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Algebras of Finite Representation Type

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ABSTRACT

In this thesis we are considering finite dimensional algebras. We prove that any basic and indecomposable finite dimensional algebra A over an algebraically closed field k is isomorphic to a bound quiver algebra. Furthermore, if A is hereditary we prove that it is isomorphic to a path algebra. Finally, we prove that a path algebra is of finite representation type if and only if the underlying graph of the quiver is a Dynkin diagram. This is done using reflection functors, which were first introduced by Bernstein, Gel'fand, Ponomarev in [4].

SAMMENDRAG

I denne oppgaven studerer vi endelig-dimensjonale algebraer. Vi beviser at enhver basisk og ikke-dekomponerbar endelig-dimensjonal algebra A over en algebraisk lukket kropp k er isomorf med en bundet quiver-algebra. Videre, hvis A er hereditær beviser vi at den er isomorf med en veialgebra. Til slutt beviser vi at en veialgebra er av endelig representasjonstype hvis og bare hvis den underliggende grafen til quiveret er et Dynkin diagram. Vi bruker refleksjonsfunktorer, først introdusert av Bernstein, Gel'fand, Ponomarev (cf. [4]), til å bevise dette.

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CONTENTS

INTRODUCTION

The main goal of this thesis is to prove Gabriel's theorem, which states that the path algebra of a quiver Q is of finite representation type if and only if the underlying graph of Q is a Dynkin diagram.

The proofs of some well-known and basic results are skipped to avoid writing a textbook in abstract algebra. The reader is supposed to be familiar with some general concepts and results from basic abstract algebra, but we will start by recalling some important notions and results from the module theory in Chapter 1. We next introduce the concepts of quivers, path algebras and representations of quivers in Chapter 2. These concepts are important tools when studying algebras and modules. We will see that the representations of a quiver Q can be used to visualise modules of the path algebra of Q. In Chapter 3 we will see that different algebras are isomorphic to path algebras, or path algebras modulo some ideal. In Chapter 4 we will introduce reflection functors and a quadratic form of a quiver, which will be important in proving Gabriel's theorem.

Throughout this thesis k will denote an algebraically closed field, and an *algebra* A will denote a finite dimensional k-algebra with an identity.

1

PRELIMINARIES

In this chapter we will build a basis to be used throughout the thesis. We will recall some important notions, and create a solid foundation of useful results.

1.1 MODULES

Definition 1.1.1. Let A be an algebra, and $M \neq (0)$ a left A-module. The module M is *indecomposable* if $M = M_1 \bigoplus M_2$ implies $M_1 = (0)$ or $M_2 = (0)$. The module M is called a *simple* A-module if $M \neq (0)$ and for any submodule $N \subset M$ either $N = M$ or $N = (0)$. The module M is called *semisimple* if it is a direct sum of simple A-modules.

Our first result is a well-known result stating that a module satisfying some finiteness condition on its chain of submodules can be uniquely written as a direct sum of submodules. This result is called the *Krull-Remak-Schmidt theorem*, and stresses the importance of indecomposable submodules. We will not prove this theorem here (cf. [3]).

Theorem 1.1.2 (Krull-Remak-Schmidt)**.** *Let $M \neq (0)$ be a noetherian and artinian module, that is there is no strictly ascending or descending infinite chain of submodules of M. Then M can be written uniquely (up to permutations and isomorphisms) as a direct sum of indecomposable submodules of M.*

Now, let us introduce some special class of algebras called *basic algebras*. Let A be an algebra. By Theorem 1.1.2 the algebra A can be decomposed uniquely as a left A-module as follows:

$$_AA \simeq P_1 \bigoplus P_2 \bigoplus \cdots \bigoplus P_n,$$

where P_i is some indecomposable submodule of $_AA$.

Definition 1.1.3. An algebra A is called *basic* if $P_i \not\simeq P_j$ whenever $i \neq j$.

Definition 1.1.4. Let A be an algebra. Then A is of *finite representation type* if there exist only a finite number of isomorphism classes of indecomposable finitely generated left A-modules.

Our next two notions are *free modules* and *projective modules*. As we will see later a projective module is a generalization of a free module.

Definition 1.1.5. Let A be an algebra, and let F be an A-module. The module F is a *free module* if F is isomorphic to a direct sum of copies of A.

Definition 1.1.6. Let A be an algebra, and let P be an A-module. The module P is said to be *projective* if for every A-epimorphism $g : X \to Y$ and every A-homomorphism $f : P \to Y$, there exists an A-homomorphism $h : P \to X$ such that $gh = f$. That is, the following diagram commutes:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] & \\
X \arrow[r, "g"] & Y \arrow[r] & 0
\end{tikzcd}$$

It is easily observed that a direct sum of projective modules is again a projective module.

Lemma 1.1.7. *Let A be an algebra, and let F be a free A-module. Then F is a projective A-module.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & A \arrow[dll, "h_i'"'] \arrow[d, bend right, "j_i"'] \\
 & F = \bigoplus_{i=1}^{n} A \arrow[u, bend right, "p_i"'] \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] \\
X \arrow[r, "g"] & Y \arrow[r] & 0
\end{tikzcd}$$

where j_i is the natural inclusion of A into coordinate i of F and p_i is the projection of coordinate i of F onto A. Since A is clearly projective as an A-module such a map h_i' must exist. Then $h = \bigoplus_{i=1}^{n} h_i$, where $h_i = h_i' \circ p_i$, so F is projective. $\square$

Lemma 1.1.8. *Let A be an algebra, and let P be an A-module. Then P is a projective module if and only if there exists a free module F such that $F \simeq P \bigoplus Q$ for some A-module Q.*

Proof. Suppose P is a projective A-module. Let $g : F \to P$ be an epimorphism, where F is a free A-module. Let $f : P \to P$ be the identity map, denoted by 1_P. Since P is projective there exists a homomorphism $h : P \to F$ such that $gh = 1_P$.

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "1_P"] \\
F \arrow[r, "g"] & P \arrow[r] & 0
\end{tikzcd}$$

Then we get $F = \operatorname{Im} h \bigoplus \ker g$, and h must be a monomorphism. That implies $\operatorname{Im} h \simeq P$, and hence $F \simeq P \bigoplus \ker g$.

Suppose there exists a free A-module F such that $F \simeq P \bigoplus Q$. That is, there exists a $\phi : F \to P \bigoplus Q$, where ϕ is an isomorphism.

Let $g : X \to Y$ be an epimorphism and $f : P \to Y$ be a homomorphism of A-modules. By Lemma 1.1.7 the module F is projective since F is free. Hence, there exists a homomorphism $h : F \to X$ such that $gh = (f, 0)\phi$. Since ϕ is an isomorphism we obtain $gh\phi^{-1} = (f, 0)$. Consider the natural inclusion $i : P \to P \bigoplus Q$. We get that $gh\phi^{-1}i = (f, 0)i = f$. Hence, P is a projective A-module.

$$\begin{tikzcd} F \arrow[r, "\phi"'] \arrow[d, dashed, "\exists h"] & P \bigoplus Q \arrow[d, "{(f,0)}"] & P \arrow[l, hook', "i"'] \arrow[dl, "f"] \\ X \arrow[r, "g"] & Y & \end{tikzcd}$$

□

Before we continue we need to establish some notation on idempotent elements. Let $e_1, e_2 \in A$ be idempotents. Then e_1, e_2 are called *orthogonal* if $e_1e_2 = e_2e_1 = 0$, and an idempotent $e \in A$ is said to be *primitive* if $e \neq e_1 + e_2$ for any nonzero, orthogonal idempotents $e_1, e_2 \in A$. It is clear that the left A-module Ae_i is indecomposable if and only if e_i is a primitive idempotent. If a set of primitive, orthogonal idempotents in an algebra A is such that they sum up to the identity of A we say that this set is *complete*. If $\{e_1, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents in ${}_AA$ we get that it is isomorphic to $Ae_1 \bigoplus \cdots \bigoplus Ae_n$.

Lemma 1.1.9. *Let A be an algebra, $\{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents in A, and ${}_AA = Ae_1 \bigoplus \cdots \bigoplus Ae_n$ be a decomposition of ${}_AA$ into indecomposable submodules. Then every projective left A-module P can be decomposed in the following way: $P = P_1 \bigoplus \cdots \bigoplus P_t$, where P_j is indecomposable and isomorphic to some Ae_s for every $j \in \{1, \dots, t\}$.*

Proof. Let P be a projective module. Then by Lemma 1.1.8 there exists some free module F such that $F = P \bigoplus Q$, for some A-module Q. By our assumption and the definition of a free module we must have $F = \bigoplus_{i=1}^{n} (Ae_i)^m \simeq P \bigoplus Q = P_1 \bigoplus \cdots \bigoplus P_t \bigoplus Q_1 \bigoplus \cdots \bigoplus Q_s$ for some m and some s. Since each Ae_i and each P_j is indecomposable the result follows from Theorem 1.1.2. $\square$

Definition 1.1.10. Let A be an algebra, and let L, M, N be A-modules. Consider the short exact sequence:

$$0 \longrightarrow L \xrightarrow{u} M \xrightarrow{r} N \longrightarrow 0.$$

The above short exact sequence is said to *split* if there exists a homomorphism $v : N \to M$ such that $rv = 1_N$.

Note that a short exact sequence splits if and only if there exists a homomorphism $s : M \to L$ such that $su = 1_L$ or equivalently $M = \operatorname{Im} u \bigoplus \ker s = \operatorname{Im} v \bigoplus \ker r$.

Lemma 1.1.11. *Let A be an algebra. Let L, M, P be A-modules such that the following is a short exact sequence*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0. \tag{1}$$

If P is a projective A-module, then the short exact sequence splits.

Proof. Suppose P is projective. Consider the identity map $1_P : P \to P$. By the definition of a projective module there exists a homomorphism $h : P \to M$ such that $gh = 1_P$. Hence, the short exact sequence (1) splits. $\square$

1.2 RADICALS

Definition 1.2.1. Let A be an algebra. The *radical* of A is the intersection of all maximal left ideals in A. We denote it by rad A, or simply $\underline{\mathrm{r}}$.

The radical of an algebra A is a left ideal as an intersection of left ideals. We will see later that $\underline{r}$ is actually a two-sided ideal in A.

Proposition 1.2.2. *Let A be an algebra. For any $a \in A$ the following are equivalent:*

(i) $a \in \operatorname{rad} A$

(ii) $1 - xa$ *is left invertible for all* $x \in A$

(iii) $aS = (0)$ *for any simple A-module S.*

Proof. $(i) \Rightarrow (ii)$: Let $a \in \operatorname{rad} A$, and suppose by contradiction that there exists some $x \in A$ such that $1 - xa$ is not left invertible. Now, consider the ideal $A(1 - xa)$. Since $1 - xa$ is not invertible we get $A(1 - xa) \subset A$, that is, $A(1 - xa)$ is a proper ideal in A. Then there must exist some maximal ideal M in A such that $A(1 - xa) \subseteq M$. This implies $1 - xa \in M$. Since $a \in \operatorname{rad} A$ we get by the definition of a radical that $a \in M$, which implies $xa \in M$. But this would imply that $1 = (1 - xa) + xa \in M$, which is a contradiction since M is maximal. Hence, $1 - xa$ is left invertible for all $x \in A$.

$(ii) \Rightarrow (iii)$: Suppose there exists a simple A-module S such that $aS \neq (0)$. Then there must exist some nonzero $s \in S$ such that $as \neq 0$. Now, consider the left A-module Aas, and note that $(0) \subset Aas \subseteq S$. Since S is simple we get $Aas = S$. Hence, there exists an $x \in A$ such that $xas = s$, which implies $(1 - xa)s = 0$.

Now, since $1 - xa$ is left invertible, we get that $s = 0$. This is a contradiction, so $1 - xa$ is non-invertible.

$(iii) \Rightarrow (i)$: Suppose $aS = (0)$ for any simple A-module S. Let M be a maximal ideal in A. Then A/M is a simple left A-module, so $a(A/M) = (0)$ by the assumption. Denote by $1_A + M$ the identity element of A/M. In particular, $a(1_A + M) = 0$, which implies that $a + M = 0$. Then we get that $a \in M$, and hence $a \in \operatorname{rad} A$, since M was a randomly chosen maximal ideal in A. □

The next few notions and results will help us see that $\underline{r} = \operatorname{rad} A$ is actually a two-sided ideal in A.

Definition 1.2.3. Let A be an algebra, and M an A-module. Then the annihilator of M is the set $\operatorname{Ann}(M) = \{a \in A \mid am = 0 \text{ for all } m \in M\}$.

Note that $\operatorname{Ann}(M)$ is a two-sided ideal in A.

Corollary 1.2.4. *Let A be an algebra. Then $\underline{r} = \operatorname{rad} A = \bigcap_S \operatorname{Ann}(S)$, where the intersection is taken over all the simple A-modules.*

Proof. Follows directly from Proposition 1.2.2. □

Hence, $\underline{r} = \operatorname{rad} A$ is a two-sided ideal in A as an intersection of two-sided ideals.

Lemma 1.2.5 (Nakayama's lemma)**.** *Let A be an algebra, M a finitely generated A-module, and $I \subseteq \operatorname{rad} A$ be an ideal in A. If $IM = M$, then $M = (0)$.*

Proof. Let M be a finitely generated A-module and let $I \subseteq \operatorname{rad} A$ be an ideal such that $IM = M$. Let $\{m_1, \dots, m_t\}$ be a minimal set of generators of M. Then for $m_1 \in M = IM$ we can write $m_1 = \sum_{i=1}^{t} \lambda_i m_i$, where $\lambda_i \in I$. Hence, $m_1 - \lambda_1 m_1 = (1 - \lambda_1) m_1 =$

$\sum_{i=2}^{t} \lambda_i m_i$. Since $\lambda_1 \in I \subseteq \operatorname{rad} A$ we have by Proposition 1.2.2 that $1 - \lambda_1$ is left invertible. Let $u \in A$ be such that $u(1 - \lambda_1) = 1_A$. Then $m_1 = \sum_{i=2}^{t} (u\lambda_i) m_i$. If $t > 1$ this implies that $\{m_2, \ldots, m_t\}$ generates M, which is a contradiction. Hence $t = 1$ and $m_1 = 0$, which implies $M = (0)$. □

Note that any algebra is an artinian ring.

Lemma 1.2.6. *Let A be an algebra. Then $\underline{r} = \operatorname{rad} A$ is nilpotent.*

Proof. Consider the following descending chain of ideals in A:

$$A \supseteq \underline{r} \supseteq \underline{r}^2 \supseteq \cdots \supseteq \underline{r}^i \supseteq \cdots$$

Since A is artinian, there exists an $m \in \mathbb{N}$ such that $\underline{r}^m = \underline{r}^{m+1} = \underline{r} \cdot \underline{r}^m$. Since $\underline{r}$ is an A-module, and $\underline{r}^m \subseteq \underline{r}$ is an ideal in A Lemma 1.2.5 implies that $\underline{r}^m = (0)$, so $\underline{r}$ is nilpotent. □

Our next result is a well-known result called the *Wedderburn-Artin* theorem. We state it here without proof (cf. [5]).

Theorem 1.2.7 (Wedderburn-Artin Theorem)**.** *For any algebra A the following are equivalent:*

(i) *The right A-module A_A is semisimple.*

(ii) *Every right A-module is semisimple.*

(iii) *The left A-module ${}_AA$ is semisimple.*

(iv) *Every left A-module is semisimple.*

(v) $\operatorname{rad} A = 0$.

(vi) *The algebra A is isomorphic to a finite direct sum of matrix rings over k.*

An algebra A satisfying one of the equivalent statements of Theorem 1.2.7 is called a *semisimple* algebra.

Corollary 1.2.8. *Let* rad A *be the radical of an algebra* A.

(i) *If* I *is a two-sided nilpotent ideal in* A, *then* $I \subseteq \operatorname{rad} A$.

(ii) *If* A/I *is semisimple, then* $I = \operatorname{rad} A$.

Proof. (i): Let I be a two-sided nilpotent ideal in A, that is, $I^m = 0$ for some $m > 0$. Let $x \in I$ and $a \in A$. Then $ax \in I$, and $(ax)^r = 0$ for some $0 < r \leq m$. Hence, $(1 + ax + (ax)^2 + \cdots + (ax)^{r-1})(1 - ax) = 1$. Then by Proposition 1.2.2 we get $x \in \operatorname{rad} A$ since $I \subseteq A$. This implies $I \subseteq \operatorname{rad} A$, since x was some random element in I.

(ii): Suppose A/I is semisimple. Then $\operatorname{rad}(A/I) = (0)$ by Theorem 1.2.7. We know from (i) that $I \subseteq \operatorname{rad} A$, we are going to show that our assumption implies $\operatorname{rad} A \subseteq I$. Consider the canonical homomorphism $\phi : A \to A/I$. The homomorphism ϕ sends $\operatorname{rad} A$ to $\operatorname{rad}(A/I)$, which is zero. Let $a \in \operatorname{rad} A$. Then $\phi(a) = 0$, so $a + I = (0)$. Hence $a \in I$, so $\operatorname{rad} A \subseteq I$. □

Next we define the *radical of a module*.

Definition 1.2.9. Let A be an algebra, and let M be a left A-module. The *radical* of M is the intersection of all maximal submodules of M. We denote it by $\operatorname{rad} M$.

Our next result is a collection of basic properties of a radical.

Proposition 1.2.10. *Let* A *be an algebra. Suppose* L, M *and* N *are finite dimensional left* A*-modules.*

(i) *An element* $m \in M$ *belongs to* $\operatorname{rad} M$ *if and only if* $f(m) = 0$ *for every* $f \in \operatorname{Hom}_A(M, S)$, *where* S *is any simple left* A*-module.*

(ii) $\operatorname{rad}(M \bigoplus N) = \operatorname{rad} M \bigoplus \operatorname{rad} N$.

(*iii*) *If* $f \in \operatorname{Hom}_A(M, N)$ *we get* $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$.

Proof. (*i*): Let $f \in \operatorname{Hom}_A(M, S)$, where S is any simple left A-module. If $f = 0$ it is clear that $f(m) = 0$ for any $m \in M$, so suppose $f \neq 0$. Since $\operatorname{Im} f \neq (0)$ is a submodule of S we must have $\operatorname{Im} f = S$ since S is simple. Hence, f is an epimorphism. Let $K = \ker f$. Then $M/K \simeq S$ since f is an epimorphism. In particular, M/K is simple, so K is a maximal submodule of M.

Suppose $m \in \operatorname{rad} M$. Then we must have $m \in K$, and we get $f(m) = 0$. Conversely, suppose $m \in M$ such that $f(m) = 0$ for every $f \in \operatorname{Hom}_A(M, S)$. Then we have $m \in \bigcap_f \ker f$, where the intersection is taken over all $f \in \operatorname{Hom}_A(M, S)$. For a submodule L of M we have that L is a maximal submodule of M if and only if M/L is a simple module. So for a maximal submodule L of M we have $M/L \simeq S \simeq M/\ker f$ for some $f \in \operatorname{Hom}_A(M, S)$. Hence, $L = \ker f$ for some f, and $m \in \operatorname{rad} M$.

(*ii*): Follows from (*i*) since for an $f \in \operatorname{Hom}_A(M \bigoplus N, S)$ we have $f = (f_1, f_2)$, where $f_1 \in \operatorname{Hom}_A(M, S)$, and $f_2 \in \operatorname{Hom}_A(N, S)$.

(*iii*): Let $m \in \operatorname{rad} M$. Consider a map $g \in \operatorname{Hom}_A(N, S)$, where S is a simple left A-module. Then by (*i*) we have that $f(m) \in \operatorname{rad} N$ if and only if $gf(m) = 0$. Since $gf \in \operatorname{Hom}_A(M, S)$ we get by (*i*) that $gf(m) = 0$. Then $f(m) \in \operatorname{rad} N$, and hence $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$. □

Lemma 1.2.11. *Let A be an algebra, and* $\operatorname{rad} A = \underline{r}$. *Let M be a finitely generated left A-module. Then* $\operatorname{rad} M = \underline{r}M$.

