eta' \times \eta} & B_1 \times_{s,t} B_1 & \xrightarrow{c} & B_1 \\
& \searrow^{(G_1\eta') \times (G_1\eta)} & \downarrow{\scriptstyle G_1 \times G_1} & & \downarrow{\scriptstyle G_1} \\
& & C_1 \times_{s,t} C_1 & \xrightarrow{c} & C_1
\end{array}$$

Then we have

$$(G \cdot \eta') \circ (G \cdot \eta) = c((G_1\eta') \times (G_1\eta)) = c(G_1 \times G_1)(\eta' \times \eta)$$
$$= G_1 c(\eta' \times \eta) = G \cdot (\eta' \circ \eta)$$

□

Lemma 1.3.15. Assume we are in the situation of Definition 1.3.12. The whiskering $G \cdot \phi$ is equal to the horizontal composition $\mathrm{id}_G \cdot \phi$. Similarly, the whiskering $\gamma \cdot F$ is equal to the horizontal composition $\gamma \cdot \mathrm{id}_F$.

Proof. This follows by observing that

$$G \cdot \mathrm{id}_F = G_1 \mathrm{id}_F = G_1 \mathrm{Id}_B F_0 = \mathrm{Id}_C G_0 F_0 = \mathrm{id}_{GF}$$

and

$$\mathrm{id}_G \cdot F = \mathrm{id}_G F_0 = \mathrm{Id}_C G_0 F_0 = \mathrm{id}_{GF}$$

□

Lemma 1.3.16. Assume we have $\mathcal{A}$-categories A, B, C, $\mathcal{A}$-functors $F, F', F'' : A \to B$ and $G, G', G'' : B \to C$, as well as $\mathcal{A}$-natural transformations $\eta : F \to F'$, $\eta' : F' \to F''$, $\gamma : G \to G'$ and $\gamma' : G' \to G''$. Then

$$(\gamma' \cdot \eta') \circ (\gamma \cdot \eta) = (\gamma' \circ \gamma) \cdot (\eta' \circ \eta)$$

Proof. First, we decompose and rearrange the the right hand side using Lemma 1.3.12 and Lemma 1.3.14:

$$\begin{aligned}(\gamma' \circ \gamma) \cdot (\eta' \circ \eta) &= [(\gamma' \circ \gamma) \cdot F] \circ [G'' \cdot (\eta' \circ \eta)] \\ &= (\gamma' \cdot F'') \circ (\gamma \cdot F'') \circ (G \cdot \eta') \circ (G \cdot \eta) \\ &= (\gamma' \cdot F'') \circ (G' \cdot \eta') \circ (\gamma \cdot F') \circ (G \cdot \eta) \\ &= (\gamma' \cdot \eta') \circ (\gamma \cdot \eta)\end{aligned}$$

□

Corollary 1.3.17. By the above lemmas, we obtain a strict 2-category $\mathsf{Cat}(\mathcal{A})$ where the objects are $\mathcal{A}$-categories, 1-morphisms are $\mathcal{A}$-functors and 2-morphisms are $\mathcal{A}$-natural transformations.

Lemma 1.3.18. There is an adjunction

$$\mathsf{Cat}(\mathcal{A}) \underset{\mathrm{disc}}{\overset{\mathrm{Ob}}{\rightleftarrows}} \mathcal{A}$$

Proof. Let B be an object of $\mathcal{A}$ and let C be an $\mathcal{A}$-category. Assume we have a functor

$$\mathrm{disc}(B) \xrightarrow{F} C$$

i.e. $F = (F_0, F_1)$ and fits into commutative diagrams

$$\begin{array}{ccc} B & \xrightarrow{F_1} & C_1 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle s,t} \\ B & \xrightarrow[F_0]{} & C_0 \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{F_0} & C_0 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ B & \xrightarrow[F_1]{} & C_1 \end{array}$$

which means that we can write $F_1 = \mathrm{Id}F_0$. Note that $s\mathrm{Id} = \mathrm{id} = t\mathrm{Id}$ and $c(F_1 \times F_1) = c(\mathrm{Id} \times \mathrm{Id})F_0 = \mathrm{Id}F_0 = F_1$, so F determines and is determined by the arrow

$$B \xrightarrow{F_0} C_0$$

Define

$$\mathrm{Hom}_{\mathsf{Cat}(\mathcal{A})}(\mathrm{disc}(B), C) \xrightarrow{\Phi} \mathrm{Hom}_{\mathcal{A}}(B, \mathrm{Ob}(C))$$

by $\Phi(F) = F_0$. Φ is clearly a bijection. We check naturality. Let B' be another object of $\mathcal{A}$ and let C' be another $\mathcal{A}$-category. say we have an arrow

$$B' \xrightarrow{g} B$$

and an $\mathcal{A}$-functor

$$C \xrightarrow{G} C'$$

We then have

$$\mathrm{Hom}_{\mathcal{A}}(g, \mathrm{Ob}(G)) \circ \Phi_{B'C'}(F) = \mathrm{Hom}_{\mathcal{A}}(g, \mathrm{Ob}(G))(F_0) = G_0F_0g$$

$$\Phi_{BC} \circ \mathrm{Hom}_{\mathsf{Cat}(\mathcal{A})}(\mathrm{disc}(g), G)(F) = \Phi_{BC}(GF \circ (g\mathrm{Id}, g)) = \Phi_{BC}((G_1F_1g\mathrm{Id}, G_0F_0g)) = G_0F_0g$$

So the following diagram commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{Cat}(\mathcal{A})}(\mathrm{disc}(B), C) & \xrightarrow{\Phi_{BC}} & \mathrm{Hom}_{\mathcal{A}}(B, \mathrm{Ob}(C)) \\ {\scriptstyle \mathrm{Hom}_{\mathsf{Cat}(\mathcal{A})}(\mathrm{disc}(g), G)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathcal{A}}(g, \mathrm{Ob}(G))} \\ \mathrm{Hom}_{\mathsf{Cat}(\mathcal{A})}(\mathrm{disc}(B'), C') & \xrightarrow[\Phi_{B'C'}]{} & \mathrm{Hom}_{\mathcal{A}}(B', \mathrm{Ob}(C')) \end{array}$$

and we have the stated adjunction. □

Remark 1.3.19. There are evident functors

$$\mathsf{Cat}(\mathcal{A}) \underset{\mathrm{Arr}}{\overset{\mathrm{Ob}}{\rightrightarrows}} \mathcal{A}$$

taking an $\mathcal{A}$-category to its object of objects and object of arrows respectively. There is also a functor

$$\mathcal{A} \xrightarrow{\mathrm{disc}} \mathsf{Cat}(\mathcal{A})$$

taking an object $A \in \mathcal{A}$ to its discrete $\mathcal{A}$-category

$$\mathrm{disc}(A) = (A, A, \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id})$$

Lemma 1.3.20. Limits of diagrams of shape J exist in $\mathsf{Cat}(\mathcal{A})$ if and only if they exist in $\mathcal{A}$.

Proof. Assume $\mathcal{A}$ has limits of diagrams of shape J. Now let

$$J \xrightarrow{D} \mathsf{Cat}(\mathcal{A})$$

be a diagram in $\mathsf{Cat}(\mathcal{A})$ of shape J. That is, D consists of a pair of J-shaped diagrams D_0, D_1 in $\mathcal{A}$, with natural transformations

$$D_1 \underset{t}{\overset{s}{\rightrightarrows}} D_0$$

and

$$D_0 \xrightarrow{\mathrm{Id}} D_1$$

We write $D^j = (D_0^j, D_1^j, s_j, t_j, \mathrm{Id}_j, c_j)$ for the value of D at $j \in J$. We construct a new $\mathcal{A}$-category $C = (C_0, C_1, s', t', \mathrm{Id}', c')$ with $C_0 = \lim D_0$ and $C_1 = \lim D_1$. s', t' and Id' are defined to be the canonical morphisms induced by composing the canonical projection morphisms by s, t and Id respectively. We construct the J shaped diagram $D_1 \times_{s,t} D_1$ by sending $j \in J$ to $D_1^j \times_{s_j,t_j} D_1^j$. Then $\lim D_1 \times_{s,t} D_1 \simeq C_1 \times_{s',t'} C_1$ and $j \mapsto c_j$ is a natural transformation

$$D_1 \times_{s,t} D_1 \xrightarrow{c} D_1$$

We define c' to be the canonical induced morphism

$$C_1 \times_{s',t'} C_1 \longrightarrow C_1$$

Naturality of s, t, Id, c implies that C satisfies the $\mathcal{A}$-category axioms. We now show that C satisfies the needed universal property. Let $B = (B_0, B_1, s'', t'', \mathrm{Id}'', c'')$ be another $\mathcal{A}$-category with $\mathcal{A}$-functors

$$B \xrightarrow{f^j} D^j$$

for all $j \in J$, i.e. a natural transformation

$$\Delta B \xrightarrow{f} D$$

Let

$$B_0 \xrightarrow{b_0} C_0$$

and

$$B_1 \xrightarrow{b_1} C_1$$

be the canonical induced morphisms. We show that $b = (b_0, b_1)$ is an $\mathcal{A}$-functor from B to C. Then uniqueness comes for free. Denote the canonical projection maps

$$C_1 \xrightarrow{p_1^j} D_1^j$$

$$C_0 \xrightarrow{p_0^j} D_0^j$$

Then observe that

$$p_0^j b_0 s'' = f_0 s'' = s_j f_1^j = s_j p_1^j b_1 = p_0^j s b_1$$

as morphisms

$$B_1 \longrightarrow D_0^j$$

so by the universal property of C_0, the following diagram commutes.

$$\begin{array}{ccc} B_1 & \xrightarrow{b_1} & C_1 \\ {\scriptstyle s''}\downarrow & & \downarrow{\scriptstyle s'} \\ B_0 & \xrightarrow[b_0]{} & C_0 \end{array}$$

The target and identity axioms follow similarly. Also observe that

$$p_1^j c'(b_1 \times_{s'',t''} b_1) = c_j(p_1^j \times_{s',t'} p_1^j)(b_1 \times_{s'',t''} b_1) = c_j(p_1^j b_1 \times_{s'',t''} p_1^j b_1)$$
$$= c_j(f_1^j \times_{s'',t''} f_1^j) = f_1^j c'' = p_1^j b_1 c''$$

as morphisms

$$B_1 \longrightarrow D_1^j$$

for all $j \in J$, so by the universal property of C_1, the following diagram commutes.

$$\begin{array}{ccc} B_1 \times_{s'',t''} B_1 & \xrightarrow{c''} & B_1 \\ {\scriptstyle b_1 \times_{s'',t''} b_1}\downarrow & & \downarrow{\scriptstyle b_1} \\ C_1 \times_{s',t'} C_1 & \xrightarrow[c']{} & C_1 \end{array}$$

So

$$B \xrightarrow{f} C$$

is a functor and we are done.

For the opposite direction, assume $\mathsf{Cat}(\mathcal{A})$ has J-shaped limits. Let

$$J \xrightarrow{D} \mathcal{A}$$

be a J-shaped diagram in $\mathcal{A}$. We pass it to $\mathsf{Cat}(\mathcal{A})$ be composing with

$$\mathcal{A} \xrightarrow{\text{disc}} \mathsf{Cat}(\mathcal{A})$$

and take its limit, which will also be on the form $\text{disc}(A')$ for some object $A' \in \mathcal{A}$. Then clearly $A' \simeq \lim D$. □

Remark 1.3.21. As the above lemma shows, existence of limits in $\mathsf{Cat}(\mathcal{A})$ is a non-issue. Colimits, however, are a major obstacle, due to the fact that limits and colimits do not, in general, commute. The problem arises when we try to define the composition map in the to-be colimit $\mathcal{A}$-category. There are known sufficient conditions for $\mathsf{Cat}(\mathcal{A})$ to have colimits, two of which cover most of our intended examples.

Lemma 1.3.22. Let $\mathcal{A}$ be a category. Sufficient conditions on $\mathcal{A}$ which guarantee that $\mathsf{Cat}(\mathcal{A})$ has finite colimits include

(1) if $\mathcal{A}$ is a topos with a natural numbers object, in particular a Grothendieck topos, or

(2) if $\mathcal{A}$ is locally finitely presentable.

Proof. For (1), see [JT91]. For (2), see [AR94]. □

Remark 1.3.23. Examples of (1) include simplicial and cubical sets. Examples of (2) include sets, categories, bicategories, strict n-categories, groups and abelian groups.

1.3.2 strict k-tuple categories

Definition 1.3.24. We define the category $\mathsf{Cat}\langle n\rangle$ of *strict n-tuple or n-fold categories* inductively in the following fashion. The category of 0-tuple categories is the category Set of sets and functions between them. Now let $n \geq 1$. The category of strict n-tuple categories is inductively given by the category of categories internal to strict $(n-1)$-categories, i.e. $\mathsf{Cat}\langle n\rangle := \mathsf{Cat}(\mathsf{Cat}\langle n-1\rangle)$

Definition 1.3.25. Denote by $\square$ the category consisting of two objects, 0 and 1, a pair of arrows $d_0, d_1 : 0 \rightrightarrows 1$, and an arrow $s : 1 \to 0$, such that $d_0 \circ s = \mathrm{id}_0 = d_1 \circ s$.

Notation 1.3.26. We may denote an object a of $\square^k$ as a k-tuple of binary numbers $(a_1, ..., a_k)$. De denote the morphisms $d_0^j, d_1^j : (a_1, ..., a_{j-1}, 0, a_{j+1}, ..., a_k) \to (a_1, ..., a_{j-1}, 1, a_{j+1}, ..., a_k)$ and $s^j : (a_1, ..., a_{j-1}, 1, a_{j+1}, ..., a_k) \to (a_1, ..., a_{j-1}, 0, a_{j+1}, ..., a_k)$. We denote $\square^k \ni 0 = (0, ..., 0)$ and $\square^k \ni 1 = (1, ..., 0)$. Given an element $a = (a_1, ..., a_k) \in \square^k$, we denote by $\omega(a)$ the number of 1s among the a_i. Finally, given an $a \in \square^k$, denote $a' = 1 - a \in \square^k$ as the k-tuple where the zeroes and ones have been reversed.

Remark 1.3.27. Unravelling the definition, we see that a strict k-fold category C has k *directions* of morphisms, such that the data of morphisms of varying dimension, along with their source, target and identity maps, assemble into a cubical diagram of sets $C : (\square^k)^{\mathrm{op}} \to \mathsf{Set}$. We denote the image of $a \in \square^k$ under this functor by C_a. We say the elements of C_a are *cells* of *dimension* $\omega(a)$. Furthermore, a pair of elements of $C_{(1,...,1)}$ may be composed along a common face; there are composition functions

$$c^j : C_{(1,...,1)} \times_{d_0^j, d_1^j} C_{(1,...,1)} \to C_{(1,...,1)}$$

for each $1 \leq j \leq k$. This composition rule is required to be unital with respect to degenerate k-cells (i.e. the image of s^j in $C_{(1,...,1)}$).

This composition rule is required to satisfy associativity rules to the effect that any k-dimensional grid of k-dimensional cells in which any pair of neighbours are composable in the above sense, in the appropriate direction, has a unique composition.

Next, unravelling the definition of a functor between strict k-fold category is given by a morphism of the underlying cubical sets, such that the composition operations at each level and in each direction are preserved.

Remark 1.3.28. There is a natural generalization of the 2-morphisms of $\mathsf{Cat}(\mathcal{A})$ in Section 1.3.1 for $\mathsf{Cat}\langle n\rangle$. This construction works for any complete base category $\mathcal{A}$, but we will only talk about Set here, since that is all we are interested in here. For the sake of a clean exposition, we first introduce some auxiliary notation.

Notation 1.3.29. Given an object $a \in \square^n$, denote by $S_a \subseteq \{1, ..., n\}$ the subset containing those $1 \leq j \leq n$ for which $a_j = 0$. For a given subset $S \subseteq \{1, ..., n\}$, denote by $Z(S)$ the set of those $a \in \square^n$ for which $S \subseteq S_a$.

We also introduce the binary operation $(a, b) \mapsto a + b$ for $a, b \in \square^n$, where $(a + b)_j := a_j + b_j \mod 2$.

Definition 1.3.30. Let C and D be n-fold categories. The functors $\mathrm{Fun}(C, D)$ assemble into an n-tuple category as follows. The set of objects $\mathrm{Fun}(C, D)_0$ is the set of functors $C \to D$ as described in Remark 1.3.27. Let $a \in \mathrm{Cube}^n$. An element $F \in \mathrm{Fun}(C, D)_a$, also called an a-morphism, is the data of, for each $b \in Z(S_{a'})$, a function

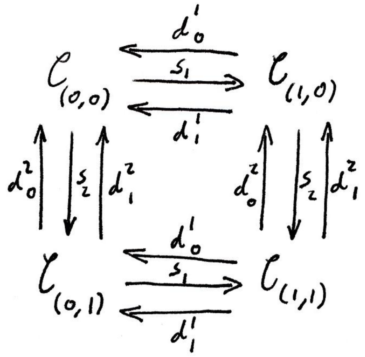
$$F_b : C_b \to D_{b+a}$$

The set of such C_b assemble into a $(n - \omega(a))$-fold category which we denote by C_0^a. We also denote by D_1^a the $(n - \omega(a))$-fold category generated by those D_c for which $c' \in Z(S_{a'})$. We then require that the F_b assemble into a functor of $(n - \omega(a))$-fold categories. Postcomposition with the face operators on a gives rise to face operations for a-morphisms, and similarly for degeneracy operations. See 1.5 for the situation written out for $n = 2$.

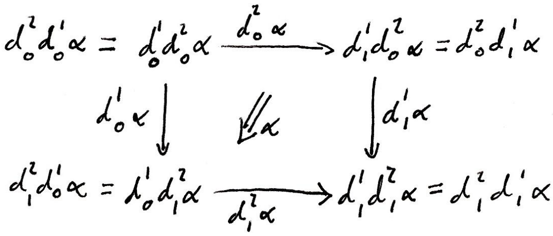
Example 1.5. Let us see in detail how this plays out for $k = 2$, i.e. a strict double category C. In this case, we have the following data:

- a set $C_{(0,0)}$ of *objects*,
- a set $C_{(0,1)}$ of *vertical* 1-*morphisms*,
- a set $C_{(1,0)}$ of *horizontal* 1-*morphisms*, and
- a set $C_{(1,1)}$ of 2-*morphisms*.

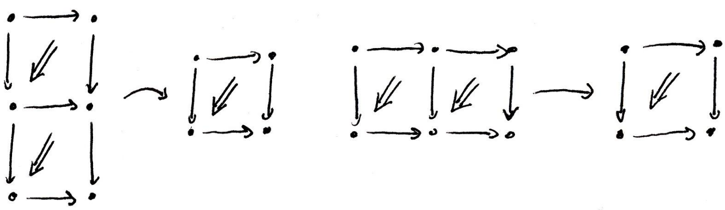
There are source, target and identity morphisms between these which fit into a cubical diagram:



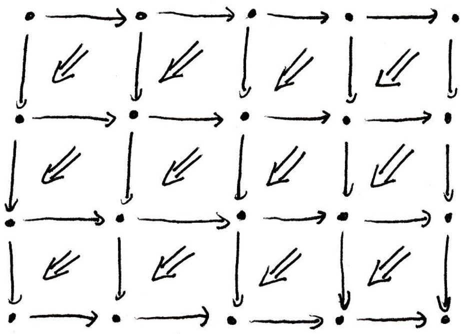
Where the face morphisms satisfy a commutation relation given by the following diagram:



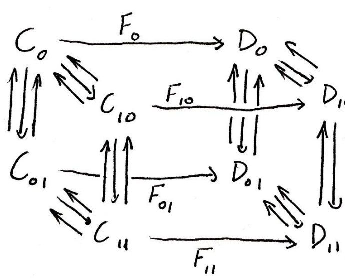
There are two directions in which to compose 2-morphisms in a double category: horizontal and vertical, represented by operations



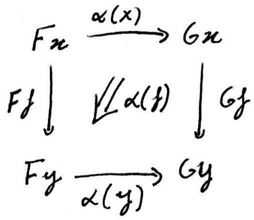
which satisfy an associativity relation to the effect that every 2-dimensional grid of composable 2-morphisms, like the following diagram, has a unique composition.