Proof. Our approach here is to prove that both $\underline{r}M \subseteq \operatorname{rad} M$ and $\operatorname{rad} M \subseteq \underline{r}M$.

Let $m \in M$, $a \in A$ and consider the homomorphism $f_m : A \to M$ defined by $f_m(a) = am$. Suppose $a \in \operatorname{rad} A$. Then it follows from Proposition 1.2.10 (iii) that $f_m(a) = am \in f_m(\text{rad}\, A) \subseteq \text{rad}\, M$, and hence $\underline{r}M \subseteq \text{rad}\, M$.

Observe that $\underline{r}(M/\underline{r}M) = (0)$, and then one easily verifies that $M/\underline{r}M$ is a left module of $A/\underline{r}$. Consider the mapping from $(A/\underline{r}, M/\underline{r}M)$ into $M/\underline{r}M$ given by

$$(a + \underline{r})(m + \underline{r}M) = am + \underline{r}M,$$

for $a \in A$, $m \in M$. Since $A/\underline{r}$ is semisimple Theorem 1.2.7 implies that $M/\underline{r}M$ is semisimple. That is,

$$M/\underline{r}M \simeq S_1 \bigoplus \cdots \bigoplus S_n,$$

where S_i is a simple left A-module for $i \in \{1, \ldots, n\}$. The radical of any simple module is zero, and therefore Proposition 1.2.10 (ii) implies $\text{rad}(M/\underline{r}M) = (0)$. Consider the canonical homomorphism $\pi : M \to M/\underline{r}M$. By Proposition 1.2.10 we get $\pi(\text{rad}\, M) \subseteq \text{rad}(M/\underline{r}M) = (0)$. That is, $\text{rad}\, M \subseteq \ker \pi = \underline{r}M$. □

1.3 LOCAL ALGEBRAS

Definition 1.3.1. An algebra A is called *local* if the set of all non-invertible elements in A is a two-sided ideal.

Lemma 1.3.2. *Let A be an algebra and $\underline{r} = \text{rad}\, A$. Consider the algebra $B = A/\underline{r}$. Then for any idempotent $\eta = g + \underline{r}$ in B there exists an idempotent $e \in A$ such that $e + \underline{r} = g + \underline{r}$. We say that the idempotents of B are* lifted *modulo $\underline{r}$.*

Proof. Cf. [1] □

Proposition 1.3.3. *An algebra A is local if and only if 0 and 1 are the only idempotents of A.*

Proof. Suppose A is local. Let $e \in A$ be an idempotent. Then $e^2 = e$, and hence $e(e-1) = 0$. Now we get three possible situations. Either

(i) e is invertible, and hence $e = 1$,

(ii) $e - 1$ is invertible, and then $e = 0$, or

(iii) both e and $e - 1$ are non-invertible. Now, since A is local, this implies that $e - (e-1) = 1$ is non-invertible, which is a contradiction.

Hence, 0 and 1 are the only idempotents of A.

Conversely, suppose 0 and 1 are the only idempotents of A. Consider the algebra $A/\underline{r}$, which is semisimple. Then by Theorem 1.2.7 there exist $n_1, \dots, n_t \in \mathbb{N}$ such that $A/\underline{r} = \bigoplus_{i=1}^{t} M_{n_i}(k)$, where $M_{n_i}(k)$ is the matrix ring of $n_i \times n_i$-matrices over k. Let I_{n_i} denote the identity element in $M_{n_i}(k)$, and consider the element $e = (I_{n_1}, 0, \dots, 0) \in A/\underline{r}$ which is clearly idempotent. Then by Lemma 1.3.2 we get that $e = a + \underline{r}$ for some idempotent $a \in A$. By our assumption we get possibilities: either $e = 0 + \underline{r}$ or $e = 1 + \underline{r}$. That is, e is either the zero element or the identity element of $A/\underline{r}$. But if $t \geq 2$ the element e is neither the zero element nor the identity element. Hence we must have $t = 1$. Then set $n_1 = n$. This implies $A/\underline{r} = M_n(k)$. Suppose $n \geq 2$. Then consider the element

$$e' = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

in $A/\underline{r}$. The element e' is an idempotent in $A/\underline{r}$. Then again, by Lemma 1.3.2 we must have that either $e' = 0 + \underline{r}$ or $e' = 1 + \underline{r}$, that is either the zero element or the identity element of $A/\underline{r}$. But e' is neither the zero element or the identity element of $A/\underline{r}$, and hence we must have $n = 1$. This implies $A/\underline{r} \simeq k$. Then, as a left A-module $A/\underline{r}$ is simple, because $\dim_k A/\underline{r} = 1$. Hence, $\underline{r}$ is a maximal left ideal in A. Similarly, $\underline{r}$ is a maximal right ideal in A. Then $\underline{r}$ is the only maximal left ideal and the only maximal right ideal of A by the definition of the radical of an algebra.

Let $a \in A$ be a non-invertible element. That is, there exists no two-sided inverse of a. However, suppose there exist $b_1, b_2 \in A$ such that $b_1 a = 1$ and $ab_2 = 1$. Then $b_2 = 1 \cdot b_2 = (b_1 a) b_2 = b_1(ab_2) = b_1 \cdot 1 = b_1$. Hence a is invertible, which is a contradiction. That is, either there exists no $b \in A$ such that $ba = 1$ or there exists no $b \in A$ such that $ab = 1$. Suppose there exists no $b \in A$ such that $ba = 1$. Then consider the left ideal $I = (a) = \{a'a \mid a' \in A\}$. Observe that $I = A$ would imply $1 \in I$, which is a contradiction. Hence $I \subset A$. Thus $I \subseteq \underline{r}$, which implies $a \in \underline{r}$. Similarly, one can show that if there exists no $b \in A$ such that $ab = 1$, then $a \in \underline{r}$. Hence, every non-invertible element in A is in $\underline{r}$. Now, what remains is to show that $\underline{r}$ is contained in the set of all non-invertible elements in A. Let $c \in A$ be an invertible element. Suppose $c \in \underline{r}$. Now, this causes $1 \in \underline{r}$, which is a contradiction. Hence the set of all non-invertible elements in A form a two-sided ideal, and A is local. □

Lemma 1.3.4. *Let A be an algebra, and let $e \in A$ be a nonzero idempotent. Then*

(i) for a left A-module M we have $\mathrm{Hom}_A(Ae, M) \simeq eM$ *as left eAe-modules.*

(ii) $\mathrm{End}_A(Ae) \simeq eAe$ *as algebras.*

Proof. (i): Let $f \in \mathrm{Hom}_A(Ae, M)$. Consider the k-linear map

$$\phi : \mathrm{Hom}_A(Ae, M) \to eM$$

defined by $f \mapsto f(e) = f(e^2) = ef(e)$. It is easily verified that ϕ is a homomorphism of left eAe-modules. Now consider the k-linear map $\phi' : eM \to \mathrm{Hom}_A(Ae, M)$ defined by $(\phi'(em))(ae) = aem$ for $a \in A$ and $m \in M$. This map can easily be shown to be a well-defined homomorphism of eAe-modules. Observe that

$$\phi(\phi'(em)) = (\phi'(em))(e) = em,$$

so ϕ' is an inverse of ϕ.

(ii): Follows directly from (i) by setting $M = Ae$. □

Lemma 1.3.5. *Let A be an algebra and M an A-module. Then M is indecomposable if and only if its endomorphism ring* $\mathrm{End}_A(M)$ *is a local ring.*

Proof. Suppose $M \simeq N \bigoplus K$, where $N, K \neq (0)$ are A-modules. Then $\mathrm{End}_A(M)$ contains the projection of $N \bigoplus K$ onto the first direct summand. This projection is an idempotent, it is nonzero, as $N \neq (0)$ by assumption, and it is not 1 since $K \neq (0)$. Hence, $\mathrm{End}_A(M)$ is not local by Proposition 1.3.3.

Conversely, suppose $\mathrm{End}_A(M)$ is not local. Then by Proposition 1.3.3 it contains a non-trivial idempotent $f : M \to M$. We then claim that $M = \ker f \bigoplus \mathrm{Im}\, f$. Let $m \in M$. Observe that $f(m - f(m)) = f(m) - f^2(m) = f(m) - f(m) = 0$, so that $m - f(m) \in \ker f$. Then we have that $m = (m - f(m)) + f(m)$, so $M = \ker f + \mathrm{Im}\, f$. Now we need to show that $\ker f \cap \mathrm{Im}\, f = (0)$. Let $m \in \ker f \cap \mathrm{Im}\, f$. This implies $f(m) = 0$ and that there exists an $m' \in M$ such that $f(m') = m$. Then $m = f(m') = f^2(m') = f(m) = 0$. Hence $\ker f \cap \mathrm{Im}\, f = (0)$, and $M = \ker f \bigoplus \mathrm{Im}\, f$. □

Lemma 1.3.6. *Let A be an algebra. An idempotent $e \in A$ is a primitive idempotent if and only if eAe is a local algebra.*

Proof. Let e be a primitive idempotent in A. It is clear that e is primitive if and only if the module Ae is indecomposable. Then by Lemma 1.3.5 we have that $\mathrm{End}_A(Ae)$ is local. Hence, by Lemma 1.3.4 we get that eAe is local.

Suppose the idempotent e is not primitive in A. Then we want to show that eAe is not local. Since e is not primitive $e = e_1 + e_2$ for some nonzero, orthogonal idempotents $e_1, e_2 \in A$. It is clear that $ee_1e \in eAe$. Observe that

$$(ee_1e)^2 = (ee_1e)(ee_1e) = ee_1(e_1 + e_2)e_1e = ee_1^3e = ee_1e,$$

so ee_1e is an idempotent in eAe. Then we need to check if ee_1e equals either 0 or e. Observe that

$$ee_1e = (e_1 + e_2)e_1(e_1 + e_2) = e_1^3 = e_1 \neq 0,$$

and $e_1 \neq e$ since $e_2 \neq 0$. So eAe is not local by Proposition 1.3.3. □

The next result classifies all the indecomposable, projective A-modules of an algebra A.

Lemma 1.3.7. *Let A be an algebra, $\{e_1, \dots, e_n\}$ be a complete set of primitive, orthogonal idempotents in A, and let P be an A-module. Then P is an indecomposable, projective A-module if and only if $P \simeq Ae_i$ for some $i \in \{1, \dots, n\}$.*

Proof. Suppose $P \simeq Ae_i$ for some $i \in \{1, \dots, n\}$. Consider the decomposition ${}_AA = \bigoplus_{i=1}^{n} Ae_i$ of A as a left A-module. The module ${}_AA$ is clearly free, and hence by Lemma 1.1.8 the module Ae_i is a projective A-module for every i. By Lemma 1.3.4 we have that $\mathrm{End}_A(Ae_i) \simeq e_iAe_i$. Since e_i is a primitive idempotent Lemma

1.3.6 implies that $\mathrm{End}_A(Ae_i)$ is local. Then, by Lemma 1.3.5 the module Ae_i is an indecomposable A-module. So, $P \simeq Ae_i$ is an indecomposable, projective A-module.

Now, let P be an indecomposable projective A-module. Then by Lemma 1.1.9 we have $P = P_1 \bigoplus \cdots \bigoplus P_t$, where $P_j \simeq Ae_s$ for some s for every $j \in \{1, \ldots, t\}$. Since P is indecomposable $t = 1$, and $P \simeq Ae_s$ for some s. $\square$

2

REPRESENTATION THEORY

2.1 QUIVERS AND PATH ALGEBRAS

In this section we will introduce some geometrical elements called *quivers*, and based on these quivers we will construct some special algebras called *path algebras*. As we will see in Chapter 3 quivers and path algebras provide a convenient way to visualize more general algebras.

Definition 2.1.1. A *quiver* $Q = (Q_0, Q_1)$ is an oriented graph where Q_0 denotes the set of vertices and Q_1 denotes the set of arrows. We always assume both Q_0 and Q_1 finite sets. That is, we are only considering *finite* quivers.

We often denote a quiver $Q = (Q_0, Q_1)$ simply by Q. To each arrow α of Q_1 we associate a pair of numbers (s, t), where $s(\alpha)$ denotes the source of α, which is the vertex where α starts, while $t(\alpha)$ denotes the target of α, which is the vertex where α ends. A *subquiver* Q' of Q is a quiver having $Q'_0 \subseteq Q_0$ and $Q'_1 \subseteq Q_1$, and for any $\alpha : i \to j \in Q_1$ such that $\alpha \in Q'_1$ we have that $s'(\alpha) = i$ and $t'(\alpha) = j$.

Let i be a vertex in Q_0. We say that i is a *sink* in Q if every arrow α directly connected to i has $t(\alpha) = i$. Similarly, i is called a *source* in Q if $s(\alpha) = i$ for every arrow α directly connected to i.

Definition 2.1.2. A *path* in $Q = (Q_0, Q_1)$ is either

(i) an oriented sequence of arrows $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$, where $t(\alpha_m) = s(\alpha_{m+1})$, $m = 1, \ldots, n-1$. These paths are called the *non-trivial paths*.

(ii) e_i for each $i \in Q_0$. These are called the *trivial paths*. We define $s(e_i) = i = t(e_i)$.

A path p is called a *cycle* if $s(p) = t(p)$. A quiver with cycles is called *cyclic*, while a quiver which contains no cycles is called *acyclic*. The underlying *graph* $\overline{Q}$ of a quiver is obtained from the quiver by forgetting about the direction of the arrows. A quiver Q is said to be *connected* if $\overline{Q}$ is connected, that is, if there is a path from any point to any other point of the graph.

Definition 2.1.3. Let Q be a quiver. The *path algebra* kQ is the algebra having as its underlying vector space the vector space with basis all the paths of Q. The elements of kQ are finite sums of the form $\sum\limits_i a_i p_i$, where $a_i \in k$ and p_i is a path in Q.

In order to define the product of two basis elements of the path algebra kQ, we first need to define the function *Kronecker delta*.

Definition 2.1.4. The *Kronecker delta* is a function of two variables, defined as follows:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Now we are ready to define the product of two basis elements of a path algebra kQ. Given two paths $p_i = \alpha_n \alpha_{n-1} \ldots \alpha_1, p_j = \beta_m \beta_{m-1} \ldots \beta_1$ of kQ. Then the product is

$$p_i p_j = \delta_{t(p_j)s(p_i)} \alpha_n \alpha_{n-1} \ldots \alpha_1 \beta_m \beta_{m-1} \ldots \beta_1.$$

That is, the product of p_i and p_j is the concatenation of the two paths if $t(\beta_m) = s(\alpha_1)$ and zero otherwise. Hence, the trivial paths e_i of kQ are orthogonal idempotents of kQ for every $i \in Q_0$. We will now see that the set of all trivial paths of a path algebra is in fact a complete set of primitive orthogonal idempotents.

Theorem 2.1.5. *Let Q be a finite quiver having n vertices. Then the set $\{e_i \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents of kQ.*

Proof. We have already seen that the e_i's are orthogonal idempotents. We need to show that e_i is also a primitive idempotent. Let e be an idempotent of e_ikQe_i. Now, study the form e must take. We know that e must be a linear combination of trivial and non-trivial paths starting and ending at i. That is, $e = \omega + be_i$, where ω is some linear combination of cycles of length ≥ 1 going through i, $b \in k$. Since e is an idempotent we get

$$0 = e^2 - e = \omega^2 + (2b-1)\omega + (b^2 - b)e_i.$$

For this to be true we must have $\omega = 0$ and $b^2 = b$. That is, $b = 0$ or $b = 1$. In the case of $b = 0$ we get $e = 0$, and in the case of $b = 1$ we get $e = e_i$. Hence, 0 and e_i are the only idempotents of e_ikQe_i. Then by Proposition 1.3.3 the algebra e_ikQe_i is local, which by Lemma 1.3.6 implies that e_i is a primitive idempotent.

Let p be a path in Q. Let $s(p) = i$, $t(p) = j$, where $i, j \in Q_0$. We must show that $(e_1 + \cdots + e_n)p = p = p(e_1 + \cdots + e_n)$:

$$(e_1 + \cdots + e_n)p = e_j \cdot p = p,$$

$$p(e_1 + \cdots + e_n) = p \cdot e_i = p.$$

Hence, $\sum_{i=1}^{n} e_i = 1_{kQ}$, so $\{e_1, \ldots, e_n\}$ is a complete set of primitive, orthogonal idempotents of kQ. $\square$

In general, we say that an algebra A is *indecomposable* if A can not be written as a direct sum of two non-zero algebras. We will now see that the decomposition of an algebra is closely related to its idempotents. An idempotent e satisfying $ae = ea$ for every $a \in A$ is called *central*.

Lemma 2.1.6. *An algebra A is indecomposable if and only if A does not contain any non-trivial central idempotents.*

Proof. If there exists such a non-trivial central idempotent e in A, then the A-modules Ae and $A(1-e)$ can be shown to be algebras, and $A \simeq Ae \bigoplus A(1-e)$ is a non-trivial decomposition as algebras.

Suppose $A = A_1 \bigoplus A_2$, where A_1, A_2 are non-zero algebras. Consider the elements $e_1 = (1,0)$ and $e_2 = (0,1)$ in A. Then $e_1 + e_2 = 1_A$, so e_1, e_2 are non-trivial orthogonal idempotents in A. Moreover, for any $a = (a_1, a_2) \in A$ we have $ae_1 = (a_1, 0) = e_1 a$ and $ae_2 = (0, a_2) = e_2 a$, so e_1, e_2 are non-trivial central idempotents of A. $\square$

Lemma 2.1.7. *Let A be an algebra, and let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents. Then A is indecomposable if and only if $\{1, \dots, n\} \neq I \cup J$ for some non-empty, disjoint sets I, J such that for $i \in I$, $j \in J$ we have $e_i A e_j = (0) = e_j A e_i$.*

Proof. Suppose two such sets I, J exist. Let $c = \sum\limits_{j \in J} e_j$. Since both I, J are non-empty we have that $c \neq 0, 1$. It is clear that c is an idempotent in A, since the e_j's are orthogonal idempotents. Also, observe that $ce_i = 0 = e_ic$ for every $i \in I$ and $ce_j = e_j = e_jc$ for every $j \in J$. Now, let $a \in A$. Then,

$$
\begin{aligned}
ca &= \left(\sum_{j\in J} e_j\right) a = \left(\sum_{j\in J} e_j a\right) \cdot 1 = \left(\sum_{j\in J} e_j a\right) \left(\sum_{j\in J} e_j + \sum_{i\in I} e_i\right) \\
&= \sum_{j,k\in J} e_j a e_k = \left(\sum_{j\in J} e_j\right) \left(\sum_{k\in J} a e_k\right) \\
&= \left(\sum_{i\in I} e_i + \sum_{j\in J} e_j\right) \left(\sum_{k\in J} a e_k\right) = a \left(\sum_{k\in J} e_k\right) = ac,
\end{aligned}
$$

using our assumption. So, c is a non-trivial central idempotent in A, and A is decomposable by Lemma 2.1.6.

Suppose A is decomposable. Then there exists a central, non-trivial idempotent $c \in A$ by Lemma 2.1.6. Observe that

$$
c = 1 \cdot c \cdot 1 = \left(\sum_{i=1}^{n} e_i\right) c \left(\sum_{j=1}^{n} e_j\right) = \sum_{i,j=1}^{n} e_i c e_j = \sum_{i=1}^{n} e_i c e_i.
$$

Let $c_i = e_i c e_i$. The element c_i is an idempotent in e_iAe_i. Since e_i is a primitive idempotent we have that e_iAe_i is local by Lemma 1.3.6. Then by Proposition 1.3.3 the elements 0 and e_i are the only idempotents of e_iAe_i. Hence, $c_i = e_i$ or $c_i = 0$. Now, let $I = \{i \mid c_i = 0\}$ and $J = \{j \mid c_j = e_j\}$. Since $c \neq 0, 1$ we have that both I, J are non-empty sets, and the sets are clearly disjoint. Observe that for $i \in I$, $j \in J$ we have $e_ic = 0 = ce_i$ and $e_jc = e_j = ce_j$. Then, since c is central we have $e_iAe_j = e_iAce_j = e_icAe_j = (0)$ and $e_jAe_i = e_jcAe_i = e_jAce_i = (0)$. □

We will now see what requirements that need to be fulfilled in order for a path algebra to be indecomposable.