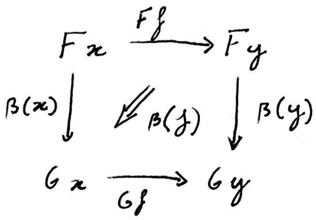


Let us now see how functors between double categories behave. Let C and D be a pair of double categories. As explained in Definition 1.3.30, a 0-morphism $F : C \to D \in \mathrm{Fun}(C, D)_0$ is a morphism between the underlying cubical sets, as in the following diagram:

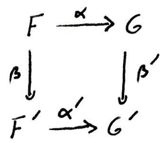


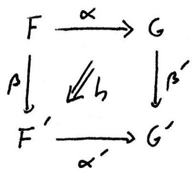
where each component map is required to commute with composition in each available direction. Next, we have 2-morphisms in two directions: horizontal and vertical 2-morphisms, also called $(1, 0)$-morphisms and $(0, 1)$-morphism respectively. Let $F, G \in \mathrm{Fun}(C, D)_0$ be functors from C to D. A $(1, 0)$-morphism $\alpha : F \to G \in \mathrm{Fun}(C, D)_{10}$ is given by, for each $x \in C_0$, a morphism $\alpha(x) : Fx \to Gx \in D_{10}$, and for each pair $x, y \in C_0$ and $f : x \to y \in C_{01}$, a $(1, 1)$-morphism $\alpha(f) \in D_{11}$ given by the left side of the below diagram. A $(0, 1)$-morphism $\beta : F \to G \in \mathrm{Fun}(C, D)_{01}$ is defined similarly by interchanging $(0, 1)$ and $(1, 0)$ in the above, pictured in the right part of the below diagram.

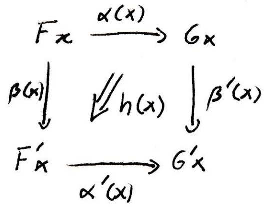




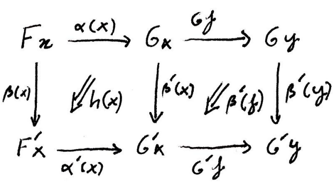
Finally, given four functors $F, F', G, G' \in \mathrm{Fun}(C, D)_0$, $(1, 0)$-morphisms $\alpha, \alpha' \in \mathrm{Fun}(C, D)_{10}$ and $(0, 1)$-morphisms $\beta, \beta' \in \mathrm{Fun}(C, D)_{01}$ fitting into the diagram below (left), a 3-morphism, or $(1, 1)$-morphism $h \in \mathrm{Fun}(C, D)_{11}$ fitting into this diagram (middle diagram below) is given by, for each $x \in C_0$, a $(1, 1)$-morphism $h(x) \in D_{11}$ fitting into the diagram below (right).

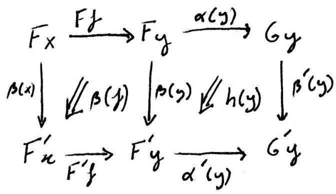




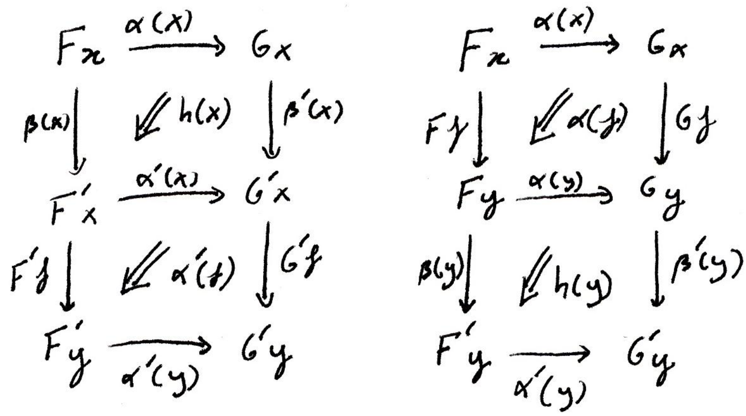


We also require, for any pair $x, y \in C_0$ and $f : x \to y \in C_{10}$, that the following horizontal compositions are equal.





Similarly, for any $f : x \to y \in C_{01}$ the following vertical compositions are equal.



Example 1.6. Let X be a topological space. In Definition 3.5.8, we define the notion of a *box* in $\mathbb{R}^k$, and the $\langle k \rangle$-space $\square(k)$ of such boxes. We define the *k-fold path category* of X, $P^k(X)$, as the topological k-fold category whose a-morphisms for $a \in 2^k$ is given by the space of morphisms $[x, y] \to X$ for $[x, y] \in \square(k)(a)$. A more precise name for this construction would be the *k-fold unreduced Moore path caregory* of X, as it is a k-fold generalization of an unreduced version of the usual Moore path space.

1.3.3 The nerve of a strict k-tuple category

(Ref. Fiore and Paoli : A Thomason model structure on the category of small n-fold categories)

Goal 1.3.31. Let C be a n-fold category. Then the classifying space BC of C is given by the geometric realization of the diagonal simplicial set associated to the n-fold nerve $N_{\bullet,\dots,\bullet}C$.

Definition 1.3.32. Let C be an n-fold category. We define the n-fold *nerve* of C, denoted $N_{\bullet,\dots,\bullet}C$, as the n-fold simplicial set whose set of $(k_1, \dots, k_n)$ multisimplices is given by the set of $(k_1, \dots, k_n)$ grids of composable n-morphisms. The multisimplicial structure maps are given by composing and inserting degenerate n-morphisms along each direction.

Definition 1.3.33. Let $X : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Set}^{\Delta^{\mathrm{op}}}$ be a bisimplicial set. We define its *realization* to be the simplicial set $|X|$ given by the coend

$$|X| = \int^{[n]\in\Delta} X_{n,*} \times \Delta[n]$$

computed in $\mathsf{Set}^{\Delta^{\mathrm{op}}}$.

Definition 1.3.34. Given an n-fold simplicial set $X : \Delta^{\mathrm{op}} \times ... \times \Delta^{\mathrm{op}} \to \mathsf{Set}$, we define its *diagonal*, $d(X)$, to be the simplicial set given by precomposing X with the diagonal functor $d : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times ... \times \Delta^{\mathrm{op}}$

Remark 1.3.35. There are now several possible ways to define the geometric realization of an n-fold simplicial set. We may take the ordinary geometric realization of its diagonal, or of the simplicial set obtained by repeated realizations of multisimplicial sets, or a combination of these. The below results show that the result is the same up to weak equivalence.

Lemma 1.3.36. The diagonal functor $d : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times ... \times \Delta^{\mathrm{op}}$ preserves homotopy colimits. I.e., for every diagram $J : \Delta^{\mathrm{op}} \times ... \times \Delta^{\mathrm{op}} \to \mathsf{Set}^{\Delta^{\mathrm{op}}}$, precomposing with d there is a weak equivalence hocolim $J \simeq$ hocolim $J \circ d$.

Lemma 1.3.37. Let $X : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Set}$ be a bisimplicial set. Denote by $d(X)$ the diagonal of X, defined as the precomposition of X with the diagonal functor $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$. Then there is a weak equivalence of simplicial sets

$$|X| := \int^{[n]\in\Delta} X_{n,*} \times \Delta[n] \simeq d(X)$$

Theorem 1.3.38. Let C be an n-fold category. Then the geometric realization of the n-fold nerve of C is weakly equivalent to the diagonal nerve $d(N_{\bullet,...,\bullet}C)$.

Remark 1.3.39. The above also applies when C is a topological strict n-fold category. In that case, we obtain an n-fold simplicial space. The classifying space is then given by the geometric realization of the diagonal, which is defined as follows.

Definition 1.3.40. Let $X : \Delta^{op} \to \mathsf{Top}$ be a simplicial topological space and write Δ_{Top} for the cosimplicial space whose n'th space is given by the standard topological n-simplex $\Delta^n_{\mathsf{Top}} = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid x_0 + ... + x_n = 1 \text{ , and } x_i \leq 0 \forall 0 \leq i \leq n\}$. The geometric realization of X, denoted $|X|$, is defined by the quotient

$$|X| = \coprod_{[n]\in\Delta} X_n \times \Delta^n_{\mathsf{Top}} / \sim$$

where the equivalence relation identifies, for each pair $(x, p) \in X_l \times \Delta^k_{\mathsf{Top}}$ and each morphism $f : [k] \to [l]$ in Δ, the points $(x, f_* p) \in X_l \times \Delta^l_{\mathsf{Top}}$ and $(f^* x, p) \in X_k \times \Delta^k_{\mathsf{Top}}$.

1.3.4 Obtaining k-categories from strict k-fold categories

Goal 1.3.41. In this section we will show how a strict k-fold category C gives rise to a k-category in a number of ways. If C is "isotropic" in the sense defined below, we show that an additional method becomes possible, which we call *folding*.

Remark 1.3.42. If C is a strict k-fold category, there is always two canonical ways to obtain a $(k-1)$-fold category from C for each $1 \leq j \leq k$, by restricting the structure to those $a \in \square^k$ for which $a_j = 0$ or $a_j = 1$.

Definition 1.3.43. Let C be a k-fold category. There is a natural action of Σ_k, the symmetric group on k letters, on $\square^k$, defined by permuting indices. We say that C is *isotropic* if this action induces an action on C in such a way that the source, target, identity and composition maps are all Σ_k-equivariant.

Remark 1.3.44. Let C be an isotropic double category. This is the same as a double category equipped with a bijection $C_{01} \to C_{10}$ and a twist autobijection $\tau : C_{11} \to C_{11}$ such that $\tau^2 = \mathrm{id}$ and these commute with all the structure maps. Then the following defines a 2-category. Let the set of objects be C_0, and let $C_1 := (C_{01} \cup C_{10})/\sim$, where we identify $f \sim \phi(f)$ for all $f \in C_{01}$, be the set of 1-morphisms. For each $f, g \in C_1$, we now define the morphism set $C_2(f, g)$ to be the union of all (11)-morphisms C_{11} whose boundary

$$\begin{array}{ccc} x_0 & \xrightarrow{f_1} & x_1 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ x_2 & \xrightarrow[g_2]{} & x_3 \end{array}$$

satisfies $\phi(f_2) \circ f_1 = f$ and $g_2 \circ \phi(g_1) = g$.

Remark 1.3.45. The same construction is available in more general circumstances, but it less well-behaved. First of all, it makes use of the *free product of categories*. Let C and D be a pair of categories with the same object sets, $\mathrm{Ob}(C) = \mathrm{Ob}(D)$. Let $F : S \to D$ be a functor from a subcategory of C. Then we can form the pushout $C \cup_{C'} D$, which acts as a categorical analogue of the amalgamated free product of monoids. Using this construction, we can construct a 2-category from the data of an arbitrary 2-fold category C equipped with functors

$F : S \to C_{10}$ and $G : S \to C_{01}$, where S is a small category. This construction behaves well when, say, one of C_{10} and C_{01} is a subcategory of the other.

Remark 1.3.46. A similar folding construction is available for k-fold categories, but the combinatorics involved would bring us too far afield, so we will not go into details about its construction.

Remark 1.3.47. The consequence of the folding construction is that whenever we talk about strict k-fold categories of cobordisms, it is in principle possible to tweak it into a strict k-category of cobordisms, but this loses information. Indeed, strict k-fold categories model homotopy types ([Lod82]), while strict k-categories do not (for example, there is no strict 3-category with the homotopy 3-type of S^2, see [Sim11, Part I, Sec. 2.7]). For this reason, we deem the question of obtaining strict k-categories of cobordisms less interesting than the k-fold case, and will not be paying further attention to strict k-categories in this text.

Chapter 2
Introduction to cobordism

Goal 2.0.48. The purpose of this chapter is to give a brief introduction to central ideas related to cobordism theory. We will assume that the reader has some basic knowledge about algebraic topology and differential topology. Standard references for background material are [Hat01] and [Hir76].

2.1 Structured manifolds

Remark 2.1.1. In this section we will look briefly at tangential structures on manifolds. We do this mainly for completeness. Everything we say later about manifolds applies also for manifolds with arbitrary tangential structure, with some added bookkeeping in order to keep track of these. However, the structures themselves will not occupy our attention, and we include this section merely for completeness. The reader may therefore safely skip this section completely.

Convention 2.1.2. We will follow the standard convention of referring to a vector bundle $F \to E \to B$ by naming only the total space E.

Definition 2.1.3. Let $G(r, n)$ be the space of r-dimensional subspaces of $\mathbb{R}^{r+n}$.

Recollection 2.1.4. There is a natural inclusion $\mathbb{R}^{n+d} \to \mathbb{R}^{n+d+1}$ given by $v \mapsto (v, 0)$, and this induces inclusions $G(r, n) \to G(r, n+1)$ and $\gamma(r, n) \to \gamma(r, n+1)$. Let $O(r)$ be the Lie group of orthogonal $r \times r$ matrices with real coefficients. The colimit of bundle maps gives rise to a model for the classifying space BO_r of $O(r)$:

$$\operatorname{colim}_{n\to\infty} G(r, n) := G(r, \infty) \simeq BO_r$$

$$\operatorname{colim}_{n\to\infty} \gamma(r, n) := \gamma(r, \infty) \simeq \gamma_r^O$$

where $\gamma_r^O \to BO_r$ is the universal r-plane bundle consisting of pairs (V, v) of an r-dimensional subspace V of $\mathbb{R}^{n+r}$, for some $n \geq 0$, and a vector $v \in V$.

Recollection 2.1.5. Let M be a smooth manifold. A rank r vector bundle $p : E \to M$ is in particular an $O(r)$-principal fiber bundle, so there exists a classifying map $\xi_p : M \to BO_r$ which is unique up to homotopy, such that the following diagram commutes:

$$\begin{array}{ccc} E & \longrightarrow & \gamma_r^O \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p^O} \\ M & \xrightarrow[\xi_p]{} & BO_r \end{array}$$

where $p^O : \gamma_r^O \to BO_r$ is the universal r-plane bundle.

Definition 2.1.6. Let X_r be a space with a fibration $f_r : B_r \to BO_r$, M a smooth manifold and $p : E \to M$ a vector bundle on M with rank r. We say that an (B_r, f_r)-structure on E is a lift of ξ_p through f_r. Two such lifts are said to be equivalent if they are homotopic.

Example 2.1. Recall that a *Whitehead tower* of a pointed space X is a factorization of the basepoint inclusion $* \to X$ into a sequence

$$* \to ... \to X^{(2)} \to X^{(1)} \to X^{(0)} \simeq X$$

such that for each n, the pointed space $X^{(n)}$ is $(n-1)$-connected, and the map $X^{(n+1)} \to X^{(n)}$ induces an isomorphism on all homotopy groups in degree $k > n$.

The Whitehead tower of BO_r starts out as

$$... \to B\mathrm{String}_r \to B\mathrm{Spin}_r \to BSO_r \simeq BSO_r \to BO_r$$

This gives us a wealth of examples of fibrations $B_r \to BO_r$ and thus of naturally occurring tangential structures on manifolds. Given a d-manifold with a smooth structure, i.e. a map $M \to BO_d$, the question of whether this structure lifts through higher orientations can be answered by obstruction theory, by looking at different universal characteristic classes.

Example 2.2. If $p : E \to M$ is a rank 1 vector bundle, also called a *line bundle*, then ξ_p lifts through $f_1 : BSO_1 \to BO_1$ if and only it is trivial, or equivalently, if the first Stiefel-Whitney class of E vanishes. To see this, recall that $BO_1 \simeq \mathbb{R}P^\infty \simeq K(\mathbb{Z}/2, 1)$, such that $[M, BO_1] \simeq H^1(M; \mathbb{Z}/2)$. Now, BSO_1 is contractible, so a line bundle is orientable if and only if ξ_p is nullhomotopic, which happens if and only if the corresponding cohomology class in $H^1(M; \mathbb{Z}/2)$ vanishes and the bundle is trivial. In particular, all line bundles on a simply connected manifold are trivial.

2.2 Remarks on embeddings of smooth manifolds

In this section we describe the classical cobordism category, which to the author's knowledge was first defined by Galatius-Madsen-Tillmann-Weiss [GMTW09].

Idea 2.2.1. The details which go into the definition of the cobordism category are somewhat technical, but informally we may describe it as having objects $(d-1)$-dimensional manifolds and d-dimensional cobordisms between these as morphisms. It is useful for technical reasons to consider manifolds not in the abstract sense, but as embedded submanifolds of high-dimensional euclidean space. We will see this equips the set of manifolds with natural topologies and a particularly pleasing choice of simplicial replacement.

Definition 2.2.2. Let M and N be manifolds, possibly with boundary. Recall that an embedding $M \to N$ is an immersion which is diffeomorphic to its image as a submanifold of N. Denote by $\mathrm{Emb}(M, N)$ the space of embeddings of M into N, equipped with the compact-open topology. If M and N have boundaries ∂M and ∂N, let $\mathrm{Emb}^\partial(M, N)$ be the subset of $\mathrm{Emb}(M, N)$ which are boundary preserving. I.e. for each $\phi : \mathrm{Emb}^\partial(M, N)$, we have $\phi(\partial M) \subseteq \partial N$.

Remark 2.2.3. For the purposes of defining the cobordism category, we are also interested in the colimit of the spaces $\mathrm{Emb}(M, \mathbb{R}^N)$ as N tends to infinity. We will take some time to talk about the nice properties this space possesses.

Definition 2.2.4. We define the infinite dimensional euclidean space $\mathbb{R}^\infty$ as the colimit

$$\mathbb{R}^\infty := \mathrm{colim}\{\mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \hookrightarrow ...\}$$

in which each arrow $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ is the inclusion $(x_1, ..., x_n) \mapsto (x_1, ..., x_n, 0)$. In other words, an element of $\mathbb{R}^\infty$ is given by a tuple of real numbers of arbitrary finite length, with scalar multiplication and addition defined degreewise.

Remark 2.2.5. $\mathbb{R}^\infty$ inherits a scalar product from its finite-dimensional constituents defined by $(x_1, x_2, ...) \cdot (y_1, y_2, ...) = \sum_{i=1}^{\infty} x_i y_i$. Note that only finitely many terms are non-zero, so the sum is well-defined. The fact that $\mathbb{R}^\infty$ is a locally compact topological vector space is clear from the definition. It is actually globally convex. Any pair of elements of $\mathbb{R}^\infty$ are contained in $\mathbb{R}^n$ for some n, and so is the line connecting them.

Remark 2.2.6. Any infinite sequential colimit is isomorphic to the colimit of any diagram degreewise isomorphic to one of its infinite subsequences. Hence we may also define $\mathbb{R}^\infty$ as

$$\mathbb{R}^\infty := \operatorname{colim}\{\mathbb{R} \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^4 \hookrightarrow \mathbb{R}^8 \hookrightarrow ...\}$$

We have $\mathbb{R}^{2^{n+1}} \simeq \mathbb{R}^{2^n} \oplus \mathbb{R}^{2^n}$ both as vector spaces and smooth manifolds, and may take the arrows in the diagram to be the inclusion into the first direct summand.

Definition 2.2.7. For $x \in \mathbb{R}^\infty$ denote by $\text{mindeg} x$ the lowest natural number n such that $x_n \neq 0$, or ∞ if $x = 0$.

Lemma 2.2.8. For $x \in \mathbb{R}^\infty$ nonzero, $\text{mindeg} x <$ is finite.

Proof. Any such x is a finite sequence of reals, so unless $x = 0$ there must be some finite degree n for which $x_n \neq 0$. □

Lemma 2.2.9. There is an isomorphism $f : R^\infty \to R^\infty \oplus R^\infty$ with the property that for one of the induced maps on summands $f_j : R^\infty \to R^\infty$ where $j = 1, 2$, the eventual image vanishes, $\bigcap_{n \geq 1} \operatorname{im}^n f_j \simeq \{0\}$.

Proof. Define the maps $u_i : \mathbb{R}^n \hookrightarrow \mathbb{R}^{2n}$ for $i = 1, 2$ such that $u_1(x_1, x_2, ..., x_n) = (x_1, 0, x_2, 0, ..., x_n, 0)$ and $u_2(x_1, x_2, ..., x_n) = (0, x_1, 0, x_2, ..., 0, x_n)$. Denote by i_j the inclusion of the j-th direct summand of $\mathbb{R}^{2n} \simeq \mathbb{R}^n \oplus \mathbb{R}^n$, that is $i_1(x_1, ..., x_n) = (x_1, ..., x_n, 0, ..., 0)$ and $i_2(x_1, ..., x_n) = (0, ..., 0, x_1, ..., x_n)$. Note that then $i_j \circ u_{j'} = u_j \circ i_{j'}$ for all $j, j' = 1, 2$. The maps f_j are then induced by the maps of diagrams

$$\begin{array}{ccccccccc}
\mathbb{R} & \xrightarrow{i_1} & \mathbb{R}^2 & \xrightarrow{i_1} & \mathbb{R}^4 & \xrightarrow{i_1} & \mathbb{R}^8 & \longrightarrow & ... \\
& \searrow^{u_j} & & \searrow^{u_j} & & \searrow^{u_j} & & \searrow^{u_j} & \\
\mathbb{R} & \xrightarrow{i_1} & \mathbb{R}^2 & \xrightarrow{i_1} & \mathbb{R}^4 & \xrightarrow{i_1} & \mathbb{R}^8 & \longrightarrow & ...
\end{array}$$

In other words, we have $f_1(x_1, x_2, ...) = (x_1, 0, x_2, 0, ...)$ and $f_2(x_1, x_2, ...) = (0, x_1, 0, x_2, ...)$. The images of f_1 and f_2 are clearly orthogonal and span $\mathbb{R}^\infty$, so they exhibit $\mathbb{R}^\infty$ as a direct sum of two copies of itself. For the second claim, for each nonzero $x \in \mathbb{R}^\infty$ and every natural number k, $\mathrm{mindeg}(f_2)^{k+1}(x) > \mathrm{mindeg}(f_2)^k(x)$. This shows that the minimal degree of any element of the eventual image is larger than any finite integer, so any such element must be trivial, and the eventual image vanishes. □

Remark 2.2.10. Any manifold admits an essentially unique embedding into $\mathbb{R}^\infty$. The abstract properties of $\mathbb{R}^\infty$ introduced in Lemma 2.2.9 are enough to establish the essential uniqueness part. For a compact manifold, a partition of unity subordinate to an atlas is usually employed to establish an embedding into $\mathbb{R}^N$ for some finite N. For non-compact manifolds, the same line of argument gives rise to an embedding into $\mathbb{R}^\infty$.

Lemma 2.2.11. Let M be an n-manifold. M admits an embedding into $\mathbb{R}^\infty$ with closed image.

Lemma 2.2.12. Let M be an n-manifold. Then the space $\mathrm{Emb}(M, \mathbb{R}^\infty)$ is contractible.

Remark 2.2.13. The proofs of Lemmas 2.2.11 and 2.2.12 are available at [Sta12].

Definition 2.2.14. Let M be an n-manifold. Denote by $\mathrm{Diff}(M)$ the topological group of diffeomorphisms of M equipped with the compact-open topology.

Lemma 2.2.15. Let M be an n-manifold. Then $\mathrm{Emb}(M, \mathbb{R}^\infty)$ admits a free action on the right by $\mathrm{Diff}(M)$.

Proof. Let $\psi \in \mathrm{Diff}(M)$ and $e \in \mathrm{Emb}(M, \mathbb{R}^\infty)$. We define the action by precomposition: $e \cdot \psi = e \circ \psi$. This is continuous since we chose the compact-open topology on both spaces and M is a manifold, and in particular locally compact Hausdorff. To see that this is a free action, note that is for any $e \in \mathrm{Emb}(M, \mathbb{R}^\infty)$ we have $e \circ \psi = e$, we have in particular $e \circ \psi(x) = e(x)$ for every $x \in M$, but since e is monic, this implies $\psi(x) = x$ and $\psi = \mathrm{id}_M$. □

Definition 2.2.16. Let M be a manifold and let U and V be submanifolds of M. We say that U and V intersect *transversally* if $T_xU + T_xV = T_xM$ (note: this is the ordinary sum, not a direct sum) at each $x \in U \cap V$. Similarly, if $U \to M$ is a smooth map, we say that f is transverse to V if $\mathrm{im}(df_x) + T_{f(x)}V = T_{f(x)}M$ for each $x \in f^{-1}(V)$.

Theorem 2.2.17. Let $\mathrm{Emb}(U, M)$ be the space of embeddings of U into M, and let V be a submanifold of M. Denote by $\mathrm{Emb}^V(U, M) \subseteq \mathrm{Emb}(U, M)$ the subset of those embeddings which are transverse to V. Then $\mathrm{Emb}^V(U, M)$ is a dense subset of $\mathrm{Emb}(U, M)$

Proof. See [Tho54] or [Hir76], Chapter 2, Theorem 2.4. □

Definition 2.2.18. Let W be a d-dimensional manifold. An embedding of W into $[a_0, a_1] \times \mathbb{R}^{n+d-1}$ is called *proper* if W intersects $\{a_0, a_1\} \times \mathbb{R}^{n+d-1}$ transversally in ∂W.

Remark 2.2.19. Analogues of Lemmas 2.2.11, 2.2.12 and 2.2.15 exist when we replace M by W and embeddings by proper embeddings.

2.3 Rudiments of spectra

Goal 2.3.1. In this section, we will give the bare minimum material on spectra required to make sense of the rest of the text. All the material in this section is taken from [Rud98] Readers with a basic knowledge of spectra may safely skip this section.

Convention 2.3.2. We assume that all spaces are pointed CW complexes. We use the notation $SX = S^1 \wedge X$ for the pointed suspension of a space X.

Definition 2.3.3.

- A *spectrum* E is a sequence $\{E_n, s_n\}_{n\in\mathbb{Z}}$, where the E_n are spaces and the s_n are embeddings of CW complexes $s_n : SE_n \to E_{n+1}$.
- Let $E = \{E_n, s_n\}$ and $F = \{F_n, t_n\}$ be spectra. Then F is called a *subspectrum* of E if F_n is a subcomplex of E_n and $s_n|_{SF_n} = t_n$ for each $n \in \mathbb{Z}$.
- Let E be a spectrum. The k'th *suspension* of E, denoted $\Sigma^k E$ is defined by $(\Sigma^k E)_n = E_{n+k}$.
- Let E be a spectrum. A *cell* of E is a sequence $\{e, Se, ..., S^k e, ...\}$ where e is a cell of E_n which is not the suspension of any cell in E_{n-1}. If e is a k-dimensional cell of E_n, then e is a $(k-n)$-dimensional cell of E.

– Let E be a spectrum and let F be a subspectrum. We say that F is *cofinal* in E if each cell of E is eventually in F. I.e. for each cell e of E, there is some k for which $S^k e \in F$.

Example 2.3. Recall from two sections ago the universal bundles $\gamma_r^O \to BO_r$. The inclusion $G(r,n) \to G(r+1,n)$ given by adding a factor $\mathbb{R}$ in the last coordinate gives rise to bundle maps

$$\begin{array}{ccc} \gamma_r^O & \xrightarrow{j_r} & \gamma_{r+1}^O \\ \downarrow & & \downarrow \\ BO_r & \xrightarrow[i_r]{} & BO_{r+1} \end{array}$$

The pullback along j_r is given by $j_r^* \simeq \gamma_r^O \oplus \epsilon^1$, where ϵ^1 is the trivial one-dimensional vector bundle. Denote by $\mathrm{Th}(\gamma)$ the one-point compactification of a vector bundle $\gamma \to B$ when B is compact. On the one-point compactifications, we then get maps

$$e_r : \mathrm{Th}(\gamma_r^{\perp O}) \wedge S^1 \to \mathrm{Th}(\gamma_{r+1}^{\perp O})$$

so we get a spectrum, which we call the *Thom spectrum* $MO = \{\mathrm{Th}(\gamma_r^{\perp O}), e_r\}$.

Lemma 2.3.4. 1. Let E be a spectrum, let F be a cofinal subspectrum of E, and let G be a subspectrum of F. Then G is cofinal in E if and only if G is cofinal in F.

2. Let E be a spectrum and let F and F' be cofinal subspectra. Then the intersection $F \cap F'$, whose n'th space is given by $F_n \cap F'_n \subseteq E_n$, is also a cofinal subspectrum.

Definition 2.3.5. Let E and E' be spectra.

– A *map* $f : E \to E'$ is a family of pointed maps $f_n : E_n \to E'_n$ such that all the squares of the following form commute.

$$\begin{array}{ccc} SE_n & \xrightarrow{s_n} & E_{n+1} \\ {\scriptstyle Sf_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ SE'_n & \xrightarrow[s'_n]{} & E'_{n+1} \end{array}$$

- Let E'' be a spectrum. Given maps $f : E \to E'$ and $f' : E' \to E''$, the *composition* $f' \circ f$ is defined by the maps $f'_n \circ f_n : E_n \to E''_n$.
- Let F be a subspectrum of E and $f : E \to E'$ be a map. Then the *restriction* of f to F, denoted $f|_F$, is given by the family of maps $f_n|_{F_n} : F_n \to E'_n$.
- Let S be the set of pairs (F, f) where F is a cofinal subspectrum of E and $f : F \to E'$ is a map. Two such pairs (F, f) and (F', f') are said to be *equivalent* if there is a pair (G, g) such that G is a cofinal subspectrum of $F \cap F'$ and $f|_G = g = f'|_G$. A equivalence class under this relation is called a *morphism* $E \to E'$.

Lemma 2.3.6. Composition of morphisms is well defined and gives rise to a category Spc of spectra.

Lemma 2.3.7.
1. Let $f, g : E \to F$ be maps of spectra and let E' be a cofinal subspectrum of E. If $f|_{E'} = g|_{E'}$, then $f = g$.
2. Let E' and E'' be cofinal subspectra of E and let $f' : E' \to F$ and $f'' : E'' \to F$ be two equivalent maps. Then $f'|_{E' \cap E''} = f''|_{E' \cap E''}$.
3. Every morphism $f : E \to F$ contains a greatest element with respect to the partial ordering.

Proof. [Rud98, Ch.II,Proposition 1.6.] □

Definition 2.3.8. Let $E = (E_n, s_n^E)$ and $F = (F_n, s_n^F)$ be spectra.

- Let $f : E \to F$ be a map of spectra. The *mapping cone*, or *cofiber* of f is defined as the spectrum Cf whose n'th space is given by Cf_n, the cofiber of $f_n : E_n \to F_n$, and with structure maps

$$SCf_n = SF_n \cup S(CE_n) \xrightarrow{s_n^F \cup Cs_n^E} F_{n+1} \cup CE_{n+1} = Cf_{n+1}$$

– Let $f : E \to F$ be a morphism of spectra. The *cofiber* of f is defined as the cofiber of g, where $g : E' \to F$ is the largest element in f.

Definition 2.3.9. – Let X be a space. Define the *infinite suspension* of X as the spectrum whose n'th space is given by $S^n X$ and whose structure maps are the indentities.

– Let $E = \{E_n, s_n\}$ be a spectrum. For each $k \in \mathbb{Z}$, the adjoints of the s_n, which we call u_n, arrange into a sequence

$$E_k \to \Omega E_{k+1} \to \Omega^2 E_{k+2} \to ...$$

Define the $(\infty - k)$*'th loop space* of E as the colimit of this sequence:

$$\Omega^{\infty-k} E := \mathrm{colim}_{n\to\infty} \Omega^{n-k} E_n$$

Definition 2.3.10. Let E be a spectrum. The morphisms $u_n : E_n \to \Omega E_{n+1}$ induces a homomorphisms on the level of homotopy groups given by

$$\pi_k(u_n) : \pi_{n+k}(E_n) \to \pi_{n+k}(\Omega E_{n+1}) \simeq \pi_{n+k+1}(E_{n+1})$$

The *homotopy groups* of E are defined as the colimit over these homomorphisms,

$$\pi_k E := \mathrm{colim}_{n\to\infty} \pi_{n+k} E_n$$

2.4 Categories of cobordisms

Definition 2.4.1. Let M and N be d-dimensional manifolds, possible with some tangential structure. A *cobordism* from M to N is a $(d+1)$-dimensional manifold W with the same structure, such that there is a structure-preserving diffeomorphism $\partial W \simeq (-M) \sqcup N$, where $(-M)$ denotes M with the opposite structure.

Lemma 2.4.2. The cobordism relation is an equivalence relation.

Proof. Let M, N and P be d-dimensional manifolds such that there are cobordisms V from M to N and W from N to P.

- Reflexivity: The cylinder $M \times I$ is a cobordism from M to M.
- Symmetry: The manifold $(-V)$ is a cobordism from N to M.
- Transitivity: Choose smooth collars about N in V and W. With respect to these collars, there exists a glueing $X : V \cup_N W$ of V and W along their common boundary component N such that the union of the collars attain the smooth structure of the open cylinder $N \times (-1, 1)$. Then X is a cobordism from M to P.

□

Definition 2.4.3. The set of equivalence classes of unoriented d-dimensional manifolds with respect to the cobordism relation, called d-dimensional unoriented *cobordism classes*, is denoted by Ω_d.

Lemma 2.4.4. Disjoint union and cartesian product equips Ω_* with the structure of a graded commutative ring.

Theorem 2.4.5 (Pontryagin-Thom theorem)**.** There is an isomorphism of graded rings

$$\Omega_* \simeq \pi_*(MO)$$

where MO is the orthogonal Thom spectrum.

Remark 2.4.6. Classical cobordism theory focuses on the structure of Ω_d and the variant $\Omega_d(X)$ of manifolds equipped with a map into a background space X, which can be seen to define a cohomology theory $\Omega_d(X) \simeq \pi_d(MO \wedge X_+)$. Results similar to the Pontryagin-Thom theorem hold for arbitrary structure groups, where we replace MO with the Thom spectrum associated to the pullback of the universal d-plane bundle along the fibrations $B_d \to BO_d$.

Remark 2.4.7. Especially in the presence of tangential structure, we can see that the cobordism relation comes equipped with natural source and targets. In the definition above, M is the source and N is the target of W (in the unstructured case these come as auxiliary data). While from the classical point of view we would be satisfied with viewing cobordism as an equivalence relation, the modern point of view is to attempt to organize these cobordisms into a *categorical structure*. In other words, we lift the focus from the manifolds themselves and make the cobordisms between them the main actors. There are several ways one might attempt to do this, depending on what one wants to accomplish. We will review the main lines of thought in this section.

2.4.1 Contracting diffeomorphism classes

Definition 2.4.8. A cobordism W from M to N can be formalized as a cospan of embeddings

$$M \xrightarrow[i_M]{} W \xleftarrow[i_N]{} N$$

where i_M is reverses the tangential structure, i_N preserves the tangential structure, and $\partial W = i_M(M) \sqcup i_N(N)$.

Definition 2.4.9. Let M and N be d-dimensional manifolds and let W and W' be a pair of cobordisms from M to N. We say that W and W' are *equivalent* as cobordisms if there is a diffeomorphism $\phi : W \to W'$ which preserves the boundary components, i.e. there is a commutative diagram.

$$\begin{array}{ccccc} & \overset{i_M}{\nearrow} & W & \overset{i_N}{\nwarrow} & \\ M & & \downarrow \phi & & N \\ & \underset{i'_M}{\searrow} & W' & \underset{i'_N}{\swarrow} & \end{array}$$

We say that ϕ is an *isomorphism of cobordisms.*

Remark 2.4.10. Notice that the transitivity of the cobordism relation depends on a choice of smooth collars for our manifolds. Indeed, in the cospan-picture the composition is given by taking pushouts in the category of smooth manifolds, which are only well-defined up to canonical isomorphisms. Because of this dependence, the glueing procedure is not strictly associative. However, it will be associative up to isomorphism of cobordisms. This motivates the following definition.

Definition 2.4.11. Let $d > 0$. The d-dimensional *oriented cobordism category* Cob_d is defined such that

- objects are $(d-1)$-dimensional manifolds $M, N, \ldots,$
- morphisms are equivalence classes of cobordisms W from M to N in the sense of Definition 2.4.9,

– composition is given by glueing cobordisms,

– and the identity maps are given by the cylinders $M \times I$.

Lemma 2.4.12. Disjoint union equips Cob_d with a symmetric monoidal structure. The unit object is given by the empty manifold.

Remark 2.4.13. One of the main perspectives of cobordism theory has been to study Cob_d in terms of symmetric monoidal functors $\mathsf{Cob}_d \to A$, called *topological field theories*, for some symmetric monoidal algebraic category A. These can be thought of as categorical analogues of group representations, in the sense that a linear group representation is simply a functor $G \to \mathsf{Vect}_k$ where we consider G as a category with one object. In this sense, cobordism in a particular dimension encodes a specific algebraic structure. We give two easy examples in the oriented case.

Example 2.4.

1. (d=1): A functor $F : \mathsf{Cob}_1^{SO} \to \mathsf{Vect}_k$ from the oriented 1-dimensional cobordism category determines and is determined by specifying a finite-dimensional vector space $V = F(\bullet)$, the image of the point.

2. (d=2): A functor $F : \mathsf{Cob}_2^{SO} \to \mathsf{Vect}_k$ determines and is determined by a finite-dimensional commutative Frobenius algebra $A = F(S^1)$. This result is due to Abrams in [Abr96].

2.4.2 Diffeomorphisms as higher structure

Another perspective is that we might want to study topological quantum field theories which are equivariant with respect to diffeomorphisms. For example, say we have an assignment taking manifolds to (projective) chain complexes and cobordisms to chain homomorphisms. Given a pair of cobordisms $W, W' : M \to N$ and the corresponding chain homomoprhisms $Z(W), Z(W') : Z(M) \to Z(N)$, we might want to assign to an isomorphism of cobordisms $\phi : W \to W'$ a homotopy from $Z(W)$ to $Z(W')$. Since (higher) homotopies give rise to an $(\infty, 1)$-category structure (in the sense of [Lur09]) on projective chain complexes, we might ask if diffeomorphisms and smooth (higher) isotopies give rise to a similar structure on cobordisms. It turns out that this is very possible, and was carried out by Lurie in [Lur].

2.4.3 Diffeomorphism spaces

Instead of incorporating the diffeomorphisms as higher morphisms in a higher category, it is also possible to incorporate them by putting a topology on the morphism spaces. Specifically, if we denote by $\mathrm{Diff}(W)$ the set of automorphisms of a cobordism $W : M \to N$, we take one element in each such isomorphism class and add a component homotopy equivalent to $B\mathrm{Diff}(W)$. It turns out that this idea cooperates really well with the notion of the *moduli space* of manifolds, resp. manifolds with boundary, making the setting categories internal to topological spaces the natural one. In this setting, it is even possible to recover strict associativity and unitality, while keeping the extra homotopical information provided by the diffeomorphism groups. This idea was explored in detail in dimension 2 by Madsen-Weiss in [MW07] and generalized to arbitrary dimension by Galatius-Madsen-Tillmann-Weiss in [GMTW09]. We will come back to this and go into the details of the construction in the next section.

This approach is equivalent to the higher-category approach, a fact which can be made precise using the language of model categories, but which will not occupy our attention here. However, the approach of topological cobordism categories has had independent interest as a way to investigate the homology groups of moduli spaces of manifolds.

2.5 Topological cobordism categories

In this section, we give a brief introduction to the cobordism categories introduced in [GMTW09].

Remark 2.5.1. We would like to define a category whose objects are given by closed $(d-1)$-dimensional smooth manifolds, and whose morphisms are cobordisms between these. However, due to the problems we addressed earlier about the glueing of manifolds only being well-defined up to a contractible choice of diffeomorphisms, we need to supplement the data with some extra structure such that associativity and unitality holds strictly.