Theorem 2.1.8. *Let Q be a finite quiver. The path algebra kQ is indecomposable if and only if Q is a connected quiver.*

Proof. Suppose Q is not connected. Then there exists some connected subquiver Q' of Q. Let Q'' denote the subquiver of Q having $Q''_0 = Q_0 \setminus Q'_0$. Then neither Q' nor Q'' is empty. Let $p \in kQ$. Then either $p \in kQ'$ or $p \in kQ''$. Let $i \in Q'_0$ and $j \in Q''_0$. Suppose $p \in kQ'$, then $e_j p = 0$, and clearly $e_j p e_i = 0$. That is, there are no paths from i to j in kQ'. Now, suppose $p \in kQ''$. Then $pe_i = 0$, and $e_j p e_i = 0$. Hence, there are no paths from i to j in kQ''. This implies $e_j kQ e_i = (0)$. Similarly, one can show that $e_i kQ e_j = (0)$. By Lemma 2.1.7 the path algebra kQ is decomposable.

Now, let Q be a connected quiver. Suppose by contradiction that kQ is decomposable. Then by Lemma 2.1.7 the set of vertices of Q splits into two non-empty, disjoint sets Q'_0, Q''_0 such that $Q_0 = Q'_0 \cup Q''_0$. Also, for $i \in Q'_0$, $j \in Q''_0$ we have $e_i kQ e_j = (0) = e_j kQ e_i$. Since Q is connected and Q'_0, Q''_0 are non-empty we can find i, j such that there exists an arrow $\alpha : i \to j$ (or $\alpha : j \to i$). Then $\alpha = e_j \alpha e_i$ is a non-zero element in $e_j kQ e_i$, which is a contradiction. Hence, kQ is indecomposable. □

Theorem 2.1.9. *Let Q be a finite quiver and A be an algebra. Let $\phi_0 : Q_0 \to A$ and $\phi_1 : Q_1 \to A$ be two maps satisfying the following conditions:*

(*i*) $1_A = \sum_{i \in Q_0} \phi_0(i)$, $\phi_0^2(i) = \phi_0(i)$ *and* $\phi_0(i)\phi_0(j) = 0$ *for all* $i \neq j, i, j \in Q_0$,

(*ii*) *for* $\alpha : i \to j$, $\alpha \in Q_1$, $i, j \in Q_0$ *we have* $\phi_1(\alpha) = \phi_0(j)\phi_1(\alpha)\phi_0(i)$.

Let $\{e_i \mid i \in Q_0\}$ be the set of trivial paths of kQ. Then there exists a unique algebra homomorphism $\phi : kQ \to A$ such that $\phi(e_i) = \phi_0(i)$ for any $i \in Q_0$ and $\phi(\alpha) = \phi_1(\alpha)$ for any $\alpha \in Q_1$.

Proof. Let ϕ_0, ϕ_1 be two maps satisfying (i) and (ii), and let $|Q_0| = n$. Since $\{e_1, \ldots, e_n\} \cup Q_1$ generates kQ, the maps ϕ_0 and

ϕ_1 induce a map $\phi : kQ \to A$. We need to show that ϕ is in fact a unique algebra homomorphism. We then need to check that ϕ preserves the identity of kQ and that it preserves the products in kQ, and we need to show that ϕ is actually unique. Let $\alpha_n \ldots \alpha_2\alpha_1$ be a path in kQ. Since ϕ is respecting ϕ_1 we get that

$$\begin{aligned} \phi(\alpha_n \ldots \alpha_2\alpha_1) &= \phi(\alpha_n)\cdots\phi(\alpha_2)\phi(\alpha_1) \\ &= \phi_1(\alpha_n)\cdots\phi_1(\alpha_2)\phi_1(\alpha_1). \end{aligned} \tag{2}$$

Hence, ϕ preserves the products of kQ. Equation (2) also shows uniqueness of ϕ, since for any homomorphism ψ and any path $\alpha_n \ldots \alpha_2\alpha_1 \in kQ$ we would have $\psi(\alpha_n \ldots \alpha_2\alpha_1) = \phi_1(\alpha_n)\cdots\phi_1(\alpha_2)\phi_1(\alpha_1) = \phi(\alpha_n \ldots \alpha_2\alpha_1)$. Now we need to show that ϕ preserves the identity.

$$\phi(1_{kQ}) = \phi(\sum_{a\in Q_0} e_a) = \sum_{a\in Q_0} \phi(e_a) = \sum_{a\in Q_0} \phi_0(a) = 1_A.$$

So, ϕ preserves the identity, and hence ϕ is a unique algebra homomorphism. $\square$

We will now define an important ideal in the path algebra kQ.

Definition 2.1.10. Let Q be a finite quiver. Let $\mathscr{J}$ = {all linear combinations of non-trivial paths}.

Lemma 2.1.11. *Let Q be a finite and connected quiver, and $|Q_0| = n$. The set $\mathscr{J}$ is an ideal in kQ, and $kQ/\mathscr{J} \simeq k^n$.*

Proof. First we need to prove that $\mathscr{J}$ is an ideal in kQ. Let $a' = a'_1\alpha_1 + \cdots + a'_m\alpha_m \in \mathscr{J}$ for some m, $b' = b'_1e_1 + \cdots + b'_ne_n+$ lin.comb. of non-trivial paths $\in kQ$. Recall that the concatenation of two non-trivial paths is either zero or a non-trivial path. Hence,

$$\begin{aligned} a'b' = (a'_1b'_1 + \cdots + a'_1b'_n)\alpha_1 + \cdots + (a'_mb'_1 + \cdots + a'_mb'_n)\alpha_m \\ + \text{lin.comb. of non-trivial paths} \in \mathscr{J} \end{aligned}$$

Hence, $\mathscr{J}$ is a right ideal of kQ. Proving that $\mathscr{J}$ is also a left ideal is done similarly.

Consider the map $\phi : kQ \to k^n$ such that

$$\phi(a_1e_1 + a_2e_2 + \cdots + a_ne_n + \text{ lin. comb. of non-trivial paths}) \\ = (a_1, a_2, \ldots, a_n),$$

where $a_i \in k$ for $i = 1, \ldots, n$. We need to show that ϕ is a ring homomorphism, that ϕ is an epimorphism and that $\ker \phi = \mathscr{J}$.

Consider $a, b \in kQ$, where $a = a_1e_1 + \cdots + a_ne_n$+linear combination of non-trivial paths, $b = b_1e_1 + \cdots + b_ne_n$+linear combination of non-trivial paths.

$$\begin{aligned} \phi(a+b) &= \phi((a_1+b_1)e_1 + \cdots + (a_n+b_n)e_n \\ &\qquad + \text{lin.comb. of non-trivial paths}) \\ &= (a_1+b_1, \ldots, a_n+b_n) \\ &= (a_1, \ldots, a_n) + (b_1, \ldots, b_n) \\ &= \phi(a) + \phi(b) \end{aligned}$$

$$\begin{aligned} \phi(ab) &= \phi(a_1b_1e_1 + \cdots + a_nb_ne_n \\ &\qquad + \text{lin.comb. of non-trivial paths}) \\ &= (a_1b_1, \ldots, a_nb_n) \\ &= (a_1, \ldots, a_n)(b_1, \ldots, b_n) \\ &= \phi(a)\phi(b) \end{aligned}$$

So, ϕ is a ring homomorphism. Now we need to check that ϕ is actually an epimorphism.

Consider an element $(x_1, \ldots, x_n) \in k^n$. Now we need to look for an element x in kQ such that $\phi(x) = (x_1, \ldots, x_n)$. Consider the element $x = x_1e_1 + \cdots + x_ne_n$ in kQ. Observe that $\phi(x) = (x_1, \ldots, x_n)$, so ϕ is an epimorphism.

The last thing we need to do is to show that $\ker \phi = \mathscr{J}$. Let $a = a_1e_1 + \cdots + a_ne_n$+linear combination of non-trivial paths be an element in kQ. Suppose $\phi(a) = (0, \ldots, 0)$. This would imply $a_1 = \cdots = a_n = 0$, which implies $a \in \mathscr{J}$. Hence, $\ker \phi = \mathscr{J}$, and $kQ/\mathscr{J} \simeq k^n$. $\square$

The ideal $\mathscr{J}$ is called the *arrow ideal* of kQ.

2.2 ADMISSIBLE IDEALS AND BOUND QUIVER ALGEBRAS

In this chapter we are going to study *bound quiver algebras*, which are path algebras modulo some ideal. In general, we do not require for the path algebra to be finite dimensional when studying these types of algebras, but in order for the bound quiver algebra to be finite dimensional we need the quotient to satisfy some requirements. In particular, the quotient needs to be an *admissible* ideal.

Definition 2.2.1. Let Q be a finite quiver and $\mathscr{J}$ be the arrow ideal of the path algebra kQ. A two-sided ideal I in kQ is called *admissible* if

$$\mathscr{J}^m \subseteq I \subseteq \mathscr{J}^2$$

for some $m \geq 2$.

If I is an admissible ideal of kQ then (Q, I) is said to be a *bound quiver* and the quotient algebra kQ/I is said to be a *bound quiver algebra*.

Theorem 2.2.2. *Let Q be a finite quiver, and let I be an admissible ideal of kQ. The set $\{\overline{e_i} = e_i + I \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents of the bound quiver algebra kQ/I.*

Proof. Consider the canonical homomorphism $\phi : kQ \to kQ/I$. Since $\phi(e_i) = \overline{e_i}$ we know by Theorem 2.1.5 that $\{\overline{e_i} = e_i + I \mid i \in$

$Q_0\}$ is a complete set of orthogonal idempotents. What remains is to show that $\overline{e_i}$ is a primitive idempotent for every $i \in Q_0$. By Lemma 1.3.6 we need to show that 0 and $\overline{e_i}$ are the only idempotents of the algebra $\overline{e_i}(kQ/I)\overline{e_i}$. Let e be an idempotent in $\overline{e_i}(kQ/I)\overline{e_i}$. We know that e must take the form $e = b\overline{e_i} + \omega + I$, where $b \in k$ and ω is some linear combination of cycles of length ≥ 1 through i. Since, by assumption, e is an idempotent we get

$$e^2 - e = \omega^2 + (2b-1)\omega + (b^2 - b)\overline{e_i} \in I. \tag{3}$$

Since I is an admissible ideal we know by definition that $I \subseteq \mathscr{J}^2$, where $\mathscr{J}$ is the arrow ideal of kQ. Hence, we must have $b^2 - b = 0$ in (3). This implies either $b = 0$ or $b = 1$.

Suppose $b = 0$. Then $e = \omega + I$, and hence ω is an idempotent in kQ/I. We also know that $\mathscr{J}^m \subseteq I$ for some $m \geq 2$, since I is an admissible ideal. This implies $\omega^m \in I$, that is, ω is nilpotent in kQ/I. Since ω is both an idempotent and nilpotent we must have that $\omega \in I$, and hence $e = 0$ in kQ/I.

Suppose $b = 1$. Then $e = \overline{e_i} + \omega + I$, or $\overline{e_i} - e = -\omega + I$. Now, both $\overline{e_i}$ and e are idempotents in $\overline{e_i}(kQ/I)\overline{e_i}$, and since $\overline{e_i}$ is the identity of $\overline{e_i}(kQ/I)\overline{e_i}$ we get that $\overline{e_i} - e$ is an idempotent in $\overline{e_i}(kQ/I)\overline{e_i}$. Hence, ω is an idempotent in kQ/I. By the same arguing as in the previous case, ω is also nilpotent in kQ/I. Hence, $\omega \in I$, and consequently $e = \overline{e_i}$. □

Theorem 2.2.3. *Let Q be a finite quiver, and let I be an admissible ideal in kQ. Then the bound quiver algebra kQ/I is indecomposable if and only if Q is a connected quiver.*

Proof. If Q is not connected, then the path algebra kQ is decomposable by Theorem 2.1.8. Then we have a non-trivial central idempotent $c \in kQ$ by Lemma 2.1.6, and by the proof of Lemma 2.1.7 the idempotent c is a sum of trivial paths in Q. Let

$\gamma = c + I \in kQ/I$. Then γ is a central idempotent in kQ/I, and we need to check if it is trivial. Since $I \subseteq \mathscr{J}^2$ we must have $c \notin I$, because otherwise I would contain a path of length zero. Hence, γ is not the zero element in kQ/I. Suppose $\gamma = 1 + I$. Then $1 - \gamma \in I$. But this again implies that I contains a path of length zero, which is contradicts $I \subseteq \mathscr{J}^2$. Hence, γ is a non-trivial central idempotent in kQ/I, and kQ/I is decomposable by Lemma 2.1.6.

Let Q be connected, and suppose by contradiction that kQ/I is decomposable. Then the proof is similar to the proof of Theorem 2.1.8. □

Next we will see how the radical of a bound quiver algebra is connected to the arrow ideal.

Lemma 2.2.4. *Let Q be a finite quiver, let $\mathscr{J}$ be the arrow ideal of kQ and I an admissible ideal of kQ. Then $\operatorname{rad}(kQ/I) = \mathscr{J}/I$.*

Proof. By the definition of an admissible ideal we have $\mathscr{J}^m \subseteq I$. Hence, $(\mathscr{J}/I)^m = (0)$, so $\mathscr{J}/I$ is a nilpotent ideal in kQ/I. Then by Corollary 1.2.8 $\mathscr{J}/I \subseteq \operatorname{rad}(kQ/I)$. By Lemma 2.1.11 we have that $(kQ/I)/(\mathscr{J}/I) \simeq kQ/\mathscr{J} \simeq k \bigoplus \cdots \bigoplus k$. Then, again by Corollary 1.2.8 we get $\mathscr{J}/I = \operatorname{rad}(kQ/I)$. □

Corollary 2.2.5. *For each $l \geq 1$, we have $\operatorname{rad}^l(kQ/I) = (\mathscr{J}/I)^l$.*

Corollary 2.2.6. *The k-vector space $\operatorname{rad}(kQ/I)/\operatorname{rad}^2(kQ/I) = (\mathscr{J}/I)/(\mathscr{J}/I)^2 \simeq \mathscr{J}/\mathscr{J}^2$.*

2.3 REPRESENTATIONS OF QUIVERS

Definition 2.3.1. A *representation* (V, f) of a quiver $Q = (Q_0, Q_1)$ over a field k is a collection of k-vector spaces $\{V_i\}_{i \in Q_0}$ and k-linear maps $f_\alpha : V_i \to V_j$ for each arrow $\alpha : i \to j$. We always assume that $\dim_k V_i < \infty$ for all $i \in Q_0$. That is, we are only considering *finite dimensional* representations.

Definition 2.3.2. Let Q be a finite quiver and $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be a representation of Q. Let $p = \alpha_t \ldots \alpha_1$ be a non-trivial path from i to j in kQ. Then we have a k-linear map from V_i to V_j defined as follows:

$$f_p = f_{\alpha_t} \cdots f_{\alpha_1}.$$

Let Q be a quiver, and $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ denote its representation. We will now see what the corresponding representation of the bound quiver (Q, I) looks like, where I is an admissible ideal in the path algebra kQ. Let $W = (W_i, g_\alpha)_{i \in Q_0, \alpha \in Q_1}$ denote the representation of (Q, I). Then $W_i = V_i$ for all $i \in Q_0$, while the linear maps are *bound by I*. That is, if $\rho = \alpha_t \ldots \alpha_1 \in I$ we have that

$$g_\rho = g_{\alpha_t} \cdots g_{\alpha_1} = 0.$$

Definition 2.3.3. Let $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $V' = (V'_i, f'_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be two representations of a quiver Q. A *homomorphism* $h : V \to V'$ is a collection of linear maps $h_i : V_i \to V'_i$ for every $i \in Q_0$, such that for all $\alpha : i \to j \in Q_1$ the following diagram commutes:

$$\begin{tikzcd} V_i \arrow[r, "h_i"'] \arrow[d, "f_\alpha"'] & V'_i \arrow[d, "f'_\alpha"'] \\ V_j \arrow[r, "h_j"'] & V'_j \end{tikzcd}$$

That is, $h_j \circ f_\alpha = f'_\alpha \circ h_i$.

Definition 2.3.4. Let V be a representation of some quiver Q. Then the representation V is called an *indecomposable representation of* Q if $V = V' \bigoplus V''$ implies $V' = (0)$ or $V'' = (0)$ for any representations V', V'' of Q.

If Q is a finite and connected quiver, there exists a connection between the isomorphism classes of representations of a bound quiver (Q, I) and the isomorphism classes of finite dimensional left kQ/I-modules. We will describe the connection here, however we will get a deeper understanding of it in section 2.4.

Lemma 2.3.5. *Let Q be a finite and connected quiver. Then there exists a one-to-one correspondence between the isomorphism classes of representations of a bound quiver (Q, I) and the isomorphism classes of finite dimensional left kQ/I-modules.*

Proof. Let $A = kQ/I$, $n = |Q_0|$ and let $\{\overline{e_1}, \ldots, \overline{e_n}\}$ be a complete set of primitive orthogonal idempotents in A. For $\alpha \in Q_1$, let $\overline{\alpha} = \alpha + I$ be the corresponding element in A.

First, we will see that every representation corresponds to a unique finite dimensional A-module. Given a representation of (Q, I), say $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$, the corresponding A-module is $M = \bigoplus\limits_{i \in Q_0} V_i$. Now we need to check that M actually has an A-module structure, and we need to show that M is annihilated by I. Let $m = (v_1, \ldots, v_n)$ be an element of M. The action of the basis elements $\overline{e_i}$ and $\overline{\alpha}$ of A on m is defined as follows:

$$\overline{e_i}m = (0, \ldots, v_i, \ldots, 0) \text{ for all } i \in Q_0$$
$$\overline{\alpha}m = (0, \ldots, f_\alpha(v_i), \ldots, 0),$$

where the nonzero element in $\overline{\alpha}m$ is placed in the j-th coordinate ($\alpha : i \to j$), and f_α is the linear map from V_i to V_j in the representation V. Hence it is easy to see that M has an A-module structure. Let $\rho \in I$. It is clear that $\rho m = (0)$ by the way the basis elements of A act on m.

Conversely, let M be a finite dimensional left A-module. Then the corresponding representation $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ has $V_i = \overline{e_i}M$ as its vector space at vertex i. Consider $\alpha : i \to j \in Q_1$. Then

$f_\alpha : V_i \to V_j$ is given by left multiplication with $\overline{\alpha} = \alpha + I$. That is, $f_\alpha(e_i m) = \overline{\alpha} e_i m$ for any element $m \in M$. Since M is an A-module, f_α is a k-linear map. Let $\rho = \sum_{x=1}^{n} b_x \omega_x \in I$, where $b_x \in k$ and $\omega_x = \alpha_{x,s} \cdots \alpha_{x,2}\alpha_{x,1}$ is a path from a to b in Q. Then

$$\begin{aligned}
f_\rho(e_a m) &= \sum_{x=1}^{n} b_x f_{\omega_x}(e_a m) \\
&= \sum_{x=1}^{n} b_x f_{\alpha_{x,s}} \cdots f_{\alpha_{x,1}}(e_a m) \\
&= \sum_{x=1}^{n} b_x \overline{\alpha_{x,s}} \cdots \overline{\alpha_{x,1}} e_a m \\
&= \left(\sum_{x=1}^{n} b_x \overline{\alpha_{x,s}} \cdots \overline{\alpha_{x,1}} \right) e_a m \\
&= \overline{\rho} e_a m \\
&= 0
\end{aligned}$$

It can easily be shown that this correspondence is one-to-one.