Definition 2.5.2. We define the d-dimensional *cobordism category* $\mathcal{C}_d$ as the topological category whose objects are given by pairs (M, a), where M is a $(d-1)$-dimensional closed submanifold of $\mathbb{R}^\infty$, and for two such pairs (M, a) and (N, b), with $b > a$, a morphism $(M, a) \to (N, b)$ is given by a d-dimensional compact submanifold W of $[a,b] \times \mathbb{R}^\infty$ such that

$$W \cap (\{a\} \times \mathbb{R}^\infty) = M \subseteq W$$

$$W \cap (\{b\} \times \mathbb{R}^\infty) = N \subseteq W$$

and such that there exists an open neighborhood U of ∂W for which there exist a pair of real numbers $\epsilon_a > 0$ and $\epsilon_b > 0$ such that

$$U = (M \times [a, a+\epsilon_a)) \sqcup (N \times (b - \epsilon_b, b])$$

We also formally add in identity morphisms as the degenerate cobordisms $M \subset [a,a] \times \mathbb{R}^\infty = \{a\} \times \mathbb{R}^\infty$. Composition of morphisms is defined by taking the union of cobordisms in $\mathbb{R} \times \mathbb{R}^\infty$.

Remark 2.5.3. Central to the problem of computing both the cobordism ring and the classifying spaces of cobordism categories is the notion of the Pontryagin-Thom collapse map, which was devised by Thom in [Tho54]. We first need some preliminary definitions.

Definition 2.5.4. Let $p : E \to B$ be a vector bundle with B paracompact. Then there is a metric on E which restricts on each fiber E_b, $b \in B$, to the euclidean norm on that fiber.

- Define the *disk bundle* $D(E) \to B$ such that $D(E)_b = \{v \in E_b \mid |v| \leq 1\}$.
- Define the *sphere bundle* associated to p as the fiber bundle $S(E) \to B$ such that $S(E)_b = \{v \in E_b \mid |v| = 1\}$.
- Define the *Thom space* of p as the space $\mathrm{Th}(p) = D(E)/S(E)$. $\mathrm{Th}(p)$ is a pointed space with basepoint given by $S(E)$.

Lemma 2.5.5. If $p : E \to B$ is a vector bundle and B is compact, then $\mathrm{Th}(p)$ is homotopy equivalent to the one-point compactification of E.

Lemma 2.5.6. Let $p : E \to B$ and $p' : E' \to B$ be vector bundles, and B paracompact. Denote by $p \oplus p'$ the Whitney sum of p and p'. Then $\mathrm{Th}(p \oplus p') \simeq \mathrm{Th}(p) \wedge \mathrm{Th}(p')$

Proof. We have

$$(D(E)/S(E)) \wedge (D(E')/S(E')) \simeq (D(E) \times D(E'))/((S(E) \times D(E')) \cup (D(E) \times S(E')))$$

$$\simeq D(E \times E')/S(E \times E')$$

□

Corollary 2.5.7. Let $p : \epsilon^n \to B$ be the trivial n-plane bundle, i.e. $\epsilon^n \simeq \mathbb{R}^n \times B$ and p is the projection. Then $\mathrm{Th}(p) \simeq S^n \wedge B_+ = S^n(B_+)$, the n-fold suspension of B_+, where B_+ denotes B appended with a disjoint basepoint.

Remark 2.5.8. Consider now the sequence of Grassmannians

$$... \to G(d,n) \to G(d,n+1) \to ...$$

induced by the inclusions $\mathbb{R}^{d+n} \to \mathbb{R}^{d+n+1}$. The pullback of the universal bundle $\gamma^\perp(d,n+1)$ along $G(d,n) \to G(d,n+1)$ is isomorphic to $\gamma^\perp(d,n) \oplus \epsilon^1$. The inclusion $\gamma^\perp(d,n) \oplus \epsilon^1 \to \gamma^\perp(d,n+1)$ is given by $((V,v),a) \to (V,(v,a))$. It follows that there is a map

$$S^1 \wedge \mathrm{Th}(\gamma^\perp(d,n)) \to \gamma^\perp(d,n+1)$$

Thus the spaces $\mathrm{Th}(\gamma^\perp(d,n))$ assemble into a spectrum.

Definition 2.5.9. Let the d-dimensional *Madsen-Tillmann spectrum* $MT(d)$ be the spectrum whose $(n+d)$'th space is given by $\mathrm{Th}(\gamma(d,n))$ and whose structure maps are as in the above remark.

Remark 2.5.10. We will now outline the Pontryagin-Thom collapse map as applied to the study of topological cobordism categories. In a nutshell, this construction relates the classifying space $B\mathcal{C}_d$ of the cobordism category to the $(\infty-1)$st loop space $\Omega^{\infty-1}MT(d)$ of the degree d Thom spectrum $MT(d)$.

Definition 2.5.11. Let $M \subseteq \mathbb{R}^{n+d}$ be a d-dimensional compact submanifold without boundary, let $M \subset U \subseteq \mathbb{R}^n$ be a tubular neighbourhood of M in $\mathbb{R}^{n+d}$, and let $\nu : U \to M$ denote the normal bundle. The projection $\mathbb{R}^{n+d} \to \mathbb{R}^{n+d}/(\mathbb{R}^{n+d} \setminus U) \simeq \mathrm{Th}(\nu)$ is called the *Pontryagin-Thom collapse map* associated to M. Note that since different choices of U are homotopic, this map is uniquely determined up to homotopy. This map extends through the inclusion $\mathbb{R}^{n+d} \hookrightarrow S^{n+d}$ by sending the basepoint of S^{n+d} to $(\mathbb{R}^{n+d} \setminus U)$. We thus get a map

$$S^{n+d} \to \mathbb{R}^{n+d}/(\mathbb{R}^{n+d} \setminus U) \simeq \mathrm{Th}(\nu)$$

Now denote by $u : U \to \gamma^\perp(d,n)$ the classifying map of ν on the level of the total spaces. Composing the above map with $\mathrm{Th}(u)$, we obtain a map

$$t_M^{n+d} : S^{n+d} \to \mathrm{Th}(\gamma^\perp(d,n)) = MT(d)_{n+d}$$

We call this map the *classifying map* of $M \subseteq \mathbb{R}^{n+d}$.

Lemma 2.5.12. The map t_M^{n+d} is stable, i.e. the following diagram commutes up to homotopy.

$$\begin{array}{ccc} S^1 \wedge S^{n+d} & \xrightarrow{t_M^{n+d}} & S^1 \wedge MT(d)_{n+d} \\ \downarrow & & \downarrow \\ S^{n+d+1} & \xrightarrow[t_M^{n+d+1}]{} & MT(d)_{n+d+1} \end{array}$$

Proof. Starting from $M \subseteq \mathbb{R}^{n+d}$, if we append an additional factor $\mathbb{R}$ onto the right hand side, the resulting normal bundle will have an extra trivial direct summand, $\nu^{n+d+1} \simeq \nu^{n+d} \oplus \epsilon^1$, from which we get that $\mathrm{Th}(\nu^{n+d+1}) \simeq S^1 \wedge \mathrm{Th}(\nu^{n+d})$, so the Ponryagin-Thom collapse map factors through the latter. In addition, the classifying map for ν^{n+d+1} factors, up to homotopy, through $\gamma^{\perp}(d,n) \oplus \epsilon^1 \to \gamma^{\perp}(d,n)$. This establishes the homotopy commutativity of the original diagram. □

Corollary 2.5.13. The Pontryagin-Thom construction assigns to each submanifold $M \subseteq \mathbb{R}^{\infty}$ a point in $\Omega^{\infty} MT(d)$.

Remark 2.5.14. The Pontryagin-Thom collapse map also applies to submanifolds with boundary. In particular, let $W : M \to N$ be a morphism of $\mathcal{C}_d$. Then W is a neatly embedded submanifold of $[a,b] \times \mathbb{R}^{n+d-1}$, and so the Pontryagin-Thom collapse map gives a map

$$[a,b]_+ \wedge S^{d+n-1} \to \mathrm{Th}(\nu) \to MT(d)_{n+d}$$

which is stable in the same way as above, and the adjoint of this map as $n \to \infty$ is a path $[a,b] \to \Omega^{\infty-1} MT(d)$. Now, if we choose once and for all a tubular neighbourhood for all $(d-1)$-dimensional submanifolds $M, N \subseteq \mathbb{R}^{d+n-1}$, say $M \subseteq U_M \subseteq \mathbb{R}^{d+n-1}$ and $N \subseteq U_N \subseteq \mathbb{R}^{d+n-1}$m we can require the tubular neighbourhoods of the morphisms behave nicely with respect to these, in particular we can choose our tubular neighbourhoods $W \subseteq U \subseteq [a,b] \times \mathbb{R}^{d+n-1}$ such that for $\epsilon_a, \epsilon_b > 0$ as in Definition 2.5.2, we have

$$U \cap [a, a+\epsilon_a) \times \mathbb{R}^{d+n-1} = [a, a+\epsilon_a) \times U_M$$

$$U \cap (b - \epsilon_b, b] \times \mathbb{R}^{d+n-1} = (b - \epsilon_b, b] \times U_N$$

such that we can glue tubular neighbourhoods by taking their unions. Then we obtain by the above assignment a functor

$$\mathcal{C}_d \to P^1(\Omega^{\infty-1} MT(d))$$

which on the level of classifying spaces gives us a map

$$\alpha : B\mathcal{C}_d \to B(P^1(\Omega^{\infty-1} MT(d))) \simeq \Omega^{\infty-1} MT(d))$$

Theorem 2.5.15. ([GMTW09]) The map

$$\alpha : B\mathcal{C}_d \to \Omega^{\infty-1} MT(d))$$

is a weak homotopy equivalence.

Remark 2.5.16. This result was subsequently generalized, using a different method of proof, in [BM]. This result will be discussed at the end of the next chapter.

Chapter 3

Manifolds with corners

3.1 Introduction

Manifolds with corners are a generalization of manifolds with boundary, where roughly for a manifold M, the relation $\partial^2 M = \emptyset$ is replaced by $\partial^k M = \emptyset$ for some $k \in \mathbb{N}$.

Definition 3.1.1. Let $k, n \in \mathbb{N}$ such that $n \geq k$. We define $\mathbb{R}^n_k := [0, \infty)^k \times \mathbb{R}^{n-k}$ to be the n-dimensional Euclidean plane with k-corners. For a point $x \in \mathbb{R}^n_k$, denote by $c(x)$ the number of factors $[0, \infty)$ which are zero. We call $c(x)$ the *depth* of x.

Definition 3.1.2. Let U and V be subsets of $\mathbb{R}^n$. We say a function $f : U \to V$ is smooth if there exist open subsets U' and V' of $\mathbb{R}^n$ such that $U \subseteq U'$ and $V \subseteq V'$ and there exists a smooth function $g : U' \to V'$, in the sense that partial derivatives of all orders exist, and such that $g|_U = f$. A usual, we say f is a diffeomorphism if it is smooth and admits a smooth inverse.

Lemma 3.1.3. Let $U \subseteq \mathbb{R}^n_k$ and $V \subseteq \mathbb{R}^n_k$ be subsets such that there is a diffeomorphism $f : U \to V$. Then for each $x \in U$ we have $c(x) = c(f(x))$.

Warning 3.1.4. The above lemma does not apply if we loosen the restriction on f to be merely a homeomorphism. Note that there is a homeomorphism $[0, \infty)^2 \to [0, \infty) \times \mathbb{R}$, but no such diffeomorphism exists. Therefore, in the theory of *topological* manifolds with corners, it is necessary to force the transition functions to respect the corner structure, which this comes for free in the smooth case.

Definition 3.1.5. An n-dimensional manifold with k-corners is a second-countable Hausdorff space which is locally homeomorphic to $\mathbb{R}^n_k$. I.e. for each $x \in M$, there is an open set $U \subseteq M$ containing x and an homeomorphism ϕ_U from U to an open subset of $\mathbb{R}^n_k$. We say (U, ϕ_U) is a *chart* of M at x. Let (U, ϕ_U) and (V, ϕ_V) be two charts of M at x. We say the two charts are *compatible* if the transition function

$$\phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \to \phi_V(U \cap V)$$

is a diffeomorphism in the sense of Definition 3.1.2. An *atlas* on M is a covering of M by pairwise compatible charts, and a maximal such atlas is called a *smooth structure with corners* on M.

Definition 3.1.6. Let $x \in M$ be a point of an n-dimensional manifold with k-corners and let (U, ϕ_U) be a chart of M at x. Denote by $c(x)$ the number of components of $[0, \infty)^k$ which are zero in $\phi_U(x)$. This is independent on the choice of chart (U, ϕ_U) by Lemma 3.1.3.

Definition 3.1.7. Let M be an n-dimensional manifold with k-corners and let $0 \leq l \leq k$. Define the subspace

$$\partial^l M := \{x \in M \mid c(x) \geq l\}$$

called the l-boundary of M.

Definition 3.1.8. Let M be an n-dimensional manifold with k-corners. A *face* of M is a union of connected component of the subset $\{x \in M \mid c(x) = 1\}$.

Remark 3.1.9. We are interested in manifolds whose corner structure is particularly well-behaved. Precisely, the structure we will impose will tailor our manifolds such that the usual embedding theorems and bundle theory from ordinary manifold theory generalize to the setting of manifolds with corners.

3.2 $\langle k \rangle$-manifolds

Definition 3.2.1. Let M be an n-dimensional manifold with k-corners. A $\langle k \rangle$-structure on M is a decomposition of the boundary of M, $\partial M \simeq \partial_1 M \cup \partial_2 M \cup ... \cup \partial_k M$, such that

1. each $\partial_i M$ is the closure of a face of M in M,

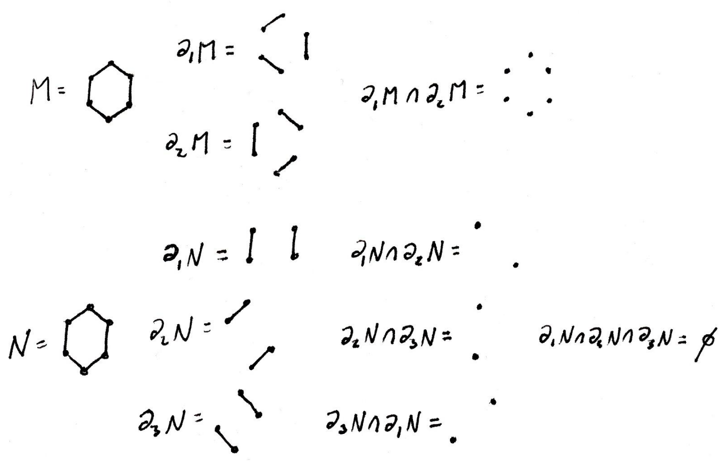
2. each $x \in M$ is contained in $c(x)$ of the $\partial_i M$'s.

3. for all $1 \le i, j \le k$ with $i \neq j$, $\partial_i M \cap \partial_j M$ is the closure of a face in $\partial_i M$ and a face in $\partial_j M$.

Example 3.1. The prototypical $\langle k \rangle$-manifolds are the spaces $\mathbb{R}^n_k$ and their one-point compactifications S^n_k. Note that the point at infinity attains maximum depth, $c(\infty) = k$.

Example 3.2. Any smooth manifold with empty boundary is a $\langle 0 \rangle$-manifold. Similarly, any manifold with boundary is a $\langle 1 \rangle$-manifold in a natural way.

Example 3.3. Let M be the 2-dimensional disk, D^2, whose boundary S^1 is subdivided into $n > 1$ pieces (i.e. an n-gon). Then M is a $\langle k \rangle$-manifold for any k such that n divides k.



Definition 3.2.2. Let 2 be the free arrow category, defined up to isomorphism as the category with two objects 0 and 1, and a single non-identity arrow $0 \to 1$. Denote by 2^k the product of k copies of 2. Explicitly, the objects of 2^k are k-tuples of binary numbers $a = (a_1, ..., a_k)$, and a single morphism denoted $a < b$ whenever $a_i \le b_i$ for all $1 \le i \le k$. For $a \in 2^k$, define the *weight* of a, denoted $w(a)$, as the number of nonzero entries in $a = (a_1, ..., a_k)$.

Definition 3.2.3. Let M be a $\langle k\rangle$-manifold and let $a \in 2^k$. Define the subspace

$$M(a) := \bigcap_{\substack{1\le i\le k \\ a_i=0}} \partial_i M$$

Example 3.4. Let $M \in \mathbb{R}^k \times \mathbb{R}^n$ be an embedded submanifold such that for each $a \in 2^k$, $\mathbb{R}^n_k(a) \subseteq \mathbb{R}^n_k \subseteq R^k \times \mathbb{R}^n$ intersects M transversally. Then assigning $M(a) := M \cap \mathbb{R}^n_k(a)$ equips M with the structure of a $\langle k\rangle$-manifold.

Remark 3.2.4. For any $\langle k\rangle$-manifold M and $a \in 2^k$, $M(a)$ is a $\langle w(a)\rangle$-manifold in a natural way.

Definition 3.2.5. The category $\langle k\rangle\mathsf{Top}$ of $\langle k\rangle$-spaces is the functor category $\mathrm{Fun}(2^k, \mathsf{Top})$.

Remark 3.2.6. Every $\langle k\rangle$-manifold M can be realized as a $\langle k\rangle$-space by Definition 3.2.3.

Definition 3.2.7. There are natural functors $-\times- : \langle k\rangle\mathsf{Top}\times\langle l\rangle\mathsf{Top} \to \langle k+l\rangle\mathsf{Top}$ given by taking $X : 2^k \to \mathsf{Top}$ and $Y : 2^l \to \mathsf{Top}$ to the composition

$$2^{k+l} \simeq 2^k \times 2^l \xrightarrow{X\times Y} \mathsf{Top}\times\mathsf{Top} \xrightarrow{-\times-} \mathsf{Top}$$

Definition 3.2.8. Consider $I = [0,1]$ as a $\langle 0\rangle$-space, such that we get a cylinder functor $-\times I : \langle k\rangle\mathsf{Top} \to \langle k\rangle\mathsf{Top}$. There are natural morphisms $i_0, i_1 : X \to X\times I$ given by the endpoint inclusions $i_0, i_1 : * \to I$. Let $f, g : X \to Y$ be a pair of morphisms of $\langle k\rangle$-spaces. A homotopy between f and g is defined as a morphism of $\langle k\rangle$-spaces $h : X\times I \to Y$, such that $hi_0 = f$ and $hi_1 = g$.

Definition 3.2.9. Let X be a $\langle k\rangle$-manifold. A neat embedding of X into $\mathbb{R}^{N+k}_k$ for some $N \geq 0$, is a morphism of $\langle k\rangle$-manifolds $e : X \to \mathbb{R}^{N+k}_k$ such that:

- for each $a \in 2^k$, the map $e(a) : X(a) \to \mathbb{R}^{N+k}_k(a)$ is an embedding,
- for each $a < b \in 2^k$, we have $X(b) \cap \mathbb{R}^{N+k}_k(a) = X(a)$, and
- these intersections are perpendicular, in the sense that there is an $\epsilon > 0$ such that

  $$X(b) \cap (\mathbb{R}^k_k(a) \times [0,\epsilon]^k(b-a) \times \mathbb{R}^N) = M(a) \times [0,\epsilon]^k(b-a)$$

  where

  $$b - a := (b_1, ..., b_k) - (a_1, ..., a_k) = (b_1 - a_1, ..., b_k - a_k)$$

[Diagram: Neat embedding of a square in $\mathbb{R}^3_2$]

Remark 3.2.10. In [Lau00], Laures showed that each $\langle k\rangle$-manifold admits a neat embedding into $\mathbb{R}^n_k$ for n large enough. Furthermore, in [Gen08], Genauer proved that the space of these embeddings is weakly contractible. This shows that any $\langle k\rangle$-manifold arises in the fashion of Example 3.4 in an essentially unique way. Given a $\langle k\rangle$-manifold M neatly embedded into $\mathbb{R}^n_k$, we can construct a submanifold $M' \subseteq \mathbb{R}^k \times \mathbb{R}^n$ such that $M(a) = M' \cap \mathbb{R}^n_k(a)$ for each $a \in 2^k$ by attaching open collars to M in the following way. For each $a \in 2^k$, we glue on the open collar $M(a) \times (-1, 0]^{\omega(a)}$. We obtain a cubical diagram of glueings, whose colimit is M'.

3.3 Vector bundles on $\langle k\rangle$-manifolds

Definition 3.3.1. Let X be a $\langle k\rangle$-space. A *$\langle k\rangle$-vector bundle* E on X is a 2^k-indexed system of vector bundles $E(a) \to X(a)$.

Remark 3.3.2. In discussing $\langle k\rangle$-vector bundles, it will be very useful to regard $\mathbb{R}$ as a $\langle 1\rangle$-space, in order to have better control over its subspaces. We will therefore consider $\mathbb{R}$ as a $\langle 1\rangle$-space with $\mathbb{R}(0) = \{0\} \subset \mathbb{R}$, $\mathbb{R}(1) = \mathbb{R}$, and $\mathbb{R}(0) \to \mathbb{R}(1)$ the inclusion $\{0\} \to \mathbb{R}$. In this sense $\mathbb{R}^n$ is a $\langle n\rangle$-space in the obvious way. Similarly, we

Example 3.5. Let M be a d-dimensional $\langle k\rangle$-manifold. We denote by TM the em tagent $\langle k\rangle$-vector bundle on M. If $a \in 2^k$ with $\omega(a) = k - 1$, we then have $TM(1)|_{M(a)} = \epsilon \oplus TM(a)$, where ϵ denotes the trivial rank 1 bundle.

Example 3.6. Let M be a d-dimensional $\langle k\rangle$-manifold equipped with a neat embedding $e : M \to \mathbb{R}^n_k$. We glue an open collar to M to obtain a manifold M' with an embedding $e' : M' \to \mathbb{R}^n$. The pullback of the tangent bundle of $\mathbb{R}^n$ then splits as $e'^*T\mathbb{R}^n \simeq \nu' \oplus TM'$, where ν' is the normal bundle of M' with respect to the embedding e'. Since then $TM'|_M = TM$, we may write $\nu = \nu'|_M \subseteq \mathbb{R}^n_k$. Since the original embedding e was neat, for any $a \in 2^k$, the restriction $\nu|_{M(a)}$ is the normal bundle of $M(a) \subseteq \mathbb{R}^{n+k-\omega(a)}_{\omega(a)}$. We may therefore extend ν to the *normal $\langle k\rangle$-bundle* ν of M by the above procedure.

Definition 3.3.3. Let X be a $\langle k\rangle$-space and let E be a $\langle k\rangle$-vector bundle on X. Then we say that E is a *geometric $\langle k\rangle$-vector bundle* if for each $a < b$ in 2^k, the pullback of $E(b)$ over $X(a)$ is naturally isomorphic to $E(a) \otimes \epsilon^c$ for some $c \geq 0$.

Remark 3.3.4. Given a $\langle k\rangle$-manifold $M \subseteq \mathbb{R}^n_k$, both the tangent and normal bundles of M are examples of geometric $\langle k\rangle$-vector bundles. Geometric bundles are well-behaved in the sense that there exists a canonical geometric $\langle k\rangle$-bundles, which we will now describe.

Definition 3.3.5. The $\langle k\rangle$-*Grassmannian* $G(d,n)\langle k\rangle$, is defined as the $\langle k\rangle$-space whose component $G(d,n)(a)$, for each $a \in 2^k$, is given by the space of $(d-k+\omega(a))$-dimensional subspaces of $\mathbb{R}^k(a) \times \mathbb{R}^{n+d-k}$, isomorphic to $G(d-k+\omega(a), n+d)$. For each $a < b$ in 2^k, the corresponding structure map $G(d,n)(a) \to G(d,n)(b)$ is given by taking each $V \in G(d,n)(a)$ to $V + \mathbb{R}(b-a) \in G(d,n)(b)$.

Definition 3.3.6. The *canonical geometric* $\langle k\rangle$*-bundle* $\gamma(d,n)\langle k\rangle$ is defined as the geometric $\langle k\rangle$-vector bundle on $G(d,n)\langle k\rangle$ whose component total space $\gamma(d,n)(a)$ for $a \in 2^k$ is given by

$$\gamma(d,n)(a) = \{(V,v) \in G(d,n)(a) \times \mathbb{R}^k(a) \times \mathbb{R}^{n+d-k} \mid v \in V\}$$

with the bundle $\gamma(d,n)(a) \to G(d,n)(a)$ given by the projection $(V,v) \mapsto V$. Given $a < b$ in 2^k, the associated bundle map $\gamma(d,n)(a<b) : \gamma(d,n)(a) \to \gamma(d,n)(b)$ sends $(V,v) \in \gamma(d,n)(a)$ to $(V + \mathbb{R}^k(b-a), \mathbb{R}^k(a<b)(v))$

Definition 3.3.7. The *canonical perpendicular geometric* $\langle k\rangle$*-bundle* $\gamma^\perp(d,n)\langle k\rangle$ is defined as the geometric $\langle k\rangle$-vector bundle on $G(d,n)\langle k\rangle$ whose component total space $\gamma^\perp(d,n)(a)$ for $a \in 2^k$ is given by

$$\gamma^\perp(d,n)(a) = \{(V,v) \in G(d,n)(a) \times \mathbb{R}^k(a) \times \mathbb{R}^{n+d-k} \mid v \in V^\perp\}$$

with the bundle $\gamma^\perp(d,n)(a) \to G(d,n)(a)$ given by the projection $(V,v) \mapsto V$. Given $a < b$ in 2^k, the associated bundle map $\gamma^\perp(d,n)(a<b) : \gamma^\perp(d,n)(a) \to \gamma^\perp(d,n)(b)$ sends $(V,v) \in \gamma^\perp(d,n)(a)$ to $(V + \mathbb{R}^k(b-a), \mathbb{R}^k(a<b)(v))$

Example 3.7. A neatly embedded $\langle k\rangle$-manifold $M^d \subseteq \mathbb{R}^n_k$ induces canonical morphisms of $\langle k\rangle$-manifolds $\xi_M : M \to G(d,n)\langle k\rangle$ and $\nu_M : M \to G(d,n)\langle k\rangle$, inducing by pullback the tangent and normal $\langle k\rangle$-vector bundles on M, respectively.

Remark 3.3.8. The canonical bundles are natural with respect to the structure maps of 2^k. I.e., there are isomorphisms

$$\gamma(d,n)(a) \simeq \gamma(d-k+\omega(a), n)$$

$$\gamma^\perp(d,n)(a) \simeq \gamma^\perp(d-k+\omega(a), n)$$

However, the isomorphism $\gamma(d,n) \simeq \gamma^\perp(n,d)$ of the ordinary canonical bundles does not extend to the $\langle k\rangle$-setting.

Definition 3.3.9. Just like in the case for ordinary Grassmannians, the canonical inclusion $\mathbb{R}^{n+d} \hookrightarrow \mathbb{R}^{n+d+1}$ induces a morphism of $\langle k \rangle$-spaces $G(d,n)\langle k \rangle \to G(d,n+1)\langle k \rangle$. The colimit as $n \to \infty$ is denoted by $BO(d)\langle k \rangle$. There are also induced bundle inclusions $\gamma(d,n) \hookrightarrow \gamma(d,n+1)$ inducing a universal vector bundle $\gamma(d) \to BO(d)$.

Remark 3.3.10. Let M be a d-dimensional $\langle k \rangle$-manifold. Then for large enough n, there is a neat embedding $e : M \to \mathbb{R}^{n+d}_k$ which is unique up to isotopy and whose normal bundle induces a morphism $\nu^n_M : M \to G(d,n)\langle k \rangle$ which is uniquely determined up to homotopy. Thus we get a morphism $\nu_M : M \to BO(d)\langle k \rangle$ which is uniquely determined up to homotopy. We call ν_M the *stable normal bundle* of M.

Definition 3.3.11. A *local $\langle k \rangle$-structure* (A,f) is a commutative diagram of $\langle k \rangle$-spaces

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & A_r & \xrightarrow{g_r} & A_{r+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_r} & & \downarrow{\scriptstyle f_{r+1}} & & \\
\cdots & \longrightarrow & BO(r)\langle k \rangle & \xrightarrow{j_r} & BO(r+1)\langle k \rangle & \longrightarrow & \cdots
\end{array}$$

Let M be a d-dimensional $\langle k \rangle$-manifold. An (A,f)-structure on M is a lift of the stable normal bundle of M through $f_d : A_d \to BO(d)\langle k \rangle$.

Example 3.8. Let $SO(d,n)\langle k \rangle$ be the $\langle k \rangle$-space whose components $SO(d,n)(a)$ for $a \in 2^k$ is given by the space of oriented $d + \omega(a)$-dimensional subspaces of $\mathbb{R}^k(a) \times \mathbb{R}^{n+d}$ and whose structure morphisms are given similarly as for $G(d,n)\langle k \rangle$. There is a natural morphism $SO(d,n)(a) \to G(d,n)(a)$ is given by forgetting the orientation. A lift of the stable normal bundle of M through $SO(d,n)\langle k \rangle$ is called an *orientation* on M.

Definition 3.3.12. An *injective $\langle k \rangle$-bundle* is a $\langle k \rangle$-vector bundle $E \to B$ such that for each $a < b$ in 2^k, the associated bundle morphism $E(a) \to E(b)$ is injective on each fiber.

Definition 3.3.13. Let $p : E \to B$ be an injective $\langle k \rangle$-bundle. We then define the associated *disk $\langle k \rangle$-bundle* as the $\langle k \rangle$-fiber bundle $D(p) : D(E) \to B$ given by $D(E)(1) = D(E(1))$, the disk bundle associated to $p(1) : E(1) \to B(1)$, and for $a \in 2^k$, $D(E)(a)$ is the preimage in $E(a)$ of $D(E)(1) \subseteq E(1)$. We define the sphere $\langle k\rangle$-bundle $S(p) : S(E) \to B$ in the same way from the sphere bundle associated to $p(1) : E(1) \to B(1)$.

Definition 3.3.14. Let $p : E \to B$ be an injective $\langle k\rangle$-bundle. Then we may define the *Thom space* $\mathrm{Th}(p)$ of p as the $\langle k\rangle$-space whose component at each $a \in 2^k$ is given by

$$\mathrm{Th}(p)(a) = \frac{D(E)(a)}{S(E)(a)}$$

and the map $\mathrm{Th}(p)(a) \to \mathrm{Th}(p)(b)$ is induced by the map $E(a) \to E(b)$.

Definition 3.3.15. Let $e : M \hookrightarrow \mathbb{R}^n_k$ be a neatly embedded $\langle k\rangle$-manifold. A *tubular neighborhood* of M is a neat embedding $e' : \nu_e \hookrightarrow \mathbb{R}^n_k$ of the normal $\langle k\rangle$-bundle of M with respect to e, such that $e'|_M = e$, i.e. e' restricts to e on the zero section.

3.4 $\langle k\rangle$-spectra

Convention 3.4.1. In this section, a space will be understood to mean a pointed CW complex.

Definition 3.4.2. Let $X : 2^k \to \mathsf{Top}_*$ be a pointed $\langle k\rangle$-space. The reduced suspension SX of X is defined by postcomposing X with the functor $S^1 \wedge - : \mathsf{Top}_* \to \mathsf{Top}_*$.

Definition 3.4.3. A $\langle k\rangle$-*spectrum* is a sequence $\{E_n, s_n\}_{n\in\mathbb{Z}}$, where E_n is a $\langle k\rangle$-space and $s_n : SE_n \to E_{n+1}$ is a morphism of $\langle k\rangle$-spaces such that $s_n(a) : SE_n(a) \to E_{n+1}(a)$ is an embedding of CW complexes.

Remark 3.4.4. A $\langle k\rangle$-spectrum may equivalently be defined as a functor from 2^k to the category of spectra and maps between them.

Remark 3.4.5. Consider once more the canonical map $j_{d,n} : G(d,n)\langle k\rangle \to G(d,n+1)\langle k\rangle$ induced by the inclusion $\mathbb{R}^{n+d} \hookrightarrow \mathbb{R}^{n+d+1}$. The pullback of $\gamma^\perp(d,n+1)\langle k\rangle$ along $j_{d,n}$ is isomorphic to $\gamma^\perp(d,n)\langle k\rangle \oplus \epsilon^1$, where ϵ^1 is the constant trivial rank one bundle. We call the resulting map

$$u_{d,n} : \gamma^\perp(d,n)\langle k\rangle \oplus \epsilon^1 \to \gamma^\perp(d,n+1)\langle k\rangle$$

Definition 3.4.6. The *Thom $\langle k\rangle$-spectrum* $MT(d)\langle k\rangle$ is such that the $(n+d)$-th space of $MT(d)(a)$ is the Thom space of $\gamma^{\perp}(d,n)(a)$ and the structure maps are the Thom maps $\mathrm{Th}(u_{d,n})$, or in terms of the spectra $MT(d)$ of [GMTW09], we have $MT(d)(a) = \Sigma^{\omega(a)-k} MT(d-k+\omega(a))$, namely, the spaces are given by:

$$\begin{aligned} MT(d)(a)_{n+d} &= \mathrm{Th}(\gamma^{\perp}(d,n)(a)) \\ &= \mathrm{Th}(\gamma^{\perp}(d-k+\omega(a),n)) \\ &= MT(d-k+\omega(a))_{n+d-k+\omega(a)} \\ &= \Sigma^{\omega(a)-k} MT(d-k+\omega(a))_{n+d} \end{aligned}$$

and the structure maps follow similarly.

Remark 3.4.7. Laures defines a procedure to pass from a $\langle k\rangle$-spectrum to an ordinary Ω-spectrum in the following way.

Definition 3.4.8. Let X be a $\langle k\rangle$-space. Define the n'th *loop space* of X as the space of based $\langle k\rangle$-maps $\mathbb{R}^n_k \to X$.

Definition 3.4.9. Let $E = \{E_n, s_n\}$ be a $\langle k\rangle$-spectrum. Define the *infinite loop spectrum* of E as the spectrum $\underline{\Omega^{\infty}_{\langle k\rangle}} E$ whose n'th space is given by

$$\operatorname{colim}_{m\to\infty} \Omega^{n+m}_{\langle k\rangle} E_m := \Omega^{\infty-n}_{\langle k\rangle} E$$

the right hand side we call the $(\infty - n)$-th loop space of E.

3.5 Cobordism categories of $\langle k\rangle$-manifolds

Idea 3.5.1. In [Gen08], Genauer defines a topological category $\bar{\mathcal{C}}^k_d$ of d-dimensional neatly embedded $\langle k-1\rangle$-manifolds and cobordisms between them, generalizing the embedded cobordism categories considered by Galatius-Madsen-Tillmann-Weiss in [GMTW09]. We will present his construction and generalize it to obtain a k-fold topological category $\mathcal{C}^k_d$. We may also fix the codimension n of the embedding, obtaining a refinement of $\mathcal{C}^k_d$ into a sequence of k-fold topological categories $\mathcal{C}^k_{d,n}$. In [BM], Bökstedt and Madsen determined the homotopy type of the k-fold topological nerve of $\mathcal{C}^n_{d,k}$. We recall their result below, but first we will spend some time on the construction of the cobordism category itself.

Definition 3.5.2. We consider $I = [0,1]$ as a $\langle 1\rangle$-manifold in the obvious way. Then the k-dimensional unit box I^k attains the structure of a $\langle k\rangle$-manifold as in Definition 3.2.7.

Remark 3.5.3. The concept of neat embedding extends to embeddings $M \hookrightarrow I^k \times \mathbb{R}^{n-k}$ by applying the conditions in Definition 3.2.9 locally at each point of the boundary.

Definition 3.5.4. Let M and N be a pair of d-dimensional $\langle k \rangle$-manifolds, possibly equipped with some tangential structure. An *abstract cobordism* from M to N is a $(d+1)$-dimensional $\langle k+1 \rangle$-manifold W such that $\partial_0 W = (-M) \sqcup N$.

Definition 3.5.5. Let M and N be a pair of d-dimensional $\langle k \rangle$-manifolds, possibly equipped with some tangential structure. An *abstract cobordism* from M to N is a $(d+1)$-dimensional $\langle k+1 \rangle$-manifold W such that $\partial_0 W = (-M) \sqcup N$.

Definition 3.5.6. Let M and N be a pair of neatly embedded d-dimensional $\langle k \rangle$-manifolds, $M, N \subseteq \mathbb{R} \times \mathbb{R}^n_k$, where $M \subseteq \{a_0\} \times \mathbb{R}^n_k$ and $N \subseteq \{a_1\} \times \mathbb{R}^n_k$, possibly equipped with some tangential structure. We assume $a_0 < a_1$ if $M \neq N$. An *embedded cobordism* from M to N is a $d+1$-dimensional $\langle k+1 \rangle$-manifold with a neat embedding $W \hookrightarrow [a_0, a_1] \times \mathbb{R}^{n-1}_k$ such that $W \cap \{0\} \times \mathbb{R}^{n-1} = M$ and $W \cap \{1\} \times \mathbb{R}^{n-1} = N$. We then have $\partial_0 W = (-M) \sqcup N$.

Remark 3.5.7. We differ between abstract and embedded cobordisms because these enjoy different properties. For example, embedded cobordisms can be organized into a strictly associative category, while this is not the case for abstract cobordisms. Indeed, composing abstract cobordisms requires a choice of smooth collars about the common boundary component. Thus the composition is only well-defined up to diffeomorphism relative the boundary. Thus abstract cobordism naturally assemble into a so-called k-fold *Segal space*.

A neat embedding of a cobordism provides a choice of smooth collar from the outset, so the composition becomes well-defined. Furthermore, we allow the interval to expand as in the Moore cylinder, so the reparameterization is unnecessary, and if we construct the Segal space of embedded cobordisms, it cill collapse into a strict k-fold topological category.

However, the two Segal spaces have weakly equivalent k-dimensional topological multinerves.

Definition 3.5.8. Let $a, b \in \mathbb{R}^k$ be two vectors in $\mathbb{R}^k$ such that for all $1 \leq j \leq k$, $a_j \leq b_j$. We then say that $a \leq b$ and this defines a partial ordering on $\mathbb{R}^k$. We denote by $[a, b]$ the box

$$[a, b] = \{(x_1, ..., x_k) \in \mathbb{R}^k \mid a_i \leq x_i \leq b_i \ \forall 1 \leq i \leq k\}$$

We call this the *box bounded by a and b*.

Recall that for $a \in 2^k$, S_a is the set of $1 \leq i \leq k$ for which $a_i = 0$. The set of boxes in $\mathbb{R}^k$, which we denote by $\square(k) = \{[x, y] \mid x, y \in \mathbb{R}^k\ , x \leq y\}$, assemble into a $\langle k\rangle$-manifold by putting $\square(k)(a) = \{[x, y] \in \square(k) \mid x_i = y_i \forall i \in S_a\}$.

Given a box $[x, y] \in \square(k)(a)$, for every $i \in S_{a'}$, there are two natural projections $s_i, t_i : \square(k)(a) \to \square(k)(a - e_i)$ given by the projections of $[x_i, y_i]$ onto x_i and y_i respectively. There is also an obvious inclusion $\mathrm{Id}_i : \square(k)(a - e_i) \to \square(k)(a)$

Definition 3.5.9. We define a strict k-tuple category $\mathcal{C}^k_{d,n}$ of embedded d-dimensional $\langle k\rangle$-manifolds in codimension n in the following manner. An a-morphism of $\mathcal{C}^k_{d,n}$ is a triple (W, x, y), where $x, y \in \mathbb{R}^k$ such that $x \leq y$ and $[x, y] \in \square(k)(a)$, and W is a neatly embedded d-dimensional $\langle k\rangle$-submanifold of $[x, y] \times \mathbb{R}^{n+d-k} \subseteq \mathbb{R}^{n+d}$.