□

We will see in Chapter 3 that every basic and indecomposable algebra can be represented as a bound quiver algebra. Therefore this connection makes representations of quivers an important tool in studying modules of algebras.

By Lemma 2.3.5 it is clear that the simple kQ/I-modules must correspond uniquely to some representation of (Q, I). It can be shown that the representations corresponding to the simple kQ/I-modules are the following. For each $i, j \in Q_0$ let $S(i)$ denote the representation $(S(i)_j, \phi_\alpha)$, where

$$S(i)_j = \begin{cases} 0 & \text{if } j \neq i \\ k & \text{if } j = i \end{cases}$$

and

$$\phi_\alpha = 0 \text{ for all } \alpha \in Q_1.$$

Throughout the thesis, we choose to let $S(i)$ denote the simple representation, having vector spaces and linear maps as described, of all quivers Q having $\overline{Q}$ as its underlying graph. We are aware that this is a small abuse of notation.

2.4 CATEGORIES AND FUNCTORS

Definition 2.4.1. A *category* $\mathscr{C}$ consists of

(i) a collection of objects, $\mathrm{Obj}(\mathscr{C})$,

(ii) for each pair $B, C \in \mathrm{Obj}(\mathscr{C})$ a set of morphisms $\mathrm{Hom}_{\mathscr{C}}(B, C)$ such that for each $B, C, D \in \mathrm{Obj}(\mathscr{C})$ there is a composition map

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(C, D) \times \mathrm{Hom}_{\mathscr{C}}(B, C) &\to \mathrm{Hom}_{\mathscr{C}}(B, D) \\ (g, f) &\mapsto g \circ f \end{aligned} \tag{4}$$

satisfying the following:

(a) for each object $B \in \mathrm{Obj}(\mathscr{C})$ there exists a morphism $1_B \in \mathrm{Hom}_{\mathscr{C}}(B, B)$ such that

$$\begin{aligned} 1_B \circ f = f &\text{ for all } f \in \mathrm{Hom}_{\mathscr{C}}(D, B) \\ g \circ 1_B = g &\text{ for all } g \in \mathrm{Hom}_{\mathscr{C}}(B, C) \end{aligned} \tag{5}$$

(b) the associative law is satisfied, that is $(f \circ g) \circ h = f \circ (g \circ h)$ for every triple $h \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, $g \in \mathrm{Hom}_{\mathscr{C}}(C, D)$, $f \in \mathrm{Hom}_{\mathscr{C}}(D, E)$ of morphisms.

Example 2.4.2. Here we present some examples of categories. Let A be an algebra, Q a finite and connected quiver and I an admissible ideal in kQ.

(i) The category of left A-modules, denoted $\mathrm{Mod}\, A$. The objects of this category are left A-modules, while the morphisms are A-homomorphisms.

(*ii*) The category of finitely generated left A-modules, denoted $\operatorname{mod} A$. The objects of this category are finitely generated left A-modules, while the morphisms are A-homomorphisms.

(*iii*) The category of representations of the bound quiver (Q, I), denoted $\operatorname{rep}_k(Q, I)$. The objects of this category are representations of (Q, I) over k, and the morphisms are homomorphisms of representations.

(*iv*) The category of representations of the quiver Q, denoted $\operatorname{rep}_k Q$. The objects of this category are representations of Q over k, and the morphisms are homomorphisms of representations.

Definition 2.4.3. Let $\mathscr{C}$ be a category. A category $\mathscr{D}$ is a *subcategory* of $\mathscr{C}$ if $\operatorname{Obj}(\mathscr{D}) \subseteq \operatorname{Obj}(\mathscr{C})$, and for every pair $B, C \in \operatorname{Obj}(\mathscr{D})$ we have $\operatorname{Hom}_{\mathscr{D}}(B, C) \subseteq \operatorname{Hom}_{\mathscr{C}}(B, C)$, and the composition in $\mathscr{D}$ is the restriction of the composition in $\mathscr{C}$. The category $\mathscr{D}$ is a *full subcategory* of $\mathscr{C}$ if $\operatorname{Hom}_{\mathscr{D}}(B, C) = \operatorname{Hom}_{\mathscr{C}}(B, C)$ for every pair $B, C \in \operatorname{Obj}(\mathscr{D})$.

Definition 2.4.4. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A *covariant functor* (or simply *functor*) $F : \mathscr{C} \to \mathscr{D}$ associates to each $B \in \operatorname{Obj}(\mathscr{C})$ an object $F(B) \in \operatorname{Obj}(\mathscr{D})$, and to each morphism $f : B \to C$ in $\mathscr{C}$ a morphism $F(f) : F(B) \to F(C)$ in $\mathscr{D}$ such that

(*i*) $F(g \circ f) = F(g) \circ F(f)$ for all composable $f, g \in \mathscr{C}$,

(*ii*) $F(1_D) = 1_{F(D)}$ for all $D \in \operatorname{Obj}(\mathscr{C})$.

Definition 2.4.5. A category $\mathscr{C}$ is *preadditive* if $\operatorname{Hom}_{\mathscr{C}}(B, C)$ is an abelian group for all $B, C \in \operatorname{Obj}(\mathscr{C})$ and the composition map

$\operatorname{Hom}_{\mathscr{C}}(C, D) \times \operatorname{Hom}_{\mathscr{C}}(B, C) \to \operatorname{Hom}_{\mathscr{C}}(B, D)$ is bilinear. That is, for $f, f_1, f_2 \in \operatorname{Hom}_{\mathscr{C}}(C, D)$ and $g, g_1, g_2 \in \operatorname{Hom}_{\mathscr{C}}(B, C)$ we have

$$(f_1 + f_2) \circ g = (f_1 \circ g) + (f_2 \circ g),$$
$$f \circ (g_1 + g_2) = (f \circ g_1) + (f \circ g_2).$$

If A is a commutative algebra and $\operatorname{Hom}_{\mathscr{C}}(B, C)$ is an A-module for all $B, C \in \operatorname{Obj}(\mathscr{C})$ and the composition map is A-bilinear, then the category $\mathscr{C}$ is called an *A-category*.

Definition 2.4.6. Let $\mathscr{C}$ and $\mathscr{D}$ be two preadditive (A-)categories. Then a functor $F : \mathscr{C} \to \mathscr{D}$ is an *additive (A-)functor* if the map $F : \operatorname{Hom}_{\mathscr{C}}(B, C) \to \operatorname{Hom}_{\mathscr{D}}(F(B), F(C))$ is a homomorphism of groups (A-modules) for all pairs $B, C \in \operatorname{Obj}(\mathscr{C})$.

Definition 2.4.7. Let $\mathscr{C}$ and $\mathscr{D}$ be two k-categories, and $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then F is called *k-linear* if F is additive, and for all objects $A, B \in Obj(\mathscr{C})$ the map $F : \operatorname{Hom}_{\mathscr{C}}(A, B) \to \operatorname{Hom}_{\mathscr{D}}(F(A), F(B))$ is a k-linear map.

Definition 2.4.8. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then F is an *equivalence of categories* if there exists a functor $H : \mathscr{D} \to \mathscr{C}$ such that $H \circ F \simeq \mathrm{id}_{\mathscr{C}}$ and $F \circ H \simeq \mathrm{id}_{\mathscr{D}}$.

The one-to-one correspondence from Lemma 2.3.5 can now be expressed as an equivalence of categories.

Theorem 2.4.9. *Let $A = kQ/I$, where Q is a finite and connected quiver, and I is an admissible ideal in kQ. Then there exists a k-linear equivalence of categories*

$$F : \operatorname{mod} A \to \operatorname{rep}_k(Q, I).$$

Proof. In Lemma 2.3.5 we described a one-to-one correspondence between the isomorphism classes of finitely generated A-modules and the isomorphism classes of representations of the bound quiver (Q, I). Now we need to define the functors $F : \operatorname{mod} A \to \operatorname{rep}_k(Q, I)$ and $H : \operatorname{rep}_k(Q, I) \to \operatorname{mod} A$ such that we get an equivalence of categories.

The action of F and H on the objects of $\operatorname{mod} A$ and $\operatorname{rep}_k(Q, I)$, respectively, are as in Lemma 2.3.5. We only need to define their actions on the morphisms of the respective categories.

Let $B, C \in \operatorname{mod} A$, and $\phi : B \to C$ be an A-homomorphism. We now want to define a morphism $F(\phi) : F(B) \to F(C)$ of $\operatorname{rep}_k(Q, I)$.

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d, "F"'] & C \arrow[d, "F"] \\ F(B) \arrow[r, dashed] & F(C) \end{tikzcd}$$

Let $i \in Q_0$, and consider the element $x = xe_i \in Be_i$. Then we have $\phi(x) = \phi(xe_i) = \phi(xe_i^2) = \phi(xe_i)e_i = \phi(x)e_i \in Ce_i$. Hence the restriction of ϕ to Be_i, let us call it ϕ_i, is a k-linear map from Be_i to Ce_i. We then define $F(\phi) = (\phi_i)_{i \in Q_0}$. Let $F(B) = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $F(C) = (V'_i, f'_\alpha)_{i \in Q_0, \alpha \in Q_1}$. Consider $\alpha : i \to j \in Q_0$. We then need to check that $f'_\alpha \phi_i = \phi_j f_\alpha$. Let $x \in Be_i$. Then

$$\phi_j f_\alpha(x) = \phi_j(\overline{\alpha} x) = \overline{\alpha}\phi(x) = \overline{\alpha}\phi_i(x) = f'_\alpha \phi_i(x).$$

Now, it is not too hard to verify that the functor F is k-linear.

Let $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$, $V' = (V'_i, f'_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be two objects in $\operatorname{rep}_k(Q, I)$, and let $(\phi_i)_{i \in Q_0}$ be a homomorphism of representations. We now want to define an A-homomorphism $H(\phi) : H(V) \to H(V')$.

$$\begin{tikzcd} V \arrow[r, "(\phi_i)_{i \in Q_0}"] \arrow[d, "H"'] & V' \arrow[d, "H"] \\ H(V) \arrow[r, dashed] & H(V') \end{tikzcd}$$

We know that $H(V) = \bigoplus_{i \in Q_0} V_i$ and $H(V') = \bigoplus_{i \in Q_0} V'_i$ as k-vector spaces, and hence there exists a k-linear map

$$\phi = (\phi_i)_{i \in Q_0} : H(V) \to H(V').$$

To complete the proof we need to show that ϕ is an A-homomorphism. It is trivially checked that for any $x, y \in H(V)$ we have $\phi(x+y) = \phi(x) + \phi(y)$. To finish the proof we only need to check if $\phi(\overline{\omega}x) = \overline{\omega}\phi(x)$ for every $\overline{\omega} = \omega + I \in A$, $x \in H(V)$. It is enough to consider one coordinate of x, say x_i for some $i \in Q_0$. We have that

$$\phi(\overline{\omega}x_i) = \phi f_\omega(x_i) = \phi_j f_\omega(x_i) = f'_\omega \phi_i(x_i) = \overline{\omega}\phi(x).$$

Hence ϕ is an A-homomorphism. One can easily verify that the functor H is k-linear.

Using the definition of the functors, observe that $FH \simeq \mathrm{id}_{\mathrm{rep}_k(Q,I)}$ and that $HF \simeq \mathrm{id}_{\mathrm{mod}\, A}$, and hence F is an equivalence of categories. □

Corollary 2.4.10. *Let $A = kQ$, where Q is a finite, connected and acyclic quiver. Then there exists a k-linear equivalence of categories*

$$F : \mathrm{mod}\, A \to \mathrm{rep}_k\, Q.$$

Proof. Since Q is acyclic, the path algebra $A = kQ$ is finite dimensional. Hence the result follows from setting $I = (0)$ in Theorem 2.4.9. □

Let A be an algebra. The next result describes the indecomposable projective left A-modules in terms of its corresponding representations.

Lemma 2.4.11. *Let (Q, I) be a bound quiver and let $A = kQ/I$. For an $i \in Q_0$, let $P(i) = A\overline{e_i}$ denote the corresponding indecomposable projective left A-module.*

(*i*) *Let* $(P(i)_j, \phi_\alpha)$ *denote the corresponding representation of the left module* $P(i)$*. Then* $P(i)_j$ *is the k-vector space with basis* $\overline{e_j}A\overline{e_i}$ *for every* $i, j \in Q_0$*, that is the set of all paths from* i *to* j*. Consider an arrow* $\alpha : j \to l \in Q_1$*, where* $j, l \in Q_0$*. Then the k-linear map* $\phi_\alpha : P(i)_j \to P(i)_l$ *is given by left multiplication with* $\overline{\alpha} = \alpha + I$*.*

(*ii*) *Let* $(P'(i)_j, \phi'_\alpha)$ *denote the representation corresponding to* $\operatorname{rad} P(i)$*. Then* $P'(i)_j = P(i)_j$ *for* $j \neq i$*,* $P'(i)_i$ *is the k-vector space with basis all cylces through* i*. The k-linear map* $\phi'_\alpha = \phi_\alpha$ *for any arrow* α *starting in* $j \neq i$ *and* $\phi'_\alpha = \phi_\alpha\big|_{P'(i)_i}$ *for any arrow starting in* i*.*

Proof. Follows from the functor defined in Theorem 2.4.9. □

3

PRESENTING ALGEBRAS AS PATH ALGEBRAS

In the previous chapter we saw that representations of quivers are useful tools for visualising modules. In this chapter we will see that quivers can also be used for visualising algebras.

3.1 BASIC ALGEBRAS AND PATH ALGEBRAS

The main purpose of this section is to show that any basic and indecomposable algebra A is isomorphic to a bound quiver algebra.

We start by associating to each basic and indecomposable algebra A a quiver Q_A. We call this quiver the *ordinary quiver of A*, and it is defined as follows:

(i) The vertices of Q_A are defined by considering a set of primitive orthogonal idempotents of A, say $\{e_1, e_2, \ldots, e_n\}$. The vertices are in one-to-one correspondence to the idempotents of A, so that $(Q_A)_0 = \{1, \ldots, n\}$.

(ii) Given two vertices $i, j \in (Q_A)_0$. The arrows $\alpha : i \to j$ of Q_A are in one-to-one correspondence to the vectors in some basis of the k-vector space $e_j(\operatorname{rad} A / \operatorname{rad}^2 A)e_i$.

Note that since A is a finite dimensional algebra, Q_A is a finite quiver. The quiver Q_A does not depend on the chosen set of primitive orthogonal idempotents. We will see later why we chose to define the vertices and arrows of Q_A in this way. It will turn out to be quite convenient.

Lemma 3.1.1. *Let A be a basic and indecomposable algebra. Then the ordinary quiver Q_A of A is connected.*

Proof. Suppose Q_A is not connected. Then $(Q_A)_0$ splits into two non-empty disjoint sets I, J such that for $i \in I, j \in J$ there is no arrow $\alpha : i \to j$ or $\alpha : j \to i$. It can be shown that $e_i A e_j = (0) = e_j A e_i$ (cf. [1]). Hence A is decomposable by Lemma 2.1.7, which is a contradiction. It follows that Q_A must be connected. □

We will now see that if a basic and indecomposable algebra A is isomorphic to kQ/I for some quiver Q and some admissible ideal I, then $Q = Q_A$.

Lemma 3.1.2. *Let Q be a finite and connected quiver, I an admissible ideal of the path algebra kQ and $A \simeq kQ/I$. Then $Q = Q_A$.*

Proof. By Theorem 2.2.2 the set $\{\overline{e_i} = e_i + I \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents of kQ/I. That is, by the way the vertices of Q_A were defined, the vertices of Q_A are in a one-to-one correspondence with the vertices of Q. Consider the way we defined the arrows of Q_A. These arrows are in a one-to-one correspondence with the arrows of Q by Corollary 2.2.6. Since a quiver is uniquely defined by its sets of vertices and arrows we conclude that $Q = Q_A$. □

Lemma 3.1.2 explains why we chose to define the vertices and the arrows of Q_A the way we did. The next theorem is the main theorem of this section.

Theorem 3.1.3. *Let A be a basic and indecomposable algebra. Then there exists an admissible ideal I in kQ_A such that $A \simeq kQ_A/I$.*

Proof. Our approach here will be to construct a homomorphism $\phi : kQ_A \to A$, to show that ϕ is onto, and that $\ker \phi$ is an admissible ideal I in kQ_A.

By Theorem 2.1.9 we know that there exists a unique algebra homomorphism $\phi : kQ_A \to A$ if we can find two maps $\phi_0 : (Q_A)_0 \to A$, $\phi_1 : (Q_A)_1 \to A$ satisfying some conditions. So we first need to construct such maps ϕ_0, ϕ_1. For each arrow $\alpha : i \to j$ in $(Q_A)_1$, choose $x_\alpha \in \operatorname{rad} A$ such that the set $\{x_\alpha + \operatorname{rad}^2 A \mid \alpha \in (Q_A)_1\}$ forms a basis of the algebra $e_j(\operatorname{rad} A/\operatorname{rad}^2 A)e_i$. Let ϕ_0 be defined by $\phi_0(i) = e_i$ for all $i \in (Q_A)_0$, and let ϕ_1 be defined by $\phi(\alpha) = x_\alpha$ for all $\alpha \in (Q_A)_1$. Now we need to check that ϕ_0, ϕ_1 satisfies condition (i) and (ii) of Theorem 2.1.9. By the way ϕ_0 was defined and by Theorem 2.1.5, the elements $\phi_0(i)$ form a complete set of primitive orthogonal idempotents in A, hence condition (i) of Theorem 2.1.9 is satisfied. If $\alpha : i \to j$ we have $\phi_0(j)\phi_1(\alpha)\phi_0(i) = e_j x_\alpha e_i = x_\alpha = \phi_1(\alpha)$, so condition (ii) of Theorem 2.1.9 is satisfied. Then by Theorem 2.1.9 there exists a unique algebra homomorphism $\phi : kQ_A \to A$ that respects ϕ_0 and ϕ_1.

Now we need to show that the homomorphism ϕ just constructed is a surjective homomorphism. Since the image of ϕ, $\operatorname{Im}\phi$, is generated by the set $\{e_i, x_\alpha \mid i \in (Q_A)_0, \alpha \in (Q_A)_1\}$, it follows from the classical Wedderburn-Malcev Theorem (cf. [5]) that ϕ is surjective.