For $i \in S_{a'}$, we define the source and target of (W, a, b) in the i-direction by intersecting W with $s_i([x, y]) \times \mathbb{R}^{n+d-k}$ and $t_i([x, y]) \times \mathbb{R}^{n+d-k}$, respectively. We define the identities similarly.

Composition is defined by taking the union of subsets of $\mathbb{R}^{n+d}$.

Remark 3.5.10. Definition 3.5.9 defines an ordinary strict k-fold category of cobordisms. The way we topologize the category varies depending on whether we limit the codimension. We first treat the simpler case of infinite codimension, and then the case of finite codimension. The former is an adaption of the treatment in [GMTW09] to the k-fold setting, while the latter is taken from [BM].

3.5.1 A topology on $\mathcal{C}^k_d$

We topologize the k-fold category $\mathcal{C}^k_d = \mathrm{colim}_{n\to\infty} C^k_{d,n}$ as follows. The topology on the set of objects is identical to the one considered in [GMTW09], but we recall its construction here. Let M be a closed $d-k$-dimensional manifold. The space of codimension infinity embeddings $\mathrm{Emb}(M, \mathbb{R}^{d-k+\infty}) = \mathrm{colim}_{n\to\infty} \mathrm{Emb}(M, \mathbb{R}^{d-k+n})$ is contractible by Whitney's embedding theorem. Now, the diffeomorphism group $\mathrm{Diff}(M)$ of M acts freely on $\mathrm{Emb}(M, \mathbb{R}^{d-k+\infty})$ on the right by precomposition, and the projection

$$\mathrm{Emb}(M, \mathbb{R}^{d-k+\infty}) \to \mathrm{Emb}(M, \mathbb{R}^{d-k+\infty})/\mathrm{Diff}(M)$$

onto orbits is a principal $\mathrm{Diff}(M)$-bundle. It follows that the tensor product

$$\mathrm{Emb}(M,\mathbb{R}^{d-k+\infty})\otimes_{\mathrm{Diff}(M)} M \to \mathrm{Emb}(M,\mathbb{R}^{d-k+\infty})/\mathrm{Diff}(M) := B_\infty(M)$$

is a principal $\mathrm{Diff}(M)$-bundle with contractible total space and fiber M. It follows that $\mathrm{Emb}(M,\mathbb{R}^{d-k+\infty})/\mathrm{Diff}(M) \simeq B\mathrm{Diff}(M)$. Since there is a one-to-one correspondence between $\mathrm{Emb}(M,\mathbb{R}^{d-k+\infty})/\mathrm{Diff}(M)$ and the set of submanifolds of $\mathbb{R}^{d-k+\infty}$ diffeomorphic to M. We thus topologize $\mathrm{Ob}\,\mathcal{C}_d^k$ such that

$$\mathrm{Ob}\,\mathcal{C}_d^k = \mathbb{R}^k \times \sqcup_M B_\infty(M)$$

where M varies over closed $(d-k)$-dimensional manifolds, one in each diffeomorphism class. In particular then $\mathrm{Ob}\,\mathcal{C}_d^k \simeq \sqcup_M B\mathrm{Diff}(M)$.

We now treat the morphism spaces. Fix an $a \in 2^k$ with $\omega(a) \geq 1$ and assume that we have already topologized the sets $C_k^d(b)$ for all $b \in 2^k$ for which $\omega(b) < \omega(a)$. Let (W,h) be a pair where W is an element of $\mathcal{C}_d^k(a)$ and h is a collar for W. That is, for each $j \in S_{a'}$, a pair of embeddings

$$h_0^j : [0,1) \times s_j(W) \to W$$

$$h_1^j : (0,1] \times t_j(W) \to W$$

We also require that the restriction of the h_ν^i for $i \neq j$ and $\nu = 0,1$ to $\partial_j W$ defines a collar of $\partial_j W$.

For $0 < \epsilon < \frac{1}{2}$, let $\mathrm{Emb}_\epsilon(W,[0,1]^{\omega(a)} \times \mathbb{R}^{d-k+n})$ be the space of neat embeddings $e : W \to [0,1]^{\omega(a)} \times \mathbb{R}^{d-k+n}$ such that for each $j \in S_{a'}$, there are embeddings

$$e_0^j : s_j(W) \to \mathbb{R}^{n-k+n}$$

$$e_1^j : t_j(W) \to \mathbb{R}^{n-k+n}$$

such that $e \circ h_0^j(\tau_0, x_0) = (\tau_0, e_0^j(x_0))$ and $e \circ h_1^j(\tau_1, x_1) = (\tau_1, e_1^j(x_1))$ for all $\tau_0 \in [0,\epsilon)$, $\tau_1 \in (1-\epsilon, 1]$, $x_0 \in s_j(W)$ and $s_1 \in t_j(W)$. We define

$$\mathrm{Emb}(W,[0,1]^{\omega(a)} \times \mathbb{R}^{d-k+\infty}) = \operatorname*{colim}_{\substack{\epsilon\to 0\\ n\to\infty}} \mathrm{Emb}_\epsilon(W,[0,1]^{\omega(a)} \times \mathbb{R}^{d-k+n})$$

This construction of $\mathrm{Emb}(W,[0,1]^{\omega(a)} \times \mathbb{R}^{d-k+\infty})$ is a more elaborate version of Definition 3.2.9. We now denote by $\mathrm{Diff}_\epsilon(W)$ the subgroup of $\mathrm{Diff}_{\langle a\rangle}(W)$ of $\langle a\rangle$-structure preserving diffeomorphisms of W which restrict to product diffeomorphisms on the ϵ-collars and we denote the colimit $\mathrm{Diff}(W) = \mathrm{colim}_{\epsilon\to 0}\,\mathrm{Diff}_\epsilon(W)$

the group of diffeomorphisms which preserve ϵ-collars for sufficiently small ϵ. Now, $\mathrm{Diff}_\epsilon(W)$ acts freely on $\mathrm{Emb}_\epsilon(W, [0,1]^{\omega(a)} \times \mathbb{R}^{d-k+n})$ on the right by precomposition, and we again get principal $\mathrm{Diff}_\epsilon(W)$-bundles

$$\mathrm{Emb}_\epsilon(W, [0,1]^{\omega(a)} \times \mathbb{R}^{d-k+n}) \to \mathrm{Emb}_\epsilon(W, [0,1]^{\omega(a)} \times \mathbb{R}^{d-k+n})/\mathrm{Diff}_\epsilon(W)$$

Taking the colimit as $n \to \infty$ and $\epsilon \to 0$, we then obtain the principal $\mathrm{Diff}(W)$-bundle

$$\mathrm{Emb}(W, [0,1]^{\omega(a)} \times \mathbb{R}^{d-k+\infty}) \to \mathrm{Emb}(W, [0,1]^{\omega(a)} \times \mathbb{R}^{d-k+\infty})/\mathrm{Diff}(W)$$

Viewing W as a left $\mathrm{Diff}(W)$-module, we form the tensor product

$$\mathrm{Emb}(W, [0,1]^{\omega(a)} \times \mathbb{R}^{d-k+\infty}) \otimes_{\mathrm{Diff}(W)} W \to \mathrm{Emb}(W, [0,1]^{\omega(a)} \times \mathbb{R}^{d-k+\infty})/\mathrm{Diff}(W)$$

As above, $\mathrm{Emb}(W, [0,1]^{\omega(a)} \times \mathbb{R}^{d-k+\infty})$ is contractible, hence the above is a principal $\mathrm{Diff}(W)$-bundle with fiber W. We conclude that

$$\mathrm{Emb}(W, [0,1]^{\omega(a)} \times \mathbb{R}^{d-k+\infty})/\mathrm{Diff}(W) := B_\infty(W) \simeq B\mathrm{Diff}(W)$$

The above describes the space of nondegerate a-morphisms of $\mathcal{C}_k^d$, i.e. those whose associated box $[x,y](a)$ has no $j \in S_{a'}$ for which $x_j = y_j$. The degenerate morphisms are added by appending the spaces of lower morphisms. Denote by $(R^k)^2_+(a)$ the set of pairs $(x,y) \in \mathbb{R}^k \times \mathbb{R}^k$ such that $x_j < y_j$ if $a_j = 1$ and $x_j = y_j$ if $a_j = 0$. We then topologize $C_d^k(a)$ such that

$$C_d^k(a) = \sqcup_{\substack{b \neq a \\ b < a}} C_d^k(b) \sqcup_W (R^k)^2_+(a) \times B_\infty(W)$$

where W varies over $d - k + \omega(a)$-dimensional $\langle a \rangle$-cobordisms (W, h), one in each diffeomorphism class.

3.5.2 Models for $\mathcal{C}_{d,n}^k$

We now show how to put a topology on $\mathcal{C}_{d,n}^k$ by a method fairly different to the above. The following is due to [BM]. Let $\Psi_d(\mathbb{R}^{d+n})$ denote the set of d-dimensional submanifolds W of $\mathbb{R}^{d+n}$ such that $\partial W = \emptyset$ and W is a closed subset of $\mathbb{R}^{n+d}$. We topologize $\Psi_d(\mathbb{R}^{d+n})$ in the following way. Given a $W \in \Psi_d(\mathbb{R}^{n+d})$, let $NW \subset \mathbb{R}^{d+n}$ be a tubular neighborhood of W with projection map $r : NW \to W$. Denote by $C^\infty(W, NW)$ be the space of smooth sections of r, equipped with the C^∞-Whitney topology. Assume we are given a metric μ on $G(d,n)$, we now define a norm on $C^\infty(W, NW)$. Given an element $s \in C^\infty(W, NW)$ and a compact subset $K \subset \mathbb{R}^{d+n}$, we write

$$||s||_K = \sup_{x \in W \cap K} (|s(x) + \mu(T_xW, ds(T_xW)))$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^{d+n}$ and ds is the differential of s. Now define

$$\Gamma_{K,\epsilon} = \{s \in C^\infty(W, NW) \mid ||s||_K < \epsilon\}$$

I.e. $\Gamma_{K,\epsilon}$ is the subset of $\mathcal{C}^\infty(W, NW)$ for which s and the zero section are sufficiently close on K. The corresponding neighborhood of W in $\Psi_d(\mathbb{R}^{d+n})$ is given by

$$\mathcal{N}_{K,\epsilon}(W) = \{V^d \mid V^d \cap K = s(W) \cap K\ ,\ s \in \Gamma_{K,\epsilon}\}$$

I.e. $\mathcal{N}_{K,\epsilon}$ is the set of submanifolds sufficiently close to W wherever it intersects K.

Theorem 3.5.11. The sets $\mathcal{N}_{K,\epsilon}(W)$ form a system of neighbourhoods for a topology on $\Psi_d(\mathbb{R}^{n+d})$. Given W and K, the set $\mathcal{N}_{K,\epsilon}(W)$ is open for sufficiently small ϵ, and this topology turns $\Psi_d(\mathbb{R}^{d+n})$ into a metrizable space

Proof. The proof of metrizability is given in section 4 of [BM] by showing that $\Psi_d(\mathbb{R}^{d+n})$ is a second countable regular space. □

Remark 3.5.12. Note that if $M \in \Psi_d(\mathbb{R}^{n+d})$ and $U \subseteq \mathbb{R}^{n+d}$ is an open subset, when the intersection $M \cap U$ is a d-dimensional submanifold of U with $\partial(M \cap U) = \emptyset$ and $M \cap U$ is a closed subset of U. In addition, if we have two open sets and such submanifolds $M \subseteq U$ and $N \subseteq V$ such that $M \cap V = N \cap U$, then the union $M \cup N$ is a d-dimensional submanifold of $U \cup V$ with empty boundary and closed as a subspace. In other words, Ψ_d extends to a sheaf of topological spaces on $\mathbb{R}^{n+d}$. This sheaf is the basis for the sheaf model of the cobordism category and was used by Madsen-Weiss in [MW07] and by Galatius-Madsen-Tillmann-Weiss in [GMTW09].

Remark 3.5.13. We are also interested in the subspaces

$$D^k_{d,n} = \{W \in \Psi_d(\mathbb{R}^{d+n}) \mid W \subset \mathbb{R}^k \times (-1,1)^{d+n-k}\}$$

of manifolds with $(d+n-k)$ compact directions and k non-compact directions. This space can also be characterized as the subspace of submanifolds for which the projection onto the last k coordinates are proper. The spaces $\Psi_d(\mathbb{R}^{n+d})$ and $D^k_{d,n}$ have basepoints given by the empty manifold.

Theorem 3.5.11 shows in particular that $\Psi_d(\mathbb{R}^{n+d})$ and $D^k_{d,n}$ are compactly generated weak Hausdorff spaces. The spaces $D^k_{d,n}$ give rise to a topological poset model for the cobordism category with very good properties. In particular, we have topological niceties like Lemma 3.5.15.

Definition 3.5.14. We define the k-fold topological poset $\mathcal{D}^k_{n,d}$. For $a \in 2^k$, the a-morphisms are given by the subspace $\mathcal{D}^k_{d,n}(a) = \square(k)(a) \times D^k_{d,n}$ consisting of those pairs $([x,y], M)$ for which M intersects the walls of $[x,y]$ transversally. Two such pairs $([x,y], M)$ and $([x',y'], M')$ are composable in the i-direction if and only if $x_j = x'_j$ and $y_j = y'_j$ for all $j \neq i$ and $y_i = x'_i$, i.e. if the boxes are composable, and $M = M'$. In that case, the composition is given by $([x'',y''], M)$, where $x''_j = x_j$ and $y''_j = y_j$ for all j. This is called the *topological poset model* of the cobordism category.

Lemma 3.5.15. Let $a \in \mathbb{R}$ and define the subset $U_a \subseteq \mathcal{D}^1_{d,n}$ of elements M such that a is a regular value of the projection $x_1 : M \to \mathbb{R}$. Then U_a is an open set in $\mathcal{D}^1_{d,n}$. The analogous subset in $\Psi_d(\mathbb{R}^{n+d})$ is not open in general.

Proof. Let $M \in U_a$. Then by a standard theorem of differential topology, there exists an open neighbourhood V of $M_a := x_1^{-1}(a)$ such that x_1 has full rank on V. Choosing a compact subset A of $x_1(V)$, let $K = A \times [-1,1]^{n+d-1} \subseteq \mathbb{R}^{n+d}$. Now let S be the set of d-planes which are perpendicular to the first coordinate axis and let $0 < \epsilon < \inf_{\substack{x \in M \cap K \\ s \in S}} (\mu(T_x M, s)$. The open set $\mathcal{N}_{K,\epsilon}(M)$ is then an open neighbourhood of M contained in U_a, which shows that U_a is open.

To see why openness fails for $\Psi_d(\mathbb{R}^{n+d})$, notice that for a general $M \in \Psi_d(\mathbb{R}^{n+d})$, $x_1^{-1}(a)$ is not a bounded subset, hence cannot be contained in any compact subset. It follows that any open neighborhood of M will contain at least one element with a critical point of x_1 at a.

□

Definition 3.5.16. We define the k-fold topological poset $\mathcal{D}^{\perp k}_{n,d}$. It is defined like $D^k_{n,d}$ in 3.5.14, but we require of our a-morphisms $([x,y], M)$ that M intersects the walls of $[x,y]$ *orthogonally*. This is called the *orthogonal topological poset model* of the cobordism category.

Remark 3.5.17. In order to properly relate $D^k_{d,n}$ to the neat embeddings into $[x,y] \times (-1,1)^{d+n-k}$, we have the following lemma.

Lemma 3.5.18 ([GRW10], Lemma 3.4.)**.** Let $f : X \to \Psi_d(\mathbb{R}^{n+d})$ be a continuous map, and let $U, V \subseteq X$ be open sets with $\bar{U} \subseteq V$. Furthermore, let $a \in \mathbb{R}$ be a regular value for the projection $x_1 : f(x) \to \mathbb{R}$ onto the first coordinate for each $x \in V$. Lastly, let $\epsilon > 0$ be a real number. Then there is a homotopy

$$f_t : X \to \Psi_d(\mathbb{R}^{n+d})$$

for $t \in [0,1]$, and for which

1. $f_0 = f$ and for each $x \in U$, there is a manifold $M \subseteq \mathbb{R}^{n+d-1}$ such that $f_1(x) \cap [a-\epsilon, a+\epsilon] = M \times [a-\epsilon, a+\epsilon]$.
2. The restriction to $[0,1] \times (X \setminus V) \to \Psi_d(\mathbb{R}^{n+d})$
3. The composition of f_t with the restriction $\Psi_d(\mathbb{R}^{n+d}) \to \Psi_d(\mathbb{R}^{n+d} \setminus ([a-2\epsilon, a+2\epsilon] \times \mathbb{R}^{n+d-1}))$ is a constant homotopy.

Proof. Fix a smooth function $\lambda : \mathbb{R} \to \mathbb{R}$ for which

$$\lambda(s) = \begin{cases} 0 & , |s| \leq 1 \\ s & , |s| \geq 2 \end{cases}$$

and such that $\lambda'(s) > 0$ for $|s| > 1$. We now construct a scaling function $\phi_\tau(s) = (1-\tau)s + \tau\epsilon\lambda(\frac{s-a}{\epsilon})$ for $\tau \in [0,1]$. Further, let $\rho : X \to [0,1]$ be a bump function with support contained in V and such that $U \subseteq \rho^{-1}(1)$. The homotopy $[0,1] \times X \to \Psi_d(\mathbb{R}^{n+d})$ is now defined by $(x,t) \mapsto (\phi_{t\rho(x)} \times \mathrm{id})^{-1}(f(x))$. $\square$

Remark 3.5.19. The homotopy in the above lemma can be seen as stretching the manifold about a regular value in order to make the manifold orthogonal to the x_1 direction. An important aspect of the homotopy is also that the tangent space of the restriction $f(x) \cap x_1^{-1}(a) = M$ is preserved. Furthermore, the lemma still holds when we exchange the fist coordinate projection with the i'th. We therefore get the following corollary.

Corollary 3.5.20. Let $X \to \Psi_d(\mathbb{R}^{n+d})$ be a continuous map, let U, V be open sets in X such that $\bar{U} \subseteq V$ and let $[a,b] \in \square(k)$ such that for each $x \in X$, $f(x)$ intersects the walls of the box $[a,b] \times \mathbb{R}^{d+n-k}$ transversally. Finally, let $0 < \epsilon < \frac{1}{4}$ be a real number. Then there is a homotopy $h : X \times [0,1] \to \Psi_d(\mathbb{R}^{d+n})$ such that

1. $h(x,0) = f(x)$,
2. $h(x,1)$ is orthogonal to the walls of the box $[a,b] \times \mathbb{R}^{d+n-k}$,
3. $h(x,t) = f(x)$ for all $x \in X \setminus V$
4. For all $x \in X$, h is a constant homotopy on the intersection $f(x) \cap \big((\mathbb{R}^{n+d} \setminus [a-2\epsilon, b+2\epsilon]) \cup \mathrm{int}([a+2\epsilon, b-2\epsilon])\big)$.

Proof. Apply Lemma 3.5.18 to each a_i and b_i for all $i \in \{1, ..., k\}$. Since $\epsilon < \frac{1}{4}$, the homotopies are orthogonal to one another and thus commute, so the resulting composed homotopy is well defined and independent of the order of application. □

Remark 3.5.21. The above observations give rise to a comparison between the topological poset models for the cobordism category.

Lemma 3.5.22. The inclusion $\mathcal{D}^{\perp k}_{d,n} \hookrightarrow \mathcal{D}^k_{d,n}$ induces a levelwise weak equivalence of simplicial spaces $N\mathcal{D}^{\perp k}_{d,n} \simeq \mathcal{D}^k_{d,n}$ on the level of their k-fold nerves.

Theorem 3.5.23 ([BM])**.** We consider $D^k_{d,n}$ as a constant k-fold simplicial space. There is then a forgetful map $N\mathcal{D}^k_{d,n} \to D^k_{d,n}$ given by discarding the data of transversal planes. This map is a levelwise weak homotopy equivalence.

In [BM], Bökstedt-Madsen prove this result as a corollary of a more general theorem, using their machinery of *abstract transversality.*

Remark 3.5.24. The following lemma follows from the definition of the topologies in the previous sections. The proof makes heavy use of model categorical techniques. In order to give a satisfying proof, we would have to go through the necessary background material on model categories, for which we do not have space or time. Rather than give an unsatisfying half-proof, we have opted to leave it as is.

Lemma 3.5.25. With the topologies of the previous sections, there are continuous closed inclusions $\mathcal{C}^k_{d,n} \to \mathcal{C}^k_{d,n+1}$ giving rise to an isomorphism of topological categories

$$\mathrm{colim}_{n\to\infty} \mathcal{C}^k_{d,n} \simeq \mathcal{C}^k_d$$

This induces a homotopy equivalence of classifying spaces

$$B\mathcal{C}^k_d \simeq \mathrm{colim}_{n\to\infty} B\mathcal{C}^k_{d,n}$$

3.6 The nerve of the cobordism multicategory

Let $(W,[x,y]) \in \mathcal{C}^k_{d,n}$ be a k-morphism, i.e $W \subseteq [x,y] \times \mathbb{R}^{d-k+n}$, and let $U \subset \mathbb{R}^{d+n}_k$ be a tubular neighborhood of W as per Definition 3.3.15. The Pontrjagin-Thom collapse map associated to U gives a map

$$[x,y]_+ \wedge S^{d-k+n} \to \mathrm{Th}(U^\perp_{d,n})$$

whose adjoint map is

$$[x,y] \to \Omega^{d-k+n}\mathrm{Th}(U^\perp_{d,n})$$

It is possible to make this into a functor of strict k-fold categories $\mathcal{C}^k_{d,n} \to P^k(\Omega^{d-k+n}\mathrm{Th}(U^\perp_{d,n}))$, where $P^k(-)$ denoted the k-fold unreduced Moore path category, see Example 1.6. There is a homotopy equivalence $B\Pi_k(\Omega^{d-k+n}\mathrm{Th}(U^\perp_{d,n})) \simeq \Omega^{d-k+n}\mathrm{Th}(U^\perp_{d,n})$, so we get a map

$$\alpha : B\mathcal{C}^k_{d,n} \to \Omega^{d+n-k}\mathrm{Th}(U^\perp_{d,n})$$

The main theorem of [BM] states that this map is a weak homotopy equivalence.

Theorem 3.6.1 ([BM])**.** Let $\gamma^\perp_{d,n}$ denote the universal normal bundle on $G(d,n)$. Then

$$B\mathcal{C}^k_{d,n} \simeq \Omega^{d+n-k}\mathrm{Th}(\gamma^\perp_{d,n})$$

Corollary 3.6.2. When $n \to \infty$, the classifying space of the infinite codimension cobordism k-fold category is

$$B\mathcal{C}^k_d \simeq \mathrm{colim}_{n\to\infty}\, \Omega^{d+n-k}\mathrm{Th}(\gamma^\perp_{d,n}) \simeq \Omega^{\infty-k}\mathsf{MT}(d)$$

Corollary 3.6.3. We have $\Omega B\mathcal{C}^k_{d,n} \simeq B\mathcal{C}^{k-1}_{d,n}$, hence $\{B\mathcal{C}^k_d\}^\infty_{k=1}$ is an Ω-spectrum model for $\mathsf{MT}(d)$

Definition 3.6.4. Denote by $\mathsf{Cob}_{\langle k\rangle}(d)$ the ordinary topological category obtained from $\mathcal{C}^k_d$ by restricting to those $a \in \square^k$ for which $a_i = 1$ for all $2 \le i \le k$.

Theorem 3.6.5 ([Gen08], Main Theorem 4.5)**.** There is a homotopy equivalence

$$B\mathsf{Cob}_{\langle k\rangle}(d) \simeq \Omega^{\infty-1}_{\langle k\rangle} MT(d)\langle k\rangle$$

3.6.1 Outline of the proof of Theorem 3.6.1

In this section, we will give an outline of the proof of Theorem 3.6.1. The techniques involved are due to Bökstedt-Madsen in [BM] and Galatius-Randall-Williams in [GRW10]. The proof proceeds in the following steps:

1. a weak homotopy equivalence $BC^k_{d,n} \to D^k_{d,n}$
2. a weak homotopy equivalence $D^k_{d,n} \to \Omega^{d+n-k}\Psi_d(\mathbb{R}^{n+d})$
3. a weak homotopy equivalence $\mathrm{Th}(\gamma^\perp(d,n)) \to \Psi_d(\mathbb{R}^{n+d})$

Lemma 3.6.6. There is a weak homotopy equivalence

$$BC^k_{d,n} \to D^k_{d,n}$$

Proof. Recall that an element of $N\mathcal{C}^k_{d,n}$ is a grid of cobordisms. We can view this as a single cobordism W embedded into a k-dimensional grid of boxes in $\mathbb{R}^{n+d}$ intersecting W transversally, even orthogonally. Similarly an element of $N\mathcal{D}^{\perp k}_{d,n}$ can be seen as an element W of $D^k_{d,n}$ along with a k-dimensional grid of embedded planes intersecting W transversally, even orthogonally. We can construct an embedding of $N\mathcal{C}^k_{d,n}$ into $N\mathcal{D}^{\perp k}_{d,n}$ by adjoining an open collar as in 3.2.10 and extending it to infinity. Now, $D^{\perp k}_{d,n}$ contracts onto the image of this inclusion. The idea here is to push everything outside of the grid of transverse planes to infinity. We can accomplish this by a homotopy $\phi_t(a,b) : \mathbb{R} \to \mathbb{R}$ defined as follows. Let $s = \frac{t}{1-t}$. We then define

$$\phi_t(a,b)(x) = \begin{cases} x & , x \in [a,b] \\ a & , x \in (a-s,a) \\ b & , x \in (b,b+s) \\ x+s & , x \in (-\infty, a-s] \\ x-s & , x \in [b+s,\infty) \end{cases}$$

A box $[a,b] \in \square(k)$ defines a homotopy $\phi_t([a,b]) : \mathbb{R}^k \to \mathbb{R}^k$ given by

$$(x_1, ..., x_k) \mapsto (\phi_t(a_1,b_1)(x_1), ..., \phi_t(a_k,b_k)(x_k))$$

and this defines a homotopy $\phi_t : N\mathcal{D}^{\perp k}_{d,n} \to \mathcal{D}^{\perp k}_{d,n}$ for which $\phi_0 = \mathrm{id}$ and the image of ϕ_1 equals the image of $N\mathcal{C}^k_{d,n}$ in $N\mathcal{D}^{\perp k}_{d,n}$. By Lemma 3.5.18 and Theorem 3.5.23

we now get a sequence of weak homotopy equivalences

$$B\mathcal{C}^k_{d,n} \to B\mathcal{D}^{\perp k}_{d,n} \to B\mathcal{D}^k_{d,n} \to D^k_{d,n}$$

□

Theorem 3.6.7. There is a weak homotopy equivalence

$$D^k_{d,n} \to \Omega^{d+n-k}\Psi_d(\mathbb{R}^{n+d})$$

Proof. We define a map $\mathbb{R} \times D^k_{d,n} \to D^{k+1}_{d,n}$ given by $(t, M) \mapsto M - t \cdot e_{k+1}$, where e_{k+1} denotes the $(k+1)$st standard basis vector of $\mathbb{R}^{k+1}$. As $t \to \pm\infty$, the image of this map approaches the basepoint $\emptyset$, so it extends to a map $S^1 \wedge D^k_{d,n} \to D^k_{d,n}$. Then the adjoint $D^k_{d,n} \to \Omega D^{k+1}_{d,n}$ is a homotopy equivalence ([GRW10, Theorem 3.13.]). □

Theorem 3.6.8. There is a weak homotopy equivalence

$$\mathrm{Th}(\gamma^\perp(d, n)) \to \Psi_d(\mathbb{R}^{n+d})$$

Proof. The map which witnesses this weak homotopy equivalence is constructed as follows. Recall first that $D^{n+d}_{d,n} = \Psi_d(\mathbb{R}^{n+d})$. If $(V, v) \in \gamma^\perp(d, n)$, we send this pair to the translated linear manifold $V - v \in \Psi_d(\mathbb{R}^{n+d})$. As $|v| \to \infty$, the value of this assignment approaches the empty manifold, so the map extends to a based map $\phi : \mathrm{Th}(\gamma^\perp(d, n)) \to \Psi_d(\mathbb{R}^{n+d})$.

Let $\Psi^o \subset \Psi_d(\mathbb{R}^{n+d})$ be the subspace of manifolds containing the origin. There is a map $S : [0, 1] \times \Psi^o \to \Psi^o$ given by

$$S(t, M) = \begin{cases} M \cdot \frac{1}{1-t} & , t < 1 \\ T_0 M & t = 1 \end{cases}$$

This defines a deformation retraction of Ψ^o onto the subspace $L \simeq G(d, n) \subset \Psi^o$ of linear manifolds intersecting the origin. Now consider the vector bundle $p : N \to \Psi^o$ whose fiber at $M \in \Psi^o$ is the fiber of the normal bundle of M at the origin, $p^{-1}(M) = \nu_M^{-1}(0)$. We then obtain a map $e : N \to \Psi_d(\mathbb{R}^{n+d})$ sending a pair (M, v) to the translated manifold $M - v$. This map extends to an open set about the zero-section, giving an embedding onto the subspace $U \subseteq \Psi_d(\mathbb{R}^{n+d})$ of manifolds with a unique point closest to the origin. Since the elements of the complement $C := \Psi_d(\mathbb{R}^{n+d}) \setminus U$ in particular do not contain the origin, we can push these manifolds off to infinity by applying the isotopy $- \cdot \frac{1}{1-t} : [0,1) \times R^{n+d} \to \mathbb{R}^{n+d}$. Extending this to a map $[0,1] \times C \to C$ for which $(1, M) = \emptyset$, we get that C is contractible. Now, since U is the image of an embedding of an open set around the zero section of N, the inclusion of the complement $e^{-1}(C) \to N$ is a cofibration. Furthermore, the inclusion of C into $\Psi_d(\mathbb{R}^{n+d})$ is the pushout of the inclusion $e^{-1}(C) \to N$, so it is itself a cofibration. This gives rise to a homotopy equivalence

$$\mathrm{Th}(p) \simeq N/e^{-1}(C) \simeq \Psi_d(\mathbb{R}^{n+d})/C \simeq \Psi_d(\mathbb{R}^{n+d})$$

The observation that $\mathrm{Th}(p) \simeq \mathrm{Th}(\gamma^{\perp}(d,n))$ finishes the proof. $\square$

Chapter 4

Manifolds with singularities

In this chapter we will be looking at manifolds with Baas-Sullivan singularities. These objects were first studied by Sullivan in [Sul67a] and [Sul67b], and their cobordism theory has been studied by Baas in [Baa73] and Perlmutter in [Perb] and [Pera].

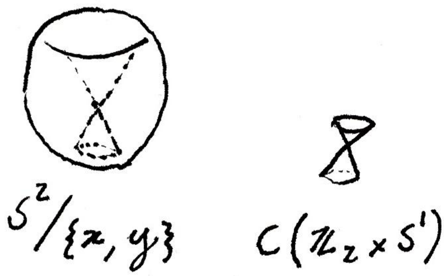
4.1 Manifolds with a single singularity type

Definition 4.1.1. Let P be a closed p-dimensional manifold. A *closed n-dimensional P-manifold* is a topological space $\bar{A}$ which is constructed as the union of an n-dimensional manifold A with $\partial A = \beta_1 A \times P$, and the space $A_1 \times CP$, where CP denotes the unreduced cone. To be precise, $\bar{A}$ is given by the pushout

$$\bar{A} = A \cup_{\beta_1 A \times P} \beta_1 A \times CP$$

Remark 4.1.2. Unravelling the definition, we see that $\bar{A}$ has two types of points; those at which $\bar{A}$ locally look like $\mathbb{R}^n$ are called *smooth points*, and those at which $\bar{A}$ locally looks like $\mathbb{R}^{n-p-1} \times CP$ are called *singular points*.

Example 4.1. Let M be the topological space given by an n-sphere with q disctinct points $\{x_1, ..., x_p\} \subset S^n$ identified, $M = S^n/\{x_1, ..., x_p\}$. This space is a closed n-dimensional P-manifold with $P = \mathbb{Z}_q \times S^{n-1}$. Let A be S^n with the interiors of q closed disks removed, then $\partial A = \mathbb{Z}_q \times S^{n-1}$, and M is homeomorphic to $A \cup_{\mathbb{Z}_q \times S^{n-1}} C(\mathbb{Z}_q \times S^{n-1})$. The below figure shows the situation for $q = 2$ and $n = 2$.



Example 4.2. Let M and N be closed n-dimensional submanifolds of $\mathbb{R}^N$ which intersect in a nonempty $(n-1)$-dimensional submanifold Q. Let S be the union of tubular neighborhoods of Q in M and N, and denote by $\bar{S}$ the closure of S in $M \cup N$. Then $\bar{S} \cong Q \times C\mathbb{Z}_4$ and $M \cup N$ is a closed n-dimensional P-manifold with $P = \mathbb{Z}_4$, given by

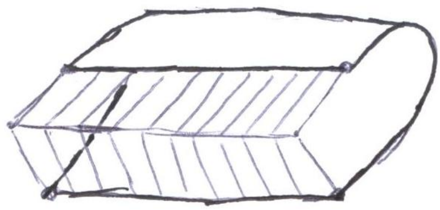
$$M \cup N = (M \cup N) \setminus S \cup_{Q \times \mathbb{Z}_4} Q \times C\mathbb{Z}_4$$

As an example, consider two copies of S^n intersecting transversally and nontrivially in $\mathbb{R}^{n+1}$. Then this intersection can be written as

$$S^n \cup S^n = D^n \times \mathbb{Z}_4 \cup_{S^{n-1} \times \mathbb{Z}_4} S^{n-1} \times C\mathbb{Z}_4$$

Remark 4.1.3. Due to the uniform structure of the singularities, it is possible to ignore the cone part $A_1 \times CP$ of a manifold $M = A \cup_{A_1 \times P} A_1 \times CP$ with singularity type P, and look instead at manifolds A equipped with a diffeomorphism $\beta : \partial A \xrightarrow{\sim} \bar{A} \cup A_1 \times P$.

The following figure illustrates the situation. The foremost line is the subset of singular points, and after removing the cone $[0, 1] \times C(\mathbb{Z}_2)$, we are left with a disk equipped with a partition of its boundary into two sets of line segments, one of which comprises the singular boundary.



Remark 4.1.4. By relaxing the requirement $\partial A = A_1 \times P$ to an inclusion $A_1 \times P \subseteq \partial A$, we obtain a notion of not necessarily closed P-manifolds. Writing $\partial A = \delta A \cup A_1 \times P$, the association $A \mapsto \delta A$ is a natural choice for a boundary operation for P-manifolds. As in the story for manifolds with tangential structure, we would like to take δA itself to be a P-manifold. When we do this, the most natural formalism this notion of manifolds with singularities fits into is the setting of $\langle k \rangle$-manifolds we introduced in the previous chapter. Definition 4.1.5 makes these ideas precise.

Definition 4.1.5. Let P be a closed manifold. A manifold of singularity type P is given by a $\langle 2 \rangle$-manifold A and a $\langle 1 \rangle$-manifold $\beta_1 A$ equipped with isomorphisms

$$\phi_1 : \partial_1 A \xrightarrow{\sim} \beta_1 A \times P$$

$$\phi_0 : \partial_1 \partial_0 A \xrightarrow{\sim} \partial_0(\beta_1 A) \times P$$

Furthermore, we require that these diffeomorphisms are compatible, in the sense that the following diagram commutes:

$$\begin{array}{ccc}
\partial_0 \partial_1 A & \xrightarrow{\partial_0 \phi_1} & \partial_0(\beta_1 A \times P) \\
{\scriptstyle \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}} \\
\partial_1 \partial_0 A & \xrightarrow[\phi_0]{} & \partial_0 \beta_1 A \times P
\end{array}$$

where we have used the equality $\partial_0 \partial_1 A = \partial_0 A \cap \partial_1 A = \partial_1 \partial_0 A$. In this way, the assignment $A \mapsto \beta_1 A$ is a natural transformation from P-manifolds to manifolds with boundary, so we may denote the above data simply by A alone, leaving $\beta_1 A$ and the isomorphisms ϕ_0 and ϕ_1 implicit in the notation. Furthermore, we say a P-manifold A is *closed* if $\partial_0 A = \emptyset$. In particular, for any P-manifold A, $\partial_0 A$ is a closed P-manifold.

Definition 4.1.6. The above definition readily generalizes to $\langle k\rangle$-manifolds with singularity type $S = \{P_1, ..., P_k\}$, where each P_i is a closed manifold, and we allow the P_i to be trivial, i.e. the point $*$. Such an S-manifold M is given by the data of, for each $a \in 2^k$, an $\langle\omega(a)\rangle$-manifold $M\langle a\rangle$, such that for each $0 \leq i \leq k$ with $a_i = 1$, there is an isomorphism of $\langle\omega(a) - 1\rangle$-manifolds

$$\beta(a, i) : \partial_i M\langle a\rangle \simeq M\langle a - e_i\rangle \times P_i$$

where e_i is the k-tuple with all zeroes except for a 1 in the i'th position. We require that, whenever $a_i = a_j = 1$, the following diagram commutes.

$$\begin{array}{ccccc}
\partial_j\partial_i M\langle a\rangle & \xrightarrow{\beta(a,i)} & \partial_j M\langle a - e_i\rangle \times P & \xrightarrow{\beta(a-e_i,j)\times \mathrm{id}} & M\langle a - e_i - e_j\rangle \times P_j \times P_i \\
\big\downarrow{\scriptstyle \mathrm{id}} & & & & \\
\partial_j M\langle a\rangle \cap \partial_i M\langle a\rangle & & & & \big\downarrow{\scriptstyle \mathrm{id}\times T} \\
\big\downarrow{\scriptstyle \mathrm{id}} & & & & \\
\partial_i\partial_j M\langle a\rangle & \xrightarrow{\beta(a,j)} & \partial_i M\langle a - e_j\rangle \times P_j & \xrightarrow{\beta(a-e_j,i)\times \mathrm{id}} & M\langle a - e_i - e_j\rangle \times P_i \times P_j
\end{array}$$

Remark 4.1.7. Although we have given the full definition of a manifold with multiple singularities (our definition is a very slight generalization of the one given in [Baa73], by omitting the requirement $P_1 = *$) we want to focus on the case $P_i = *$ for all $0 \leq i \leq k-1$ (i.e. $S = \{*, ..., *, P\}$. However, everything we say in the rest of the chapter admits a suitable generalization to arbitrary singularity types.

4.1.1 Mapping spaces

Definition 4.1.8. Let $M = (A, B, \beta)$ and $N = (A', B', \beta')$ be P-manifolds. A P-morphism $f : M \to N$ is a pair of smooth maps $f_1 : A \to A'$ and $f_0 : B \to B'$ such that the following diagram commutes.

$$\begin{array}{ccc} \partial_1 A & \xrightarrow{\partial_1 f_1} & \partial_1 A' \\ {\scriptstyle \beta_1}\big\downarrow & & \big\downarrow{\scriptstyle \beta_1'} \\ B \times P & \xrightarrow[f_0 \times \mathrm{id}_P]{} & B' \times P \end{array}$$

Definition 4.1.9. Let M and N be P-manifolds. The set $C^\infty_P(M,N)$ of P-morphisms $M \to N$ is a subset of the space $C^\infty_{\langle 2\rangle}(M,N)$ of morphisms of $\langle 2\rangle$-manifolds. We give $C^\infty_P(M,N)$ the subspace topology.

Definition 4.1.10. A P-morphism $M \to N$ is a P-diffeomorphism if it is a diffeomorphism considered as a morphism of $\langle 2\rangle$-manifolds. We denote the space of P-diffeomorphisms by $\mathrm{Diff}^P(M,N)$. Also, we denote the space of self-diffeomorphisms of a P-manifold M by $\mathrm{Diff}^P(M) := \mathrm{Diff}^P(M,M)$. This is a topological group with the product given by composition.