Lastly, we need to show that $\ker\phi$ is an admissible ideal in kQ_A. Let $\mathscr{J}$ be the arrow ideal in kQ_A. Because of the way ϕ was constructed we know that $\phi(\mathscr{J}) \subseteq \operatorname{rad} A$, and hence $\phi(\mathscr{J}^l) \subseteq \operatorname{rad}^l A$ for every $l \geq 1$. Since $\operatorname{rad} A$ is nilpotent $\operatorname{rad}^m A = (0)$ for some $m \geq 1$, and hence $\mathscr{J}^m \subseteq \ker\phi$. Next we claim that $\ker\phi \subseteq \mathscr{J}^2$. Let $x \in \ker\phi$. Now, study the form x takes.

$$x = \sum_{i \in (Q_A)_0} b_i e_i + \sum_{\alpha \in (Q_A)_1} b_\alpha \alpha + j, \tag{6}$$

where $b_i, b_\alpha \in k$, $\{e_i \mid i \in (Q_A)_0\}$ is the set of trivial paths in kQ_A and $j \in \mathscr{J}^2$. In order for our claim to hold we must show that

$b_i = b_\alpha = 0$ for any $i \in (Q_A)_0$, $\alpha \in (Q_A)_1$. Since $x \in \ker \phi$ we must have

$$0 = \phi(x) = \sum_{i \in (Q_A)_0} b_i \phi_0(i) + \sum_{\alpha \in (Q_A)_1} b_\alpha x_\alpha + \phi(j).$$

We know that $x_\alpha \in \operatorname{rad} A$ for all $\alpha \in (Q_A)_1$ by the definition of ϕ, so we get that

$$\sum_{i \in (Q_A)_0} b_i \overline{e_i} = - \sum_{\alpha \in (Q_A)_1} b_\alpha x_\alpha - \phi(j) \in \operatorname{rad} A.$$

Since $\operatorname{rad} A$ is nilpotent by Lemma 1.2.6, and since $\{\overline{e_i} = e_i + I \mid i \in (Q_A)_0\}$ is a set of primitive orthogonal idempotents by Theorem 2.2.2, we get that $b_i = 0$ for every $i \in (Q_A)_0$. Hence,

$$\sum_{\alpha \in (Q_A)_1} b_\alpha x_\alpha = -\phi(j) \in \operatorname{rad}^2 A.$$

and,

$$\sum_{\alpha \in (Q_A)_1} b_\alpha (x_\alpha + \operatorname{rad}^2 A) = 0$$

in $\operatorname{rad} A / \operatorname{rad}^2 A$. But, by construction, the set $\{x_\alpha + \operatorname{rad}^2 A \mid \alpha \in (Q_A)_1\}$ is a basis of $e_i(\operatorname{rad} A / \operatorname{rad}^2 A)e_j$. Hence, we must have $b_\alpha = 0$. Now, since $b_a = b_\alpha = 0$ we see from equation (6) that $x = j \in \mathscr{J}^2$, and since x was some arbitrary element in $\ker \phi$, we get that $\mathscr{J}^m \subseteq \ker \phi \subseteq \mathscr{J}^2$. Hence, the ideal $I = \ker \phi$ is an admissible ideal in kQ_A. $\square$

3.2 HEREDITARY ALGEBRAS

In section 3.1 we saw that any basic, indecomposable algebra A is isomorphic to a bound quiver algebra. We will here study which requirements that need to be fulfilled for a basic and indecomposable algebra A to be isomorphic to a path algebra. That is, under which circumstances is A isomorphic to kQ for some finite, connected, acyclic quiver Q? We will here see that A is of this form if and only if it is *hereditary*.

Definition 3.2.1. An algebra A is called *left hereditary* if every left ideal of A is projective as an A-module.

A *right hereditary* algebra is defined similarly. A well-known result from homological algebra states that in the case of a left and right noetherian algebra, an algebra is left hereditary if and only if it is right hereditary. In particular, this applies to all the algebras we will consider, and therefore we will just call them hereditary.

Lemma 3.2.2. *Let A be a hereditary algebra. Then every submodule of a free A-module is isomorphic to a direct sum of left ideals in A.*

Proof. Let L be a free A-module, and let $\{e_\lambda \mid \lambda \in \Lambda\}$ be its basis. Consider a submodule M of L. We then need to show that M is isomorphic to a direct sum of left ideals in A. We may assume, without loss of generality, that the index set Λ is a well-ordered set. Then for each $\lambda \in \Lambda$, let $L_\lambda = \bigoplus\limits_{\mu<\lambda} Ae_\mu$. Observe that $L_0 = 0$ and $L_{\lambda+1} = Ae_\lambda \bigoplus L_\lambda$. Let $x \in M \cap L_{\lambda+1}$. Then x is of the form $x = ae_\lambda + y$, where $a \in A, y \in L_\lambda$, and this representation is unique. Thus, we may define an A-homomorphism

$$f_\lambda : M \cap L_{\lambda+1} \to A$$

given by $x \mapsto a$. Hence, we can construct a short exact sequence

$$0 \longrightarrow M \cap L_\lambda \longrightarrow M \cap L_{\lambda+1} \xrightarrow{f_\lambda} \operatorname{Im} f_\lambda \longrightarrow 0 \qquad (7)$$

Because $\operatorname{Im} f_\lambda$ is a left ideal in A, it is projective since A is hereditary. Then by Lemma 1.1.11 the short exact sequence (7) splits. Hence, there exists an $f'_\lambda : \operatorname{Im} f_\lambda \to M \cap L_{\lambda+1}$ such that

$$M \cap L_{\lambda+1} = \ker f_\lambda \bigoplus \operatorname{Im} f'_\lambda.$$

Then there exists a submodule N_λ of $M \cap L_{\lambda+1}$ such that $N_\lambda \simeq \operatorname{Im} f'_\lambda$ and

$$M \cap L_{\lambda+1} = \ker f_\lambda \bigoplus N_\lambda = M \cap L_\lambda \bigoplus N_\lambda.$$

We start by proving that $M = \sum_{\lambda \in \Lambda} N_\lambda = N$, and then we will complete the proof by proving that this sum is direct. Since $L = \bigcup_{\lambda \in \Lambda} L_\lambda$ we have for each $x \in L$ a least $\lambda \in \Lambda$ such that $x \in L_{\lambda+1}$. Denote this index by μ_x. Suppose by contradiction that $N \subset M$, that is, N is a proper subset of M. Then there exists an element $x \in M$ such that $x \notin N$. Let μ denote the least μ_x such that $x \in M$, but $x \notin N$. Choose an element y such that $\mu_y = \mu$. Then $y \in M$, but $y \notin N$. Hence, $y \in M \cap L_{\mu+1}$, and y takes the form $y = u + v$, where $u \in M \cap L_\mu$, $v \in N_\mu$. Therefore $u = y - v \in M$. Since y was chosen such that $y \notin N$ we must have $u \notin N$ to avoid a contradiction. But since $u \in M \cap L_\mu$ we get that $\mu_u < \mu$, so $u \in N$. This is a contradiction, so $M = N$, or $M = \sum_{\lambda \in \Lambda} N_\lambda$.

Now, what remains is proving that $M = \sum_{\lambda \in \Lambda} N_\lambda$ is a direct sum. Suppose $x_1 + \cdots + x_n = 0$ for $x_i \in N_{\lambda_i}$. We can assume, without loss of generality, that $\lambda_1 < \cdots < \lambda_n$. We must show that then $x_i = 0$ for every i. We get that $x_1 + \cdots + x_{n-1} = -x_n \in (M \cap L_{\lambda_n}) \cap N_{\lambda_n} = (0)$, so $x_n = 0$. Continue similarly to see that $x_i = 0$ for every i. Hence, $M = \bigoplus_{\lambda \in \Lambda} N_\lambda$. □

Proposition 3.2.3. *Let A be an algebra. The following are equivalent:*

(i) *The algebra A is hereditary.*

(ii) *Every submodule of a projective left A-module is projective.*

(iii) *The radical $\underline{r} = \operatorname{rad} A$ is a projective left A-module.*

Proof. Here we will only prove $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$. For the proof of $(iii) \Rightarrow (i)$, see [2].

$(i) \Rightarrow (ii)$: Suppose A hereditary. Let P be a projective left A-module, and let $Q \subseteq P$ be a submodule of P. We want to show that Q is a projective module. By Lemma 1.1.8 there exists a free module F and some A-module R such that $F = P \bigoplus R$. Hence, P is a submodule of the free module F. Since Q is a submodule of P we must have that Q is a submodule of F as well. Then by Lemma 3.2.2 we have $Q \simeq I_1 \bigoplus \cdots \bigoplus I_n$, where I_j is a left ideal in A for $j = 1, \ldots, n$. The ideals I_j are all projective modules since A is hereditary. Hence, Q is a projective module.

$(ii) \Rightarrow (iii)$: Suppose every submodule of a projective left A-module is projective. Then we need to find a projective left module P such that $\underline{r} \subseteq P$ is a submodule. The radical $\underline{r} \subseteq A$ is an ideal in A, and hence $\underline{r}$ is an A-module. Consider ${}_AA$, that is A considered as a left A-module. Then $\underline{r}$ is a submodule of ${}_AA$. Since ${}_AA$ is a projective left A-module, $\underline{r}$ is a projective left A-module. $\square$

Lemma 3.2.4. *Let A be an basic, indecomposable and hereditary algebra. Then the ordinary quiver Q_A of A is acyclic.*

Proof. By Lemma 3.1.1 we have that Q_A is connected, so we can find $i, j \in (Q_A)_0$ such that there exists an arrow $\alpha : i \to j$. Then by definition we get that $e_j(\underline{r}/\underline{r}^2)e_i \neq (0)$. Let $\overline{\alpha}$ be a nonzero element in $e_j\underline{r}e_i$. Then we have a nonzero A-homomorphism

$$f_\alpha : Ae_i \to Ae_j$$

defined by left multiplication with $\overline{\alpha}$.

Since Ae_j is an indecomposable projective module Proposition 3.2.3 implies that $\operatorname{Im} f_\alpha$ is projective. Hence the short exact sequence

$$0 \to \ker f_\alpha \to Ae_i \to \operatorname{Im} f_\alpha \to 0$$

splits by Lemma 1.1.11, and $Ae_i \simeq \ker f_\alpha \bigoplus \operatorname{Im} f_\alpha$. Since Ae_i is indecomposable and $\operatorname{Im} f_\alpha \neq (0)$ we must have $\ker f_\alpha = (0)$. Hence, f_α is a monomorphism. Since A is basic we know that f_α is not an isomorphism.

Now suppose there exists a cycle in Q_A going through i. Then clearly $f = f_{\alpha_t} \cdots f_{\alpha_1}$ is a monomorphism since f_{α_s} is a monomorphism for every $s \in \{1, \ldots, t\}$. That is, $f : Ae_i \to Ae_i$ is a monomorphism, but not an isomorphism, which is a contradiction. Therefore, Q_A is acyclic. □

Lemma 3.2.5. *Let Q be a finite, connected and acyclic quiver. Then the path algebra kQ is hereditary.*

Proof. By Proposition 3.2.3 it is enough to show that $\underline{r} = \operatorname{rad}(kQ)$ is a projective kQ-module.

It is clear that $\underline{r} = \underline{r} \cdot 1_{kQ} = \underline{r}(e_1 + \cdots + e_n) = \underline{r}e_1 \bigoplus \cdots \bigoplus \underline{r}e_n$. Now, if we can show that $\underline{r}e_i$ is projective for every $i \in \{1, \ldots, n\}$ we get that $\underline{r}$ is projective. The set of all non-trivial paths starting in i, $\mathscr{B} = \{p \mid s(p) = i\}$, is a basis of $\underline{r}e_i$. Let $\alpha_1, \ldots, \alpha_t$ be the arrows in Q_1 such that $s(\alpha_j) = i$ for $j = 1, \ldots, t$. Then any element $p \in \mathscr{B}$ is of form $p = q\alpha_j$, $j \in \{1, \ldots, t\}$, where q is any path satisfying $s(q) = t(\alpha_j)$. Hence we have

$$\underline{r}e_i = \bigoplus_{j=1}^{t} kQe_{t(\alpha_j)}\alpha_j \simeq \bigoplus_{j=1}^{t} kQe_{t(\alpha_j)}.$$

Since $kQ = kQ \cdot 1_{kQ} = kQ(e_1 + \cdots + e_n) = kQe_1 \bigoplus \cdots \bigoplus kQe_n$ as a kQ-module, we get from Lemma 1.1.8 that kQe_i is a projective module for $i = 1, \ldots, n$. Hence, $\underline{r}e_i$ is a projective module for every i as a direct sum of projective modules. So, $\underline{r}$ is projective, and hence kQ is hereditary by Proposition 3.2.3. □

Lemma 3.2.6. *Let Q be a finite, connected and acyclic quiver and $I \subseteq kQ$ an admissible ideal. Then kQ/I is not hereditary if $I \neq (0)$.*

Proof. Let $A = kQ/I$. We identify with each of the idempotents $e_i \in A$ the residue class of the trivial path at i, $\overline{e_i} = e_i + I$. In Lemma 2.4.11 we saw that the indecomposable projective modules $P(i) = A\overline{e_i}$ can be described in terms of its corresponding representation in the following way: $P(i) = (P(i)_j, \phi_\alpha)$. The k-vector space $P(i)_j$ is the k-vector space having as its basis all paths $\overline{\omega} = \omega + I$ where $\omega \in kQ$ is a path from i to j. Let $\alpha : j \to l \in Q_1$. Then the k-linear map $\phi_\alpha : P(i)_j \to P(i)_l$ is given by left multiplication with $\overline{\alpha} = \alpha + I$. The dimension of $e_j kQ e_i$, $\dim_k(e_j kQ e_i)$, equals the number of paths from i to j in Q, denote this number by $\omega(i,j)$. Hence $\dim_k e_j P(i) = \omega(i,j) - \dim_k e_j I e_i$. We are going to use this equation to prove that if A is hereditary we must have $I = (0)$.

If A is hereditary, suppose by contradiction that $I \neq (0)$. Since Q is acyclic we can number the vertices of Q in such a way that if there exists an arrow from x to y we have $x > y$. (Such a numbering is called an *admissible numbering*.) Then there is a least number i such that there exists some $j \in Q_0$ with $e_j I e_i \neq (0)$. By Lemma 2.4.11 we get that $\operatorname{rad} P(i) \neq (0)$. Since A is hereditary $\operatorname{rad} A$ is projective by Proposition 3.2.3. Hence $\operatorname{rad} P(i)$ is projective, and then by Lemma 1.1.9 we get that $\operatorname{rad} P(i) \simeq P(j_1)^{n_1} \bigoplus \cdots \bigoplus P(j_t)^{n_t}$ for some $t \geq 1$, where $j_1, \ldots, j_t \in Q_0$ and $n_1, \ldots, n_t \in \mathbb{N}$. It can be shown that $\{j_1, \ldots, j_t\}$ is the set of all successors of i, that is, all vertices which is such that there exists an arrow $\alpha : i \to j_s$, $s \in \{1, \ldots, t\}$. This implies that $i > j_s$, and by the minimality of i we know that $e_j I e_{j_s} = (0)$. It is also possible to show that n_s is the number of arrows from i to j_s in Q_1 for $s = 1, \ldots, t$. We have that

$$\dim_k e_j P(j_s) = \dim_k e_j A e_{j_s} = \omega(j_s, j),$$

for every j and every s. It follows that

$$\begin{aligned}\dim_k e_j(\operatorname{rad} P(i)) = \sum_{m=1}^{t} n_m \dim_k e_j P(j_m) &= \sum_{m=1}^{t} n_m \omega(j_m, j) \\ &= \omega(i,j) > \omega(i,j) - \dim_k e_j I e_i = \dim_k e_j P(i)\end{aligned}$$

which is a contradiction since $\operatorname{rad} P(i) \subseteq P(i)$. Hence, $I = (0)$. $\square$

Theorem 3.2.7. *Let A be a basic and indecomposable algebra. Then $A \simeq kQ_A$ if and only if A is hereditary.*

Proof. Suppose A is hereditary. By Theorem 3.1.3 we get $A \simeq kQ_A/I$ for some admissible ideal I in kQ_A. Since A is hereditary, basic and indecomposable Lemma 3.1.1 and Lemma 3.2.4 implies that Q_A is finite, connected and acyclic. Then $I = (0)$ by Lemma 3.2.6, and thus $A \simeq kQ_A$.

Conversely, suppose $A \simeq kQ_A$. Then A is hereditary by Lemma 3.2.5. $\square$

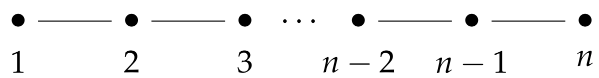
4

ALGEBRAS OF FINITE REPRESENTATION TYPE

4.1 DYNKIN DIAGRAMS

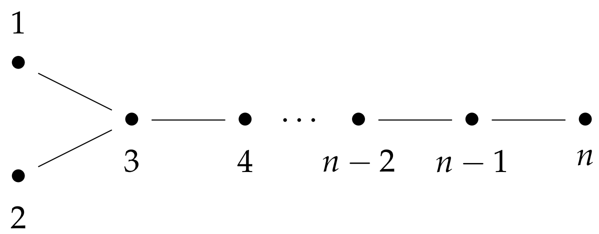
In this thesis certain quivers will be of particular interest. We will be particularly interested in the quivers whose underlying graph is a *Dynkin diagram*. As we will see in section 4.4 a path algebra is of finite representation type if and only if its underlying graph is a Dynkin diagram. We will present the Dynkin diagrams here.

A_n



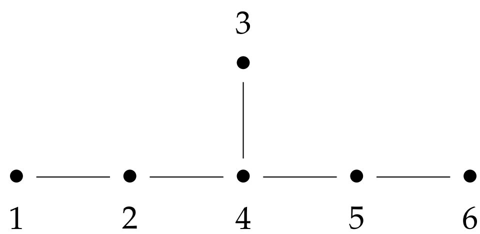
$n \geq 1$

D_n

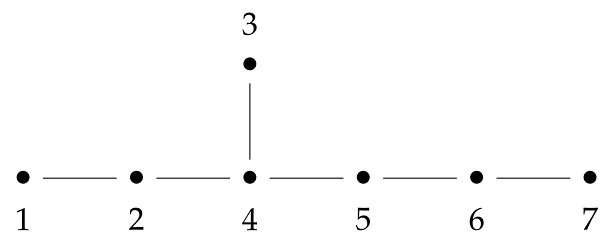


$n \geq 4$

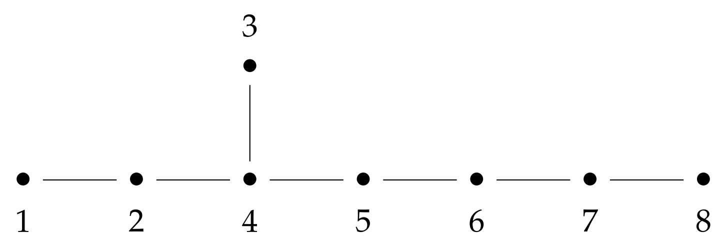
E_6



E_7



E_8



We shall see in Theorem 4.2.8 that the requirements for a path algebra to be of finite representation type only depends on the underlying graph of the path algebra. It follows that the orientation of the underlying quiver is insignificant. Motivated by this fact we will now describe a way to express different quivers having the same underlying graph.

Definition 4.1.1. Let $Q = (Q_0, Q_1)$ be a finite and connected quiver having n vertices. For every vertex $i \in Q_0$, let $\sigma_i Q = Q' = (Q'_0, Q'_1)$ be the quiver having $Q'_0 = Q_0$, but all arrows in Q_1 having i either as its source or target are reversed in Q'_1. Denote the set of vertices having i either as its source or target by $\mathcal{E}_i$. There exists a bijection $Q_1 \to Q'_1$ such that each $\alpha \in Q_1$ corresponds to some $\alpha' \in Q'_1$, where α' is described in the following way:

(i) if $s(\alpha) \neq i$ and $t(\alpha) \neq i$, then $t(\alpha') = t(\alpha)$ and $s(\alpha') = s(\alpha)$,

(ii) if $s(\alpha) = i$ or $t(\alpha) = i$, then $s(\alpha') = t(\alpha)$ and $t(\alpha') = s(\alpha)$.

We call the quiver $Q' = \sigma_i Q$ the *reversed quiver of Q with respect to vertex i*.

In the proof of Lemma 3.2.6 we defined an admissible numbering of the vertices of a quiver. We will now study a further property of the admissible numbering. Let $a_1, \ldots, a_n$ be an admissible numbering of the vertices of an acyclic quiver Q, having $a_i < a_j$ for $i < j$. Then we have that

(i) a_1 is a sink in Q, and

(ii) a_i is a sink in $\sigma_{a_{i-1}} \ldots \sigma_{a_1} Q$ for every $2 \leq i \leq n$.

The set $\{a_1, \ldots, a_n\}$ is called an *admissible sequence of sinks* in Q. Note that $\{a_1, \ldots, a_n\}$ is an admissible sequence of sinks in Q if and only if $a_1, \ldots, a_n$ is an admissible numbering. Similarly, we have that

(i) a_n is a source in Q, and

(ii) a_i is a source in $\sigma_{a_{i+1}} \ldots \sigma_{a_n} Q$ for every $1 \leq i \leq n-1$.