Definition 4.1.11. Consider a pair of n-dimensional P-manifold, say $M = (A, A_1) = A \cup_{A_1 \times P} A_1 \times CP$ and $N = (B, B_1) = B \cup_{B_1 \times P} B_1 \times CP$. We define their disjoint union to be

$$M \sqcup N = (A \sqcup B) \cup_{(A_1 \sqcup B_1) \times P} (A_1 \sqcup B_1) \times CP$$

In other words, $M \sqcup N = (A \sqcup B, A_1 \sqcup B_1)$.

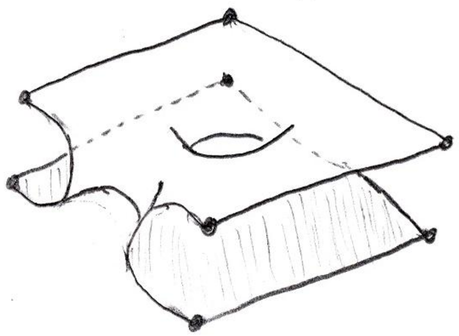
Definition 4.1.12. Let $M = (A, A_1)$ and $N = (B, B_1)$ be n-dimensional closed P-manifolds in the sense of Definition 4.1.5. A *P-cobordism* from M to N is a P-manifold W such that $\partial_0 W = (-M) \sqcup N$.

Remark 4.1.13. By the commutative diagram in Definition 4.1.5, it follows that if $M = (A, A_1)$ and $N = (B, B_1)$ are n-dimensional closed P-manifolds and W is a P-cobordism from M to N, then $\partial_1 W \cong W_1 \times P$ is a cobordism from $A_1 \times P$ to $B_1 \times P$.

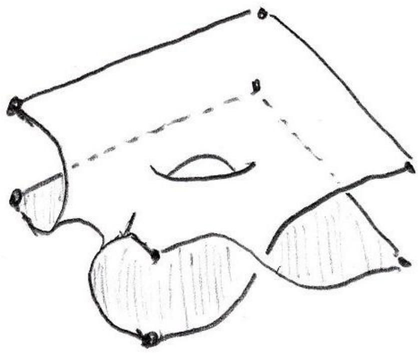
Remark 4.1.14. We would like to impose slightly more structure on our P-cobordisms in order to make them more susceptible to analysis by homotopy theoretical methods. Namely we would like to require that the cobordism $W_1 \times P : A_1 \times P \to B_1 \times P$ in Remark 4.1.13 arises as the product of a cobordism $W_1 : A_1 \to B_1$ and the manifold P, thus eliminating the possibility of twisting in P.

Definition 4.1.15. A P-cobordism whose singular boundary arises as in Remark 4.1.14 is called a *regular* P-cobordism.

Example 4.3. The following is an example of a regular P-cobordism.



The following is an example of a P-manifold which is not a regular P-cobordism, as the twist cobordism $T : \mathbb{Z}_2 \to \mathbb{Z}_2$ is not a product of $\mathbb{Z}_2$ with a cobordism $* \to *$.



Definition 4.1.16. Let M be an m-dimensional P-manifold. Fix once and for all an embedding $i_P : P \to \mathbb{R}^{p+l}$ for some $l \geq 0$. We define the space of neat P-embeddings $M \hookrightarrow \mathbb{R}^{n+m}_2$ as the subspace $\mathrm{Emb}^P(M, \mathbb{R}^{n+m}_2) \subset \mathrm{Emb}(M, \mathbb{R}^{n+m}_2)$ of the space of neat embeddings such that for each $e \in \mathrm{Emb}^P(M, \mathbb{R}^{n+m}_2)$, the restriction to $\partial_1 M \simeq \beta_1 M \times P$ is equal to $e|_{\partial_1 M} = e_1 \times i_P$ for some embedding $e_1 : \beta_1 M \to \mathbb{R}^{n+m-p-l-1}$.

Remark 4.1.17. Let M be an m-dimensional P-manifold. Then there is a natural free right action of $\mathrm{Diff}^P(M)$ on $\mathrm{Emb}^P(M, \mathbb{R}_2^{n+m})$ given by precomposition. For large n, $\mathrm{Emb}^P(M, \mathbb{R}_2^{n+m})$ becomes contractible, so the moduli space

$$\mathrm{colim}_{n\to\infty} \mathrm{Emb}^P(M, \mathbb{R}_2^{n+m})/\mathrm{Diff}^P(M) := B_\infty(M)$$

is homotopy equivalent to $B\mathrm{Diff}_P(M)$.

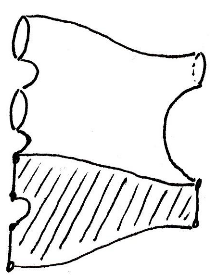
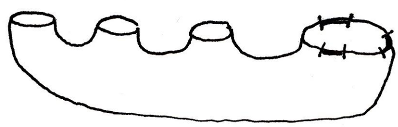
Remark 4.1.18. Definition 4.1.16 holds also for embedded cobordisms, that is, neat embeddings into $[x, y] \times \mathbb{R}^{n+m-2}$ for $x, y \in \mathbb{R}^2$ with $x \leq y$. In this case we have a free action on $\mathrm{Emb}^P(M, [x, y] \times \mathbb{R}^{n+m-2})$ by the subgroup $\mathrm{Diff}^P(M; \partial\Box^2) \subseteq \mathrm{Diff}^P(M)$ of P-diffeomorphisms fixing the boundary decompositions. This is a P-singular adaptation of the situation discussed in Section 3.5.1. The moduli space

$$\mathrm{colim}_{n\to\infty} \mathrm{Emb}^P(M, [x, y] \times \mathbb{R}^{n+m-2})/\mathrm{Diff}^P(M; \partial\Box^2) := B_\infty(M; \Box^2)$$

now becomes homotopy equivalent to $B\mathrm{Diff}^P(M; \partial\Box^2)$.

4.2 Higher categorical structure from singularities

In this section we will consider manifolds with singularity of type P. We will explore the different ways in which such manifolds assemble into k-fold categories. The question of how the introduction of singularities introduces a new categorical level, either in terms of k-categories or k-fold categories, is a question which has been emphasized by Nils Baas in private communication. Although determining what the singularities add to the equation is a very interesting problem, we restrict ourselves in the following to the structure inherited on P-manifolds from their corner structure.

Definition 4.2.1. Let P be a closed p-dimensional manifold. We fix an embedding $i_P : P \to \mathbb{R}^{p+m}$. We define the k-fold category of P-cobordisms $\mathcal{C}_{d,n}^{k-1}(P)$ for $d \geq p+1$ in a fashion similar to how we defined $\mathcal{C}_d^k$, i.e. an a-morphism for $a \in 2^k$ is given by a neat P-embedding $M^d \to [x,y] \times \mathbb{R}^{d+n-\omega(a)}$ where $[x,y] \in \square(k)(a)$. That is, we assign the singular boundary to the last coordinate, such that $s_k M = \beta M \times i_P(P) \subseteq s_k([x,y]) \times \mathbb{R}^{n+d-\omega(a)-m-p} \times \mathbb{R}^{m+p}$, such that $i_P(P) \subseteq \mathbb{R}^{m+p}$ and $\beta M \subseteq s_k([x,y]) \times \mathbb{R}^{n+d-\omega(a)-m-p}$. We topologize these as a subspace of $\mathcal{C}_{d,n}^k$.

Definition 4.2.2. We can extract a $(k-1)$-fold category which we denote $\mathsf{Cob}_{d,n}^{k-1}(P)$ from $\mathcal{C}_{d,n}^{k-1}(P)$ by restricting to composition in the non-singular (i.e. the first $(k-1)$) directions.

Remark 4.2.3. It follows from Definition 4.2.1 that the $(1,1)$-morphisms of $\mathcal{C}_{d,n}^1(P)$ are d-dimensional P-manifolds in the sense of Definition 4.1.5. Furthermore, the $(1,0)$-morphisms are closed $(d-1)$-dimensional P-manifolds.

Remark 4.2.4. Notice that if we remove the requirement that our P-cobordisms are regular, then we can define the 2-fold category of *twisted* P-cobordisms as the full subcategory of $\mathcal{C}_d^2$ for which each horizontal morphism is of the form $V \times P$ but whose embedding does not need to be constant on the factor P. In particular, there is an action on the set of such cobordisms by the H-space $\Omega\mathrm{Diff}(P)$.

4.2.1 The nerve of $\mathsf{Cob}_P(d)$

Remark 4.2.5. A consequence of Theorem 3.6.1 is that a compact d-dimensional submanifold $M \subseteq \mathbb{R}^{d+n}$ corresponds up to homotopy to a map $f_M : S^{n+d} \to \mathrm{Th}(\gamma^\perp(d,n))$. Furthermore, the empty manifold corresponds to the fixed map at the basepoint. It follows then that the map f_M is nullhomotopic if and only if the manifold M is nullcobordant. One perspective on the theory of manifolds with singularities is that we are taking a prescribed p-dimensional compact manifold P and forcing it to be nullcobordant, by adding cones $N \times CP$ for all $d-p$-dimensional manifolds N. The corresponding procedure in the thom space is to take the homotopy cofiber of a certain composition of maps which we describe below.

In [Perb], Perlmutter defines the spectrum $MT_P(d)$ as a generalization of the Madsen-Tillmann spectrum $MT(d)$ in the following way. From the embedding $i_P : P \to \mathbb{R}^{p+m}$ in 4.1.16, we obtain the Pontryagin-Thom collapse map $c_P : S^{p+m} \to \mathrm{Th}(\gamma^\perp(p,m))$. There is a morphism of vector bundles

$$\begin{array}{ccc} \gamma^\perp(d-p,n-m)\times\gamma^\perp(p,m) & \xrightarrow{\hat{\mu}} & \gamma^\perp(d,n) \\ \downarrow & & \downarrow \\ G(d-p,n-m)\times G(p,m) & \xrightarrow[\mu]{} & G(d,n) \end{array}$$

where $\hat{\mu}((V,v),(U,u)) = (V\times U,(v,u))$, which gives rise to a map of Thom spaces

$$\mathrm{Th}(\hat{\mu}) : \mathrm{Th}(\gamma^\perp(d-p,n-m)) \wedge \mathrm{Th}(\gamma^\perp(p,m)) \to \mathrm{Th}(\gamma^\perp(d,n))$$

Precomposing this map with c_P, we obtain a map

$$\tau_{d,n} = \mathrm{Th}(\hat{\mu}) \circ (\mathrm{id} \wedge c_P) : \mathrm{Th}(\gamma^\perp(d-p,n-m)) \wedge S^{p+m} \to \mathrm{Th}(\gamma^\perp(d,n)) \quad (4.1)$$

Since $\mathrm{Th}(\gamma^\perp(d-p,n-m)) \wedge S^{p+m} \subseteq \mathrm{Th}(\gamma^\perp(d,n)$ defines a cofinal subspectrum, this map extends to a morphism of spectra

$$\tau_P : MT(d-p) \to MT(d)$$

The embedding $G(d,n) \to G(d+1,n)$ also gives rise to a morphism of spectra $\hat{j}_d : \Sigma^{-1}MT(d) \to MT(d+1)$. The cofiber of the composition

$$\Sigma^{-1}MT(d-p) \xrightarrow{\Sigma^{-1}\tau_P} \Sigma^{-1}MT(d) \xrightarrow{\hat{j}_d} MT(d+1)$$

is called $MT_P(d+1)$.

Theorem 4.2.6 ([Perb], Theorem 1.1)**.** There is a weak homotopy equivalence

$$B\mathsf{Cob}_P(d+1) \simeq \Omega^{\infty-1}MT_P(d+1)$$

4.2.2 A poset model for $\mathsf{Cob}^k_{d,n}(P)$

Goal 4.2.7. Just like $\mathbb{R}^{d+n}$ is the natural background space for d-dimensional manifolds without boundary, the natural background space for d-dimensional manifolds M with boundary is the space $\mathbb{R}^{d+n-1}\times\mathbb{R}_+$. In this section we define a moduli space of manifolds with Baas-Sullivan singularities of type P and construct a poset model for $\mathsf{Cob}^k_{d,n}(P)$, with a view toward a generalization of Theorem

4.2.6 in line with Bökstedt-Madsen's generalization (Theorem 3.6.1) of Theorem 2.5.15. Of course, we may swap $\mathbb{R}_1^{d+n}$ with $\mathbb{R}_k^{d+n}$ in the case of d-dimensional $\langle k\rangle$-manifolds, and in this way the material which follows admits a natural extension to manifolds with more than one singularity type, although we will only cover here the case of a single singularity type. Due to constraints on time, we have had to leave many results as conjectures, although we believe most of them should follow in a relatively straightforward manner from the proofs of the analogous results in the previous chapter.

Definition 4.2.8. We fix once and for all a p-dimensional closed compact submanifold $P \subseteq (-1,1)^{p+m}$, and let $d > p$ and $n \geq m$. Let $\Psi_{d,n}(P)$ be the set of d-dimensional submanifolds $M \subseteq \mathbb{R}^{d+n-1} \times \mathbb{R}_+$ such that M is closed as a subset, and $\partial M \subseteq \mathbb{R}^{d+n-1} \times \{0\}$. The latter factor splits as $\mathbb{R}^{d+n-p-m-1} \times \mathbb{R}^{p+m}$, and we require that $\partial M = \beta_1 M \times P$ for a $\beta_1 M \in \Psi_{d-p-1}(\mathbb{R}^{d+n-m-p-1})$. By adjoining an open collar $[0,-\infty) \times \partial M$, we can topologize $\Psi_{d,n}(P)$ as a subspace of $\Psi_d(\mathbb{R}^{d+n})$.

Definition 4.2.9. For $0 \leq k \leq n+d-m-p-1$, let $D^k_{d,n}(P)$ be the subspace of $\Psi_{d,n}(P)$ consisting of those manifolds M such that $M \subseteq \mathbb{R}^k \times (-1,1)^{n+d-k-1} \times [0,1)$. As before, the empty manifold serves as a basepoint for both $\Psi_{d,n}(P)$ and $D^k_{d,n}(P)$.

Lemma 4.2.10. Let $[a,b] \in \square(k)$ and denote by $U_a \subseteq D^k_{d,n}(P)$ the subset of P-manifolds M such that M intersects the walls of $[a,b]$ transversally. Then U_a is a open subset.

Proof. The openness of U_a follows in the same manner as in Corollary 3.5.20. □

Conjecture 4.2.11. There is a weak homotopy equivalence $D^k_{d,n}(P) \simeq \Omega D^{k+1}_{d,n}$

Remark 4.2.12. We expect Conjecture 4.2.11 to be an easy consequence of the proof of [GRW10, Theorem 3.13.], but time has not allowed a detailed analysis.

Definition 4.2.13. Let $\mathcal{D}^k_{d,n}(P)$ be the k-fold topological poset whose space of top level morphisms is given by the subspace $\mathcal{D}^k_{d,n}(P) \subseteq \square(k) \times D^k_{d,n}(P)$ consisting of those pairs $([a,b], M)$ for which M intersects the walls of $[a,b]$ transversally. Two such pairs $([x,y], M)$ and $([x',y'], M')$ are composable in the i-direction if and only if $x_j = x'_j$ and $y_j = y'_j$ for all $j \neq i$ and $y_i = x'_i$, i.e. if the boxes are composable, and $M = M'$. In that case, the composition is given by $([x'', y''], M)$, where $x''_j = x_j$ and $y''_j = y_j$ for all j. This defines a k-fold topological poset structure.

Definition 4.2.14. We also define the topological poset $\mathcal{D}^{\perp k}_{d,n}(P)$ as the sub-poset of $\mathcal{D}^k_{d,n}(P)$ consisting of those $([a,b],M)$ for which M intersects the walls of $[a,b]$ orthogonally.

Lemma 4.2.15. The inclusion $\mathcal{D}^{\perp k}_{d,n}(P) \hookrightarrow \mathcal{D}^k_{d,n}(P)$ induces a weak equivalence of simplicial spaces $N\mathcal{D}^{\perp k}_{d,n}(P) \to N\mathcal{D}^k_{d,n}(P)$.

Proof. Let $M \in \mathcal{D}^k_{d,n}(P)$ such that M intersects the walls of the box $[x,y]$ transversally. The homotopies we defined in Lemma 3.5.18 and Corollary 3.5.20 fix the walls of the box $[x,y]$. We can extend this homotopy to the box $[(x,0),(y,2)]$ where the last coordinate is along the factor $[0,1)$. can be applied along the first $n+d-m-p-1$ coordinates of $\mathbb{R}^{n+d-1}$. We thus obtain the result in the same way as for Lemma 3.5.22. □

Conjecture 4.2.16. There is a weak homotopy equivalence $B\mathcal{D}_{d,n}(P) \simeq D_{d,n}(P)$.

Conjecture 4.2.17. The inclusion

$$N\mathsf{Cob}^k_{d,n}(P) \to N\mathcal{D}^{\perp k}_{d,n}(P)$$

defined by attaching infinite open collars gives rise to a weak homotopy equivalence

$$B\mathsf{Cob}^k_{d,n}(P) \simeq B\mathcal{D}^{\perp k}_{d,n}(P)$$

Conjecture 4.2.18. There is a weak homotopy equivalence

$$B\mathcal{D}^k_{d,n}(P) \simeq \Omega^{n+d-k} C\tau_{d,n}$$

where $C\tau_{d,n}$ is the homotopy cofiber of the map defined in (4.1).

Remark 4.2.19. When $k = 1$ and $n = \infty$, and assuming Conjecture 4.2.17, we regain Perlmutter's theorem (Theorem 4.2.6). This conjecture generalizes Perlmutter's theorem by drawing a comparison with the Bökstedt-Madsen theorem (Theorem 3.6.1) and Galatius-Madsen-Tillmann-Weiss theorem (the case when $k = 1$ and $n = \infty$).

4.2.3 The nerve of $\mathcal{C}^k_{d,n}(P)$

Remark 4.2.20. We can extend the definition of $D^k_{d,n}(P)$ and $\mathcal{D}^k_{d,n}(P)$ to obtain topological $(k+1)$-fold poset models for the full $(k+1)$-fold category $\mathcal{C}^k_{d,n}(P)$, as follows.

Definition 4.2.21. Define the poset model of $\mathcal{C}^k_{d,n}(P)$ as the $(k+1)$-fold topological poset $\bar{\mathcal{D}}^k_{d,n}(P)$ whose top level morphisms are pairs $([x,y],M) \in \square(k+1)\times\mathbb{R}^{n+d}$ such that M is closed as a subset and has empty boundary, and M intersects the walls of $[x,y]$ transversally. We further require that the intersection $(s_{k+1}([x,y]) \cup t_{k+1}([x,y])) \cap M$ split as

$$\beta M \times P \subseteq (s_{k+1}([x,y]) \cup t_{k+1}([x,y])) \times \mathbb{R}^{n+d-k-p-m-1} \times \mathbb{R}^{p+m}$$

where $\beta M \subseteq (s_{k+1}([x,y]) \cup t_{k+1}([x,y])) \times \mathbb{R}^{n+d-k-p-m-1}$ and $P \subseteq \mathbb{R}^{p+m}$. Composition is defined by glueing boxes as for $\mathcal{D}^k_{d,n}(P)$.

Remark 4.2.22. The chief problem for calculating $B\mathcal{C}^k_{d,n}(P)$ is that it is not obvious what the homotopy type of $B\bar{D}^k_{d,n}(P)$ is. One of the nice properties of $\mathcal{D}^k_{d,n}(P)$ is that every d-dimensional submanifold $M \in D^k_{d,n}(P)$ appears as a top-level morphism. This is not the case for $\bar{D}^k_{d,n}(P)$. Indeed, we cannot apply [BM, Theorem 1.8] unless we restrict to a specific subspace of $D^k_{d,n}$. It is also not clear what, if any, alterations are necessary on the thom space of the universal bundle to account for this modification.

Although we do not treat this problem in detail here, it is an interesting problem which the author wishes to return to in the future, and may serve as a signpost for future development in the field.

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