The set $\{a_n, \ldots, a_1\}$ is then called an *admissible sequence of sources* in Q.

4.2 REFLECTION FUNCTORS

Motivated by the previous section we now define some functors, called *right reflection functors* and *left reflection functors*, between the category of representations of a quiver Q and the category of representations of the reversed quiver with respect to some sink/source of Q.

Definition 4.2.1. Let Q be a finite and connected quiver, let a be a sink in Q and $Q' = \sigma_a Q$. Let $\mathcal{E}_a = \{\alpha_1, \ldots, \alpha_n\}$. The *left reflection functor* $\mathcal{S}_a^+ : \operatorname{rep}_k Q \to \operatorname{rep}_k Q'$ is a functor defined as follows. Let $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be an object in $\operatorname{rep}_k Q$. Then $\mathcal{S}_a^+(V) = W = (W_i, g_\alpha)_{i \in Q'_0, \alpha \in Q'_1}$, where

(i) $W_i = V_i$ for all $i \neq a$. To define W_a, first consider the mapping $h : \bigoplus_{\alpha_i \in \mathcal{E}_a} V_{s(\alpha_i)} \to V_a$ defined by $h(v_1, \dots, v_n) = f_{\alpha_1}(v_1) + \cdots + f_{\alpha_n}(v_n)$. Then $W_a = \ker h$.

$$\begin{tikzcd}
V_{s(\alpha_1)} \arrow[dr, "f_{\alpha_1}"] & \\
\vdots & V_a \\
V_{s(\alpha_n)} \arrow[ur, "f_{\alpha_n}"'] &
\end{tikzcd}$$

(ii) $g_\alpha = f_\alpha$ for all $\alpha \notin \mathcal{E}_a$. If $\alpha \in \mathcal{E}_a$ then $g_\alpha = \pi \circ \iota$, where π is the projection and ι the embedding defined by the following sequence:

$$W_a \xrightarrow{\iota} \bigoplus_{\alpha_i \in \mathcal{E}_a} V_{s(\alpha_i)} \xrightarrow{\pi} V_{s(\alpha)}$$

Let $\phi = \{\phi_i\}_{i \in Q_0} : V \to V'$ be a morphism in $\operatorname{rep}_k Q$, where $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $V' = (V'_i, f'_\alpha)_{i \in Q_0, \alpha \in Q_1}$. Then $\mathcal{S}_a^+(\phi)$ is defined to be $\rho = \{\rho_i\}_{i \in Q_0} : \mathcal{S}_a^+(V) = W \to \mathcal{S}_a^+(V') = W'$, where $\rho_i = \phi_i$ for all $i \neq a$, and ρ_a is the unique k-linear map such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & W_a \arrow[r, "\iota"] \arrow[d, dashed, "\rho_a"'] & \bigoplus_{\alpha \in \mathcal{E}_a} V_{s(\alpha)} \arrow[r, "h"] \arrow[d, "\bigoplus_{\alpha \in \mathcal{E}_a} \phi_{s(\alpha)}"] & V_a \arrow[d, "\phi_a"] \\
0 \arrow[r] & W'_a \arrow[r, "\iota'"] & \bigoplus_{\alpha \in \mathcal{E}_a} V'_{s(\alpha)} \arrow[r, "h'"] & V'_a
\end{tikzcd}$$

In a similar way we define the *right reflection function* $\mathcal{S}_a^-$:

Definition 4.2.2. Let Q' be a finite and connected quiver, let a be a source in Q' and $Q = \sigma_a Q'$. Let $\mathcal{E}_a = \{\alpha_1, \dots, \alpha_n\}$. The *right reflection functor* $\mathcal{S}_a^- : \operatorname{rep}_k Q' \to \operatorname{rep}_k Q$ is a functor defined as follows. Let $W = (W_i, g_\alpha)_{i \in Q'_0, \alpha \in Q'_1}$ be an object in $\operatorname{rep}_k Q'$. Then $\mathcal{S}_a^-(W) = V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$, where

(i) $V_i = W_i$ for all $i \neq a$. Consider the mapping $\tilde{h} : W_a \to \bigoplus_{\alpha_i \in \mathcal{E}_a} W_{t(\alpha_i)}$ defined by $\tilde{h}(w) = (g_{\alpha_1}(w), \ldots, g_{\alpha_n}(w))$. Then $V_a = \bigoplus\limits_{\alpha_i \in \mathcal{E}_a} W_{t(\alpha_i)} / \operatorname{Im} \tilde{h} = \operatorname{coker} \tilde{h}$.

$$\begin{tikzcd}[row sep=small]
W_{t(\alpha_1)} & \\
\vdots & W_a \arrow[ul, "g_{\alpha_1}"'] \arrow[dl, "g_{\alpha_n}"] \\
W_{t(\alpha_n)} &
\end{tikzcd}$$

(ii) $f_\alpha = g_\alpha$ for all $\alpha \notin \mathcal{E}_a$. If $\alpha \in \mathcal{E}_a$, then $f_\alpha = \tau \circ \nu$, where τ is the projection and ν the embedding defined by the following sequence:

$$W_{t(\alpha)} \xrightarrow{\nu} \bigoplus_{\alpha_i \in \mathcal{E}_a} W_{t(\alpha_i)} \xrightarrow{\tau} V_a$$

Let $\phi' = \{\phi'_i\}_{i \in Q'_0} : W \to W'$ be a morphism in $\operatorname{rep}_k Q'$, where $W = (W_i, g_\alpha)$ and $W' = (W'_i, g'_\alpha)$. Then $\mathcal{S}_a^-(\phi')$ is defined to be $\rho' = \{\rho'_i\}_{i \in Q'_0} : \mathcal{S}_a^-(W) = V \to \mathcal{S}_a^-(W') = V'$, where $\rho'_i = \phi'_i$ for all $i \neq a$, and ρ'_a is the unique k-linear map such that the following diagram commutes:

$$\begin{tikzcd}
W_a \arrow[r, "\tilde{h}"] \arrow[d, "\phi'_a"] & \bigoplus\limits_{\alpha \in \mathcal{E}_a} W_{t(\alpha)} \arrow[r, "h"] \arrow[d, "\bigoplus\limits_{\alpha \in \mathcal{E}_a} \phi'_{t(\alpha)}"] & V_a \arrow[r] \arrow[d, dashed, "\rho'_a"] & 0 \\
W'_a \arrow[r, "\tilde{h}'"] & \bigoplus\limits_{\alpha \in \mathcal{E}_a} W'_{t(\alpha)} \arrow[r, "h'"] & V'_a \arrow[r] & 0
\end{tikzcd}$$

Note that $\mathcal{S}_a^{+/-}(V_1 \bigoplus V_2) = \mathcal{S}_a^{+/-}(V_1) \bigoplus \mathcal{S}_a^{+/-}(V_2)$. Since the definitions of the reflection functors $\mathcal{S}_a^+$ and $\mathcal{S}_a^-$ are quite technical we will illustrate them with a small example.

Example 4.2.3. Consider the quiver $Q : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3$. Observe that vertex 3 is a sink in Q and vertex 1 is a source in Q. Let $Q' =$

$\sigma_3 Q$, $Q'' = \sigma_1 Q$. Let $\mathcal{S}_3^+ : \operatorname{rep}_k Q \to \operatorname{rep}_k Q'$ be the left reflection functor at the sink 3 and $\mathcal{S}_1^- : \operatorname{rep}_k Q \to \operatorname{rep}_k Q''$ be the right reflection functor at the source 1. Consider the representation

$$V: \quad k \xrightarrow{1_k} k \xrightarrow{1_k} k.$$

Then

$$\mathcal{S}_3^+(V): \quad k \xrightarrow{1_k} k \xleftarrow{0} 0,$$

and

$$\mathcal{S}_1^-(V): \quad 0 \xleftarrow{0} k \xrightarrow{1_k} k.$$

We will now study some further properties of the functors $\mathcal{S}_a^+$ and $\mathcal{S}_a^-$. Before stating the first result regarding the properties of these functors we will make an important observation.

Let a be a sink in a finite and connected quiver Q. Then a is clearly a source in the reversed quiver $\sigma_a Q$. Hence there exists a functor $\mathcal{S}_a^- \mathcal{S}_a^+ : \operatorname{rep}_k Q \to \operatorname{rep}_k Q$. To study how this functor acts on the objects of $\operatorname{rep}_k Q$ we construct for each object V in $\operatorname{rep}_k Q$ a morphism of functors

$$i_V^a = ((i_V^a)_1, \ldots, (i_V^a)_n) : \mathcal{S}_a^- \mathcal{S}_a^+(V) \to V,$$

where $n = |Q_0|$. We shall now describe how to construct i_V^a, and later we will use some properties of this morphism to gain information about our functors $\mathcal{S}_a^-$ and $\mathcal{S}_a^+$.

Let $\mathcal{S}_a^+(V) = W$, and $\mathcal{S}_a^-(W) = U$. For a vertex $i \neq a$ we get by the definition of $\mathcal{S}_a^+$ and $\mathcal{S}_a^-$ that $U_i = \mathcal{S}_a^-(W_i) = W_i = \mathcal{S}_a^+(V_i) = V_i$, and therefore we set

$$(i_V^a)_i = 1_{V_i}.$$

In the case of $i = a$ we first observe from Definition 4.2.1 and Definition 4.2.2 that $\operatorname{Im} \tilde{h} = W_a = \ker h$.

$$W_a \xrightarrow{\tilde{h}} \bigoplus_{\alpha \in \mathcal{E}_a} V_{s(\alpha)} \xrightarrow{h} V_a$$

Hence $\mathcal{S}_a^-\mathcal{S}_a^+(V_a) = \mathcal{S}_a^-(W_a) = \operatorname{coker}\tilde{h} = \bigoplus_{\alpha\in\mathcal{E}_a} V_{s(\alpha)} / \operatorname{Im}\tilde{h} = \bigoplus_{\alpha\in\mathcal{E}_a} V_{s(\alpha)} / \ker h$. We then take $(i_V^a)_a$ to be the natural mapping

$$(i_V^a)_a : \bigoplus_{\alpha\in\mathcal{E}_a} V_{s(\alpha)} / \ker h \to V_a.$$

Now, one should verify that i^a is actually a natural transformation. Some properties of i^a are collected in the next result.

Proposition 4.2.4. *Let Q be a finite and connected quiver and $V = (V_i, f_\alpha)_{i\in Q_0,\alpha\in Q_1}$ an object in $\operatorname{rep}_k Q$. Consider the morphism $i_V^a : \mathcal{S}_a^-\mathcal{S}_a^+(V) \to V$ just defined. Then*

(i) i_V^a is a monomorphism.

(ii) if i_V^a is an isomorphism, then the dimensions of the vector space $W_i = \mathcal{S}_a^+(V_i)$ is

$$\dim W_i = \begin{cases} -\dim V_a + \sum\limits_{\alpha\in\mathcal{E}_a} \dim V_{s(\alpha)} & \text{for } i = a, \\ \dim V_i & \text{for } i \neq a. \end{cases} \tag{8}$$

(iii) the object $\operatorname{coker} i_V^a$ is concentrated at vertex a, that is $(\operatorname{coker} i_V^a)_i = 0$ for $i \neq a$, while $(\operatorname{coker} i_V^a)_a = V_a / \operatorname{Im}(i_V^a)_a$.

(iv) $V \simeq \mathcal{S}_a^-\mathcal{S}_a^+(V) \bigoplus \operatorname{coker} i_V^a$ as representations.

(v) if the object V has the form $\mathcal{S}_a^-(X)$ for some $X \in \operatorname{Obj}(\operatorname{rep}_k Q')$, where $Q' = \sigma_a Q$, then i_V^a is an isomorphism.

Proof. (i): We examine if $\ker i_V^a = (0)$. In order for an element $V \in \operatorname{rep}_k Q$ to be in the kernel of i_V^a we must have $V_i = (0)$ for every $i \neq a$. It is easy to see that $\ker(i_V^a)_a = (0)$, and hence i_V^a is a monomorphism.

(ii): Suppose i_V^a is an isomorphism. This implies that

$$\begin{aligned}\dim V_a = \dim \mathcal{S}_a^- \mathcal{S}_a^+(V_a) &= \dim \left(\bigoplus_{\alpha \in \mathcal{E}_a} V_{s(\alpha)} / \ker h \right) \\ &= \dim \sum_{\alpha \in \mathcal{E}_a} V_{s(\alpha)} - \dim \ker h \\ &= \dim \sum_{\alpha \in \mathcal{E}_a} V_{s(\alpha)} - \dim W_a.\end{aligned}$$

Since it is obvious that $\dim W_i = \dim V_i$ for $i \neq a$ from the definition the result follows.

(iii): We have that $(\operatorname{coker} i_V^a)_i = V_i / \operatorname{Im}(i_V^a)_i$. Hence for $i \neq a$ we have that $(\operatorname{coker} i_V^a)_i = (0)$ since $(i_V^a)_i = 1_{V_i}$ for $i \neq a$. Yet for $i = a$ we have $(\operatorname{coker} i_V^a)_a = V_a / \operatorname{Im}(i_V^a)_a$.

(iv): Observe that we have a short exact sequence

$$0 \longrightarrow \mathcal{S}_a^- \mathcal{S}_a^+(V) \xrightarrow{i_V^a} V \longrightarrow \operatorname{coker} i_V^a \longrightarrow 0.$$

The above short exact sequence splits, and hence $V \simeq \mathcal{S}_a^- \mathcal{S}_a^+(V) \bigoplus \operatorname{coker} i_V^a$.

(v): Let $V = \mathcal{S}_a^-(X)$. We need to show that in this case i_V^a is an epimorphism. It can be shown that V and $\mathcal{S}_a^- \mathcal{S}_a^+(V)$ have the same dimension when considered modules by Corollary 2.4.10. Then, since i_V^a is a monomorphism by (i) we get that i_V^a is an epimorphism, and hence an isomorphism. □

Similarly, for a source a we can construct a morphism of functors

$$p_V^a : V \to \mathcal{S}_a^+ \mathcal{S}_a^-(V).$$

Let $\mathcal{S}_a^-(V) = W$ and $\mathcal{S}_a^+(W) = U$. We set

$$(p_V^a)_i = 1_{V_i}$$

for $i \neq a$. When $i = a$ we have that $\mathcal{S}_a^+\mathcal{S}_a^-(V_a) = \mathcal{S}_a^+(W_a) = \ker h = \operatorname{Im} \tilde{h} = \bigoplus_{\alpha_i \in \mathcal{E}_a} V_{t(\alpha_i)} / \ker \tilde{h}$. Then we take $(p_V^a)_a$ to be the mapping

$$(p_V^a)_a : V_a \to \bigoplus_{\alpha_i \in \mathcal{E}_a} V_{t(\alpha_i)} / \ker \tilde{h}.$$

Then, considering the proof of Proposition 4.2.4 it is not difficult to show that the following result holds.

Proposition 4.2.5. *Let Q be a finite and connected quiver and $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be an object in $\operatorname{rep}_k Q$. Consider the morphism $p_V^a : V \to \mathcal{S}_a^+\mathcal{S}_a^-(V)$ just defined. Then*

(i) *p_V^a is an epimorphism.*

(ii) *if p_V^a is an isomorphism, then the dimension of the vector space $W_i = \mathcal{S}_a^-(V_i)$ is*

$$\dim W_i = \begin{cases} -\dim V_a + \sum_{\alpha \in \mathcal{E}_a} \dim V_{t(\alpha)} & \text{for } i = a, \\ \\ \dim V_i & \text{for } i \neq a. \end{cases}$$

(iii) *the object $\ker p_V^a$ is concentrated at vertex a.*

(iv) *$V \simeq \mathcal{S}_a^+\mathcal{S}_a^-(V) \bigoplus \ker p_V^a$ as representations.*

(v) *if the object V has the form $\mathcal{S}_a^+(X)$ for some object $X \in \operatorname{rep}_k Q'$, where $Q' = \sigma_a Q$, then p_V^a is an isomorphism.*

We are now going to use Proposition 4.2.4 and Propostion 4.2.5 to prove the next result regarding the properties of $\mathcal{S}_a^+$ and $\mathcal{S}_a^-$.

Theorem 4.2.6. *Let Q be a finte and connected quiver and let $V = (V_i, \phi_\alpha) \in \operatorname{rep}_k Q$ be an indecomposable representation.*

(i) *If a is a sink in Q we have to possible cases:*

(a) *$\mathcal{S}_a^+(V) = 0$ if and only if $V \simeq S(a)$.*

(b) *$\mathcal{S}_a^+(V)$ is an indecomposable representation in $\operatorname{rep}_k Q'$, where $Q' = \sigma_a Q$, $\mathcal{S}_a^- \mathcal{S}_a^+(V) = V$ and the dimension of the vector space $W_i = \mathcal{S}_a^+(V_i)$ is*

$$\dim W_i = \begin{cases} -\dim V_a + \sum\limits_{\alpha \in \mathcal{E}_a} \dim V_{s(\alpha)} & \textit{for } i = a, \\ \dim V_i & \textit{for } i \neq a. \end{cases} \tag{9}$$

(ii) *If a is a source in Q we have two possible cases:*

(a) *$\mathcal{S}_a^-(V) = 0$ if and only if $V \simeq S(a)$.*

(b) *$\mathcal{S}_a^-(V)$ is an indecomposable representation in $\operatorname{rep}_k Q'$, where $Q' = \sigma_a Q$, $\mathcal{S}_a^+ \mathcal{S}_a^-(V) = V$ and the dimension of the vector space $W_i = \mathcal{S}_a^-(V_i)$ is*

$$\dim W_i = \begin{cases} -\dim V_a + \sum\limits_{\alpha \in \mathcal{E}_a} \dim V_{t(\alpha)} & \textit{for } i = a \\ \dim V_i & \textit{for } i \neq a. \end{cases} \tag{10}$$

Proof. (i): Let $V \in \operatorname{rep}_k Q$ be an indecomposable object, and let a be a sink in Q. By Proposition 4.2.4 (iv) we have $V \simeq \mathcal{S}_a^- \mathcal{S}_a^+(V) \bigoplus \operatorname{coker} i_V^a$, but V is indecomposable by assumption, which implies that either

(a) $V = \operatorname{coker} i_V^a$. Then by Proposition 4.2.4 (iii) we get $V_i = (0)$ for every $i \neq a$, and since V is indecomposable we must have $V_a \simeq k$. That is $V \simeq S(a)$. It is also clear by the definition of $\mathcal{S}_a^+$ that if $V \simeq S(a)$ then $\mathcal{S}_a^+(V) = (0)$.

Or,

(b) $V = \mathcal{S}_a^- \mathcal{S}_a^+(V)$. It is then clear that since $\operatorname{coker} i_V^a = (0)$ the morphism i_V^a is an epimorphism, and by Proposition

4.2.4 (i) the morphism i_V^a is a monomorphism, so i_V^a is an isomorphism. Hence, by Proposition 4.2.4 (ii) the dimension of $\mathcal{S}_a^+(V_i) = W_i$ is as given in (9). Let $W = \mathcal{S}_a^+(V)$, we want to show that W is indecomposable. By Proposition 4.2.5 (v) the morphism $p_V^a : W \to \mathcal{S}_a^+\mathcal{S}_a^-(W)$ is an isomorphism. Suppose $W = W_1 \bigoplus W_2$. Then $V = \mathcal{S}_a^-(W) = \mathcal{S}_a^-(W_1) \bigoplus \mathcal{S}_a^-(W_2)$. Since V is indecomposable we must have that one of the terms is (0), suppose without loss of generality that $\mathcal{S}_a^-(W_2) = (0)$. Then we have that $p_V^a(W_2) \subset \mathcal{S}_a^+\mathcal{S}_a^-(W_2) = \mathcal{S}_a^+(0) = (0)$, which implies that $W_2 = (0)$, and W is indecomposable.

(ii): Proven the same way as (i). □

Corollary 4.2.7. *Let Q be a finite and connected quiver, and let $\{a_1, \ldots, a_n\}$ be an admissible sequence of sinks.*

(i) *For any $1 \leq i \leq n$, let $S(a_i) \in \operatorname{rep}_k(\sigma_{a_{i-1}} \cdots \sigma_{a_2}\sigma_{a_1} Q)$. Then the representation $\mathcal{S}_{a_1}^- \cdots \mathcal{S}_{a_{i-1}}^-(S(a_i))$ is either (0) or an indecomposable object in $\operatorname{rep}_k Q$.*

(ii) *Let $V \in \operatorname{rep}_k Q$ be an indecomposable object, and $\mathcal{S}_{a_n}^+ \cdots \mathcal{S}_{a_2}^+\mathcal{S}_{a_1}^+(V) = (0)$. Then for some i we have*

$$V \simeq \mathcal{S}_{a_1}^-\mathcal{S}_{a_2}^- \ldots \mathcal{S}_{a_{i-1}}^-(S(a_i))$$

as representations.

Proof. Follows directly from consecutive use of Theorem 4.2.6. □

Our next result states that once we have classified the indecomposable objects of $\operatorname{rep}_k Q$, for a finite, connected and acyclic quiver Q, we have a way to classify the indecomposable objects of all categories $\operatorname{rep}_k Q'$, where Q' is some quiver having the same underlying graph as Q.

Theorem 4.2.8. *Let Q and Q' be two finite, connected and acyclic quivers with no multiple arrows having the same underlying graph $\overline{Q}$.*

(i) *There exists an admissible sequence of sinks $\{a_1, \dots, a_n\}$ in Q such that $\sigma_{a_n} \cdots \sigma_{a_2}\sigma_{a_1} Q = Q'$.*

(ii) *Let $\operatorname{ind} Q$ and $\operatorname{ind} Q'$ be complete sets of indecomposable objects in respectively $\operatorname{rep}_k Q$ and $\operatorname{rep}_k Q'$. Let $\mathcal{M} \subset \operatorname{ind} Q$ be the set of objects of the form $\mathcal{S}^-_{a_1}\mathcal{S}^-_{a_2} \cdots \mathcal{S}^-_{a_{i-1}}(S(a_i))$ for $1 \leq i \leq n$ and $\mathcal{M}' \subset \operatorname{ind} Q'$ be the set of objects of the form $\mathcal{S}^+_{a_n} \cdots \mathcal{S}^+_{a_{i+1}}(S(a_i))$ for $1 \leq i \leq n$. Then the functor $\mathcal{S}^+_{a_n} \cdots \mathcal{S}^+_{a_2}\mathcal{S}^+_{a_1}$ sets up a one-to-one correspondence between $\operatorname{ind} Q \setminus \mathcal{M}$ and $\operatorname{ind} Q' \setminus \mathcal{M}'$.*

Proof. (i): It is sufficient to consider two quivers Q and Q' that differ at only one arrow, say α. Since, in particular, Q is connected and contains no multiple arrows it is clear that $Q/\langle\alpha\rangle$ splits into two connected quivers. Let $\tilde{Q}$ be the component of $Q/\langle\alpha\rangle$ containing the vertex $t(\alpha)$ with respect to Q. Let $\{a_1, \dots, a_n\}$ be an admissible sequence of sinks in $\tilde{Q}$, and therefore also in Q. Observe that in $\sigma_{a_n} \cdots \sigma_{a_1}\tilde{Q}$ we have changed the direction of each arrow in $\tilde{Q}$ twice, except from the direction of α which has been changed only once. Hence $\sigma_{a_n} \cdots \sigma_{a_1} Q = Q'$.

(ii): Let $\{a_1, \dots, a_n\}$ be an admissible sequence of sinks in Q such that $\sigma_{a_n} \cdots \sigma_{a_1} Q = Q'$ (which we know exists by (i)). Then $\{a_n, \dots, a_1\}$ is an admissible sequence of sources in Q'. Let $\phi^+_i = \mathcal{S}^+_{a_i} \cdots \mathcal{S}^+_{a_2}\mathcal{S}^+_{a_1}$ and $\phi^-_i = \mathcal{S}^-_{a_i} \cdots \mathcal{S}^-_{a_{n-1}}\mathcal{S}^-_{a_n}$ for $i = 1, \dots, n$. To prove (b) we now want to show that

$$\phi^+_n : \operatorname{ind} Q \setminus \mathcal{M} \to \operatorname{ind} Q' \setminus \mathcal{M}'$$

is both injective and surjective.

Suppose $V_1, V_2 \in \operatorname{ind} Q \setminus \mathcal{M}$ such that $\phi_n^+(V_1) = \phi_n^+(V_2)$. We want to show that this implies $V_1 = V_2$. By Corollary 4.2.7 (ii) we get that $\phi_i^+(V) \neq (0)$ for any $1 \leq i \leq n$. Hence, by repeated use of Theorem 4.2.6 $(i)(b)$ we get $\mathcal{S}_{a_n}^- \phi_n^+(V_1) = \mathcal{S}_{a_n}^- \phi_n^+(V_2)$, which implies $\phi_{n-1}^+(V_1) = \phi_{n-1}^+(V_2)$. By proceeding similarly we get $\mathcal{S}_{a_1}^- \mathcal{S}_{a_1}^+(V_1) = \mathcal{S}_{a_1}^- \mathcal{S}_{a_1}^+(V_2)$, which implies $V_1 = V_2$ by Theorem 4.2.6. Hence ϕ_n^+ is injective.

Now, let $W \in \operatorname{ind} Q' \setminus \mathcal{M}'$. Then again, by repeated use of Theorem 4.2.6, we have that $\phi_i^-(W)$ is an indecomposable object in $\operatorname{rep}_k \sigma_{a_i} \cdots \sigma_{a_n} Q'$ for $1 \leq i \leq n$. In particular, $\phi_1^-(W) \in \operatorname{rep}_k Q$. Consecutive use of Theorem 4.2.6 gives $\phi_n^+(\phi_1^-(W)) = W$. Hence ϕ_n^+ is onto. $\square$

The proof can be extended to quivers with multiple arrows. In particular, Theorem 4.2.8 implies that if kQ is of finite representation type, then kQ' is of finite representation type for every Q' having $\overline{Q}$ as its underlying graph.

Next we will introduce some combination of reflection functors that takes the category $\operatorname{rep}_k Q$ into itself. These functors are called *Coxeter functors*.

Definition 4.2.9. Let Q be an finite, connected and acyclic quiver and let $\{a_1, \ldots, a_n\}$ be an admissible sequence of sinks in Q. Let $C^+, C^- : \operatorname{rep}_k Q \to \operatorname{rep}_k Q$ denote the functors $\mathcal{S}_{a_n}^+ \cdots \mathcal{S}_{a_2}^+ \mathcal{S}_{a_1}^+$ and $\mathcal{S}_{a_1}^- \cdots \mathcal{S}_{a_{n-1}}^- \mathcal{S}_{a_n}^-$ respectively. The functors C^+, C^- are called the *Coxeter functors* of $\operatorname{rep}_k Q$.

Let us check that C^+, C^- are well-defined. That is, we need to check that they do not depend on the choice of admissible numbering. First, observe that if both vertices a_i and a_j are sinks in some quiver Q, then there is no arrow joining a_i and a_j, and thus the functors $\mathcal{S}_{a_i}^+$ and $\mathcal{S}_{a_j}^+$ commute. That is, $\mathcal{S}_{a_i}^+ \mathcal{S}_{a_j}^+ = \mathcal{S}_{a_j}^+ \mathcal{S}_{a_i}^+$.

Now, let $\{a_1, \ldots, a_n\}$ and $\{a'_1, \ldots, a'_n\}$ be two admissible sequences of sinks in a quiver Q. Suppose $a_1 = a'_m$. Then a'_m is a sink in Q. Hence, there is no arrow adjoining a'_m and a'_i for $a'_i < a'_m$ by the definition of an admissible numbering. This implies that there is also no arrow adjoining a_1 and a'_i for $a'_i < a'_m$, which by consecutive use of the observation in the previous paragraph implies that $\mathcal{S}^+_{a'_m} \cdots \mathcal{S}^+_{a'_1} = \mathcal{S}^+_{a'_{m-1}} \cdots \mathcal{S}^+_{a'_1}\mathcal{S}^+_{a_1}$, and more generally $\mathcal{S}^+_{a'_n} \cdots \mathcal{S}^+_{a'_m} \cdots \mathcal{S}^+_{a'_1} = \mathcal{S}^+_{a'_n} \cdots \mathcal{S}^+_{a_{m+1}}\mathcal{S}^+_{a_{m-1}} \cdots \mathcal{S}^+_{a'_1}\mathcal{S}^+_{a_1}$. The same argument can be applied to the vertices $a_2, \ldots, a_n$ to obtain $\mathcal{S}^+_{a'_n} \cdots \mathcal{S}^+_{a'_1} = \mathcal{S}^+_{a_n} \cdots \mathcal{S}^+_{a_1}$. This shows that C^+ is well-defined. The same type of argument can be used to prove that C^- is well-defined.

4.3 QUADRATIC FORM OF A QUIVER

In this section we introduce some notions and prove some results needed for the proof of Gabriel's Theorem. Throughout this section, let Q denote a finite, connected and acyclic quiver, and $\mathbf{x} = (x_i)$ denote a vector in $\mathbb{Q}^n$, where $n = |Q_0|$ and $i \in Q_0$, unless stated otherwise. We start by introducing some notation on vectors.

Definition 4.3.1. A vector $\mathbf{x}$ is called

(i) *integral* if $x_i \in \mathbb{Z}$ for all $i \in Q_0$.

(ii) *positive* if $\mathbf{x}$ is not the zero vector, and $x_i \geq 0$ for all $i \in Q_0$. If a vector $\mathbf{x}$ is positive we write $\mathbf{x} > 0$. We write $\mathbf{x} < 0$ if $\mathbf{x}$ is non-positive.

Definition 4.3.2. The *quadratic form q_Q of a quiver Q* is defined by

$$q_Q(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)}x_{t(\alpha)}.$$

Let $\langle\ ,\ \rangle : \mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Q}$ be the corresponding symmetric bilinear form given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i \in Q_0} x_i y_i - \frac{1}{2} \sum_{\alpha \in Q_1} (x_{s(\alpha)} y_{t(\alpha)} + x_{t(\alpha)} y_{s(\alpha)}).$$

In this thesis we will only apply the quadratic form q_Q to integral vectors, in particular dimension vectors, which are to be defined later. So in our case, q_Q is an *integral quadratic form*.

Remark 4.3.3. (i) Observe that $q_Q(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle$. This is also clear by the definition of a bilinear form.

(ii) The bilinear form $\langle\ ,\ \rangle$ is called symmetric because $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

The next definition collects some classifications of the quadratic form q_Q.

Definition 4.3.4. The quadratic form q_Q is called

(i) *positive definite* if $q_Q(\mathbf{x}) > 0$ for every $\mathbf{x} \neq 0$.

(ii) *positive semidefinite* if $q_Q(\mathbf{x}) \geq 0$ for every $\mathbf{x} \in \mathbb{Z}^n$.

(iii) *weakly positive* if $q_Q(\mathbf{x}) > 0$ for all $\mathbf{x} > 0$.

Definition 4.3.5. A vector $\mathbf{x}$ is called a *root of* q_Q if $q_Q(\mathbf{x}) = 1$.

The quadratic form q_Q will be very important in the proof of Gabriel's Theorem. In fact, if q_Q is positive definite, there is a one-to-one correspondence between the positive roots of q_Q and the isomorphism classes of indecomposable objects of $\operatorname{rep}_k Q$. We will study this one-to-one correspondence later.

Lemma 4.3.6. *If q_Q is a weakly positive integral quadratic form on $\mathbb{Z}^n$, then it has only finitely many positive roots.*

Proof. Cf. [1]. □

Definition 4.3.7. Let $\zeta_a : \mathbb{Q}^n \to \mathbb{Q}^n$ be the linear transformation defined for each $a \in Q_0$ by

$$(\zeta_a(\mathbf{x}))_i = \begin{cases} x_i & \text{for } i \neq a, \\ -x_a + \sum_{\alpha \in \mathcal{E}_a} x_{e(\alpha)} & \text{for } i = a, \end{cases}$$

where $e(\alpha)$ is the vertex connected to α that is not a. The linear transformation ζ_a is called a *reflection* at a.

For every $a \in Q_0$ denote by $\mathbf{e}_a$ the vector in $\mathbb{Q}^n$ such that

$$(e_a)_i = \begin{cases} 0 & \text{for } i \neq a \\ 1 & \text{for } i = a \end{cases}.$$

Observe that $\zeta_a(\mathbf{e}_a) = -\mathbf{e}_a$ for every $a \in Q_0$.

Corollary 4.3.8. *The reflection at a can be expressed in the following way:* $\zeta_a(\boldsymbol{x}) = \boldsymbol{x} - 2\langle \boldsymbol{x}, \boldsymbol{e}_a \rangle \boldsymbol{e}_a$.

Proof. Can be easily verified from the definition of $\langle\ ,\ \rangle$. $\square$

Proposition 4.3.9. *Let ζ_a be a reflection. Then*

(*i*) *ζ_a is a group homomorphism.*

(*ii*) $\langle \zeta_a(\boldsymbol{x}), \zeta_a(\boldsymbol{y}) \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}^n$.

(*iii*) $\zeta_a^2 = 1$, *and thus ζ_a is an automorphism of $\mathbb{Q}^n$.*

Proof. (*i*): Simply verify.

(*ii*): We use Corollary 4.3.8:

$$\begin{aligned} \langle \zeta_a(\mathbf{x}), \zeta_a(\mathbf{y}) \rangle &= \langle \mathbf{x} - 2\langle \mathbf{x}, \mathbf{e}_a \rangle \mathbf{e}_a, \mathbf{y} - 2\langle \mathbf{y}, \mathbf{e}_a \rangle \mathbf{e}_a \rangle \\ &= \langle \mathbf{x}, \mathbf{y} \rangle - 2\langle \mathbf{y}, \mathbf{e}_a \rangle \langle \mathbf{x}, \mathbf{e}_a \rangle - 2\langle \mathbf{x}, \mathbf{e}_a \rangle \langle \mathbf{e}_a, \mathbf{y} \rangle \\ &\quad + 4\langle \mathbf{x}, \mathbf{e}_a \rangle \langle \mathbf{y}, \mathbf{e}_a \rangle \cdot 1 \\ &= \langle \mathbf{x}, \mathbf{y} \rangle. \end{aligned}$$

(iii): We use Corollary 4.3.8:

$$\begin{aligned}\zeta_a^2(\mathbf{x}) &= \zeta_a(\mathbf{x} - 2\langle \mathbf{x}, \mathbf{e}_a \rangle \mathbf{e}_a) = \zeta_a(\mathbf{x}) - 2\langle \mathbf{x}, \mathbf{e}_a \rangle \zeta_a(\mathbf{e}_a) \\ &= \mathbf{x} - 2\langle \mathbf{x}, \mathbf{e}_a \rangle \mathbf{e}_a + 2\langle \mathbf{x}, \mathbf{e}_a \rangle \mathbf{e}_a \\ &= \mathbf{x}.\end{aligned}$$

□

Definition 4.3.10. The subgroup $\mathcal{W}$ of the automorphism group on $\mathbb{Q}^n$ generated by the reflections ζ_a for every $a \in Q_0$ is called the *Weyl group of* q_Q. A root $\mathbf{x}$ of q_Q is called a *Weyl root* if there exists an $\omega \in \mathcal{W}$ such that $\mathbf{x} = \omega(\mathbf{e}_a)$ for some $a \in Q_0$.

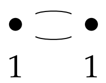
The next result shows that the quivers Q having q_Q positive definite is of special interest for us.

Theorem 4.3.11. *Let Q be a quiver, not necessarily acyclic. Then q_Q is positive definite if and only if $\overline{Q}$ is a Dynkin diagram.*

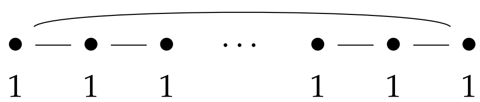
Proof. The proof is divided into four parts. In the first part we investigate the shape of Q, then in part two we establish a new quadratic form, which we will use in part three and four to investigate the size of Q.

(i) Let Q be a quiver, such that $\overline{Q}$ has one of the following graphs as a subgraph.

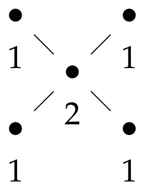
G_1



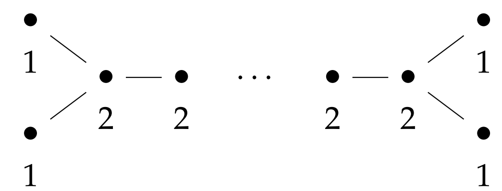
G_2



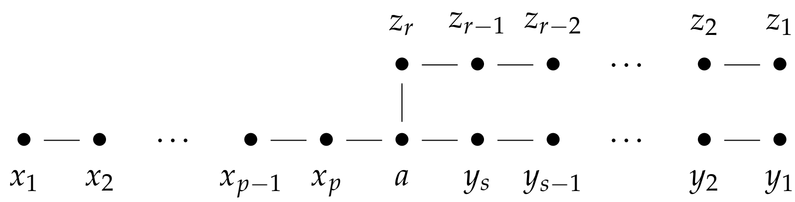
G_3



G_4



The numberings of the vertices of G_1, G_2, G_3, G_4 are chosen such that we can construct a vector $\mathbf{y} \in \mathbb{Q}^{|Q_0|}$ having $q_Q(\mathbf{y}) \leq 0$. The vector $\mathbf{y}$ is constructed in the following way: let the number on each vertex be the element in the corresponding coordinate of $\mathbf{y}$, and let the remaining coordinate be filled with zeroes. Then, as predicted, $q_Q(\mathbf{y}) \leq 0$, and q_Q is neither positive definite nor positive semidefinite. Since we are searching for the cases where q_Q is positive definite this tells us quite a lot about the shape of $\overline{Q}$. We can conclude by considering G_1 and G_2 that $\overline{Q}$ must be acyclic, from G_3 we find that each vertex can not have more than three edges, and from G_4 we see that there can not be more than one vertex having three edges. Hence $\overline{Q}$ must be of form



(11)

where $p, s, r \in \mathbb{Z}^+ \cup 0$.

(*ii*) For each $t \geq 0$ consider the quadratic form in $t+1$ variables $x_1, \ldots, x_{t+1}$:

$$\begin{aligned} C_t(x_1, \ldots, x_{t+1}) &= -x_1x_2 - \cdots - x_tx_{t+1} \\ &\qquad + x_1^2 + \cdots + x_t^2 + \frac{t}{2(t+1)}x_{t+1}^2 \\ &= \sum_{i=1}^{t} \frac{i}{2(i+1)} \left(x_{i+1} - \frac{i+1}{i} x_i \right)^2 . \end{aligned}$$

From the above formula it can be observed that C_t is positive semidefinite, that the dimension of the null space of C_t is 1 and that for any nonzero vector $\mathbf{v}$ such that $C_t(\mathbf{v}) = 0$ we have that all coordinates of $\mathbf{v}$ are nonzero.

(*iii*) Let $x_1, \ldots, x_p, y_1, \ldots, y_s, z_1, \ldots, z_r, a$ be the vertices of Q, as in the graph (11). Let $\mathbf{x} = (x_1, \ldots, x_p)$, $\mathbf{y} = (y_1, \ldots, y_s)$ and $\mathbf{z} = (z_1, \ldots, z_r)$. Then, by part (*ii*),

$$\begin{aligned} q_Q(\mathbf{x}, \mathbf{y}, \mathbf{z}, a) &= C_p(x_1, \ldots, x_p, a) + C_q(y_1, \ldots, y_s, a) \\ &\quad + C_p(z_1, \ldots, z_r, a) \\ &\quad + \left(1 - \frac{p}{2(p+1)} - \frac{s}{2(s+1)} - \frac{r}{2(r+1)}\right) a^2. \end{aligned}$$

We now want to investigate what requirements the integers p, s, r need to fulfill in order for the quadratic form q_Q to be positive definite. Since C_t is positive semidefinite it is clear that q_Q is positive semidefinite if and only if $\frac{p}{2(p+1)} + \frac{s}{2(s+1)} + \frac{r}{2(r+1)} < 1$. From part two of the proof we know that if $C_t(\mathbf{v}) = 0$ all coordinates of $\mathbf{v}$ is nonzero. Hence, q_Q is positive definite if and only if $\frac{p}{2(p+1)} + \frac{s}{2(s+1)} + \frac{r}{2(r+1)} < 1$, which is equivalent to $\frac{1}{p+1} + \frac{1}{s+1} + \frac{1}{r+1} > 1$.

(*iv*) Suppose without loss of generality that $p \leq s \leq r$, and let $\delta = \frac{1}{p+1} + \frac{1}{s+1} + \frac{1}{r+1}$. Suppose $\delta > 1$, we then want to study the possible values of p, s, r. We can see immediately that we must have $p \leq 2$ for $\delta > 1$. Hence we have the following cases:

(a) $p = 0$, s and r arbitrary positive integers. Then graph (11) coincides with the Dynkin diagram A_n for $n \geq 1$.

(b) $p = 1$, $s = 1$ and $r \geq 1$. Then the graph (11) coincides with the Dynkin diagram D_n for $n \geq 4$.

(c) $p = 1$, $s = 2$ and $r = 2, 3, 4$. Then the graph (11) coincides respectively with the Dynkin diagrams E_6, E_7 and E_8.

□

Corollary 4.3.12. *Let Q be a quiver whose underlying graph is a Dynkin diagram. Then the integral quadratic form q_Q has only finitely many positive roots.*

Proof. By Theorem 4.3.11 we get that q_Q is positive definite. It is clear that if q_Q is positive definite, then it is in particular weakly positive. Then by Lemma 4.3.6 we get that q_Q has only finitely many positive roots. □

Lemma 4.3.13. *If the quadratic form q_Q is positive definite, then the Weyl group $\mathcal{W}$ is finite.*

Proof. Let S_1 denote the set of all positive roots of q_Q, and consider the map $f : \mathcal{W} \to S_1^n$ defined by $\omega \mapsto (\omega e_a)_{a \in Q_0}$. The map f can be shown to be well-defined. Observe that $f(\omega) = 0$ implies that column a of ω must be (0) for every a, that is $\ker f = (0)$. Hence, f is injective. Since q_Q is positive definite the set S_1 is finite by Corollary 4.3.12, and hence $\mathcal{W}$ is finite since f is injective. □

Lemma 4.3.14. *Let Q be a quiver whose underlying graph is a Dynkin diagram, let $\boldsymbol{x}$ be a positive root of q_Q, and let a be a vertex of Q_0. Then either $\zeta_a(\boldsymbol{x})$ is positive or $\boldsymbol{x} = \boldsymbol{e}_a$.*

Proof. Since $\mathbf{x}$ is a root of q_Q we have that $q_Q(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle = 1$. Then by Proposition 4.3.9 (ii) we get $q_Q(\zeta_a(\mathbf{x})) = \langle \zeta_a(\mathbf{x}), \zeta_a(\mathbf{x}) \rangle = \langle \mathbf{x}, \mathbf{x} \rangle = 1$, and hence $\zeta_a(\mathbf{x})$ is also a root of q_Q. By Theorem 4.3.11 the quadratic form q_Q is positive definite, and hence:

$$\begin{aligned} q_Q(\mathbf{x} \pm \mathbf{e}_a) &= \langle \mathbf{x} \pm \mathbf{e}_a, \mathbf{x} \pm \mathbf{e}_a \rangle = q_Q(\mathbf{x}) + q_Q(\mathbf{e}_a) \pm 2\langle \mathbf{x}, \mathbf{e}_a \rangle \\ &= 1 + 1 \pm 2\langle \mathbf{x}, \mathbf{e}_a \rangle = 2(1 \pm \langle \mathbf{x}, \mathbf{e}_a \rangle) \geq 0. \end{aligned}$$

This implies $-1 \leq \langle \mathbf{x}, \mathbf{e}_a \rangle \leq 1$. Since $\langle \mathbf{x}, \mathbf{e}_a \rangle \in \mathbb{Z}$ we only have three possibilities; $\langle \mathbf{x}, \mathbf{e}_a \rangle = 1$, $\langle \mathbf{x}, \mathbf{e}_a \rangle = 0$ or $\langle \mathbf{x}, \mathbf{e}_a \rangle = -1$. If $\langle \mathbf{x}, \mathbf{e}_a \rangle = 1$ we get that $q_Q(\mathbf{x} - \mathbf{e}_a) = 0$, which implies $\mathbf{x} = \mathbf{e}_a$. Now, if $\langle \mathbf{x}, \mathbf{e}_a \rangle \leq 0$ we have that $\zeta_a(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{x}, \mathbf{e}_a \rangle > 0$ since $\mathbf{x} > 0$. $\square$

In particular, the previous lemma shows that if Q is a quiver whose underlying graph is a Dynkin diagram, the reflection ζ_a sends roots of q_Q onto roots of q_Q.

Let Q be a quiver, and let $a_1, \ldots, a_n$ be some numbering of its vertices. An element $c = \zeta_{a_n} \cdots \zeta_{a_1}$ of the Weyl group $\mathcal{W}$ is called a *Coxeter transformation*. Since $\zeta_{a_i}^2 = 1$ we have that $c^{-1} = \zeta_{a_1} \cdots \zeta_{a_n}$.

Lemma 4.3.15. *Let Q be a quiver whose underlying graph is a Dynkin diagram, and let c be its Coxeter transformation. Then*

(i) *there exists no nonzero vector $\boldsymbol{x} \in \mathbb{Q}^n$ such that $c(\boldsymbol{x}) = \boldsymbol{x}$.*

(ii) *if $\boldsymbol{x} \neq 0$, then there exists some integer $s \geq 0$ such that the vector $c^{s-1}(\boldsymbol{x}) > 0$, but $c^s(\boldsymbol{x}) < 0$. Also, there exists some integer $t \geq 0$ such that $c^{-t-1}(\boldsymbol{x}) > 0$, but $c^{-t}(\boldsymbol{x}) < 0$.*

Proof. (i): Suppose by contradiction that $\mathbf{x} \neq 0$ is such that $c(\mathbf{x}) = \mathbf{x}$. By the definition of the reflections ζ_a for $a \in Q_0$, the reflections $\zeta_{a_n}, \dots, \zeta_{a_2}$ do not change the a_1th coordinate of x, so we have that $(\zeta_{a_1}(\mathbf{x}))_{a_1} = (c(\mathbf{x}))_{a_1} = x_{a_1}$. Hence, $\zeta_{a_1}(\mathbf{x}) = \mathbf{x}$. The same argument holds for $1 \leq i \leq n$, that is $\zeta_{a_i}(\mathbf{x}) = \mathbf{x}$. Then by Corollary 4.3.8 we get $\zeta_{a_i}(\mathbf{x}) = \mathbf{x} - 2\langle \mathbf{e}_{a_i}, \mathbf{x}\rangle \mathbf{e}_{a_i} = \mathbf{x}$. Hence, we must have $\langle \mathbf{e}_{a_i}, \mathbf{x}\rangle = 0$ for all $i \in \{1, \dots, n\}$. Since $\mathbf{e}_{a_i} \neq 0$ it is clear by the definition of $\langle\ ,\ \rangle$ that $\mathbf{x} = 0$, which is a contradiction.

(ii): By Theorem 4.3.11 the quadratic form q_Q is positive definite. Then by Lemma 4.3.13 the Weyl group $\mathcal{W}$ is finite. Hence there must exist some integer h such that $c^h = 1$. Suppose all the vectors $\mathbf{x}, c(\mathbf{x}), \dots, c^{h-1}(\mathbf{x})$ are positive. Then clearly $\mathbf{y} = \mathbf{x} + c(\mathbf{x}) + \cdots + c^{h-1}(\mathbf{x})$ is positive, and in particular nonzero. Then observe that $c(\mathbf{y}) = \mathbf{y}$, which contradicts (i). Hence there exists a least integer s such that $c^{s-1}(\mathbf{x}) > 0$ and $c^s(\mathbf{x}) < 0$, $0 \leq s \leq h-1$. Similarly, find the integer t as required. □

Lemma 4.3.16. *Let Q be a quiver whose underlying graph is a Dynkin diagram, and let c be its Coxeter transformation. Let $\mathbf{x}$ denote a positive root of the quadratic form q_Q. Then*

(i) $c(\mathbf{x}) < 0$ if and only if $\mathbf{x} = \zeta_1 \cdots \zeta_{i-1}(\mathbf{e}_i)$ for some $1 \leq i \leq n$. We denote $\mathbf{p}_i = \zeta_1 \cdots \zeta_{i-1}(\mathbf{e}_i)$.

(ii) $c^{-1}(\mathbf{x}) < 0$ if and only if $\mathbf{x} = \zeta_n \cdots \zeta_{i+1}(\mathbf{e}_i)$ for some $1 \leq i \leq n$. We denote $\mathbf{q}_i = \zeta_n \cdots \zeta_{i+1}(\mathbf{e}_i)$.

Proof. (i): Suppose $c(\mathbf{x}) = \zeta_n \cdots \zeta_1(\mathbf{x})$ is not a positive vector. Then there must exist a least integer $1 \leq i \leq n$ such that $\zeta_{i-1} \cdots \zeta_1(\mathbf{x}) > 0$, but $\zeta_i \cdots \zeta_1(\mathbf{x}) < 0$. By the remark following Lemma 4.3.14 we have that $\zeta_1(\mathbf{x})$ is a root of q_Q. Preceding similarly we get that $\zeta_{i-1} \cdots \zeta_1(\mathbf{x})$ is a root. Since, by assumption, $\zeta_i \cdots \zeta_1(\mathbf{x}) < 0$ we must have $\zeta_{i-1} \cdots \zeta_1(\mathbf{x}) = \mathbf{e}_i$ by Lemma 4.3.14. Hence

$$\mathbf{x} = (\zeta_{i-1} \cdots \zeta_1)^{-1}(\mathbf{e}_i) = \zeta_1 \cdots \zeta_{i-1}(\mathbf{e}_i).$$

Now suppose $\mathbf{x} = \zeta_1 \cdots \zeta_{i-1}(\mathbf{e}_i)$. Then $c(\mathbf{x}) = \zeta_n \cdots \zeta_i \zeta_{i-1} \cdots \zeta_1 \zeta_{i-1} \cdots \zeta_1(\mathbf{e}_i)$. The reflection ζ_i is the only reflection affecting the element in coordinate i, and ζ_i appears only once. Hence $c(\mathbf{x}) = -\mathbf{e}_i < 0$.

(ii): The proof is the similar to the proof of (i). □

We can use the last two results to collect all positive roots of the quadratic form of a quiver whose underlying graph is a Dynkin diagram. This will become important because of the predicted one-to-one correspondence between these roots and the isomorphism classes of indecomposable objects of $\operatorname{rep}_k Q$.

Proposition 4.3.17. *Let Q be a quiver whose underlying graph is a Dynkin diagram, and let c be its Coxeter transformation.*

(i) *Let m_i denote the least integer such that $c^{-m_i-1}(\boldsymbol{p}_i) < 0$, where $\boldsymbol{p}_i$ is as in Lemma 4.3.16. Then the set*

$$\{c^{-s}(\boldsymbol{p}_i) \mid 1 \leq i \leq n, 0 \leq s \leq m_i\}$$

equals the set of all positive roots of q_Q.

(ii) *Let n_i denote the least integer such that $c^{n_i+1}(\boldsymbol{q}_i) < 0$, where $\boldsymbol{q}_i$ is as in Lemma 4.3.16. Then the set*

$$\{c^{t}(\boldsymbol{q}_i) \mid 1 \leq i \leq n, 0 \leq t \leq n_i\}$$

equals the set of all positive roots of q_Q.

Proof. (i): Observe that since $c^{-s}(\mathbf{p}_i) = (\zeta_n \cdots \zeta_1)^{-s}\zeta_1 \cdots \zeta_{i-1}(\mathbf{e}_i) > 0$ we must have $c^{-s}(\mathbf{p}_i) = \mathbf{e}_i$.

Hence it is clear that $c^{-s}(\mathbf{p}_i)$ is a positive root of q_Q. Then it remains to show that all positive roots of q_Q is of this form. Let $\mathbf{x}$ be a positive root of q_Q. By Lemma 4.3.15 there exists some integer s such that $c^{s-1}(\mathbf{x}) > 0$, but $c^s(\mathbf{x}) < 0$. Hence, recalling the remark following Lemma 4.3.14, it is clear that $c^{s-1}(\mathbf{x})$ is also a positive root of q_Q. Then by Lemma 4.3.16 we get $c(c^{s-1}(\mathbf{x})) = c^s(\mathbf{x}) < 0$ if and only if $c^{s-1}(\mathbf{x}) = \mathbf{p}_i$ for some $1 \leq i \leq n$. Hence we must have $\mathbf{x} = c^{-s+1}(\mathbf{p}_i)$, and $s - 1 \leq m_i$.

(ii): The proof is similar to the proof of (i). □

4.4 GABRIEL'S THEOREM

Now we are almost ready to state and prove Gabriel's theorem. We only need one last definition, and a few more results.

Definition 4.4.1. Let $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be a representation of a finite, connected and acyclic quiver Q. The *dimension vector* $\mathbf{dim}\ V$ is defined to be

$$\mathbf{dim}\ V = \begin{bmatrix} \dim V_1 \\ \vdots \\ \dim V_n \end{bmatrix} \in \mathbb{Z}^n,$$

where $n = |Q_0|$.

Using the above definition and the notation established in the previous section, we will now reformulate Theorem 4.2.6. This reformulation connects the reflection functors and $\mathcal{S}_a^{+/-}$ and the reflections ζ_a for a sink/source a.

Theorem 4.4.2. *Let Q be a finite, connected and acyclic quiver, and let $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be an indecomposable representation in* $\operatorname{rep}_k Q$.

(i) *If a is a sink in Q we have two possible cases:*

(a) $\mathcal{S}_a^+(V) = 0$ *if and only if* $V \simeq S(a)$ *and* $\boldsymbol{dim}\,\mathcal{S}_a^+(V) \neq \zeta_a(\boldsymbol{dim}\,V) < 0$.

(b) $\mathcal{S}_a^+(V)$ *is indecomposable and* $\boldsymbol{dim}\,\mathcal{S}_a^+(V) = \zeta_a(\boldsymbol{dim}\,V)$.

(ii) *If* a *is a source in* Q *we have two possible cases:*

(a) $\mathcal{S}_a^-(V) = 0$ *if and only if* $V \simeq S(a)$ *and* $\boldsymbol{dim}\,V \neq \zeta_a(\boldsymbol{dim}\,V) < 0$.

(b) $\mathcal{S}_a^-(V)$ *is indecomposable and* $\boldsymbol{dim}\,\mathcal{S}_a^-(V) = \zeta_a(\boldsymbol{dim}\,V)$.

Corollary 4.4.3. *Let* Q *be a finite, connected and acyclic quiver, let* $\{a_1, \ldots, a_n\}$ *be an admissible sequence of sinks and let* V *be an indecomposable object in* $\operatorname{rep}_k Q$. *Let* $\boldsymbol{m_j} = \zeta_{a_j} \cdots \zeta_{a_1}(\boldsymbol{dim}\,V)$ *and* $W_j = \mathcal{S}_{a_j}^+ \cdots \mathcal{S}_{a_1}^+(V)$.

(i) *If* $b \leq i \leq n$ *and* $\boldsymbol{m_i} > 0$, *then* $\boldsymbol{m_b} > 0$, W_b *is an indecomposable object in* $\operatorname{rep}_k Q$ *and* $\boldsymbol{dim}\,W_b = \boldsymbol{m_b}$.

(ii) *If* $c(\boldsymbol{dim}\,V) > 0$, *then* $C^+(V)$ *is an indecomposable object in* $\operatorname{rep}_k Q$ *and* $\boldsymbol{dim}\,(C^+(V)) = c(\boldsymbol{dim}\,V)$.

Proof. (i): Let $b \leq i \leq n$ and suppose $\mathbf{m_i} = \zeta_{a_i} \cdots \zeta_{a_1} > 0$. Suppose $\mathbf{m_b} < 0$. This would imply $\mathbf{m_i} < 0$ since $\zeta_{a_i} \cdots \zeta_{a_{b+1}}$ leave the coordinates $a_1, \ldots, a_b$ unchanged. This is a contradiction, and it is clear that we must have $\mathbf{m_b} > 0$. The fact that W_b is indecomposable in $\operatorname{rep}_k Q$ and that $\mathbf{dim}\,W_b = \mathbf{m_b}$ follows from consecutive use of Theorem 4.4.2.

(ii): Follows from (i) by setting $i = n$. □

A similar statement holds for $\{a_1, \ldots, a_n\}$ an admissible sequence of sources.

The next two results will be indispensable in our proof of Gabriel's Theorem.

Lemma 4.4.4. *Let Q be a quiver such that the path algebra kQ is of finite representation type. Then the quadratic form q_Q is positive definite.*

Tit's proof. Consider the representations $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1} \in \operatorname{rep}_k Q$ having $\mathbf{dim}\, V = \mathbf{x} = (x_i)_{i \in Q_0}$ and let $|Q_0| = n$. Then $V_i \simeq k^{x_i}$ for all $i \in Q_0$. If we fix a basis on each vector space V_i the representation V is completely determined by the set of matrices $\{M_\alpha\}_{\alpha \in Q_0}$, where M_α is the matrix corresponding to the linear map $f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$. Let g_i denote any non-singular $x_i \times x_i$-matrix over k and let $\mathscr{B}_i$ be the fixed basis of V_i. Then g_i takes the basis $\mathscr{B}_i$ to some other basis $\mathscr{B}'_i$ of V_i. Consider the diagram

$$\begin{tikzcd} V_{s(\alpha)} \arrow[r, "M_\alpha"] \arrow[d, "g_{s(\alpha)}"] & V_{t(\alpha)} \arrow[d, "g_{t(\alpha)}"] \\ V_{s(\alpha)} & V_{t(\alpha)} \end{tikzcd} \tag{12}$$

Let M be the manifold of all sets of matrices M_α over k for $\alpha \in Q_1$, and let G be the group of all sets of non-singular matrices g_i over k for $i \in Q_0$. By diagram (12) it is clear that the action of G on M must be $M'_\alpha = g_{t(\alpha)} M_\alpha g^{-1}_{s(\alpha)}$. The group G permutes the elements of M. Let M_α be an element of M. Then G makes M_α move in a fixed path, this path is called the *orbit* of M_α, or M_α's orbit in G.

We next claim that two objects of $\operatorname{rep}_k Q$ with the given dimension vector $\mathbf{x}$ are isomorphic if and only if the sets of matrices $\{M_\alpha\}_{\alpha \in Q_1}$ corresponding to them lie in the same orbit in G. Let V, V' be objects in $\operatorname{rep}_k Q$ such that $\dim V = \mathbf{x} = \dim V'$. Then

$V_i \simeq V'_i \simeq k^{x_i}$ for all $i \in Q_0$. It is clear that $V \simeq V'$ if and only if the following diagram commutes for every $\alpha \in Q_1$.

$$\begin{tikzcd} V_{s(\alpha)} \arrow[r, "M_\alpha"] \arrow[d, "g_{s(\alpha)}"] & V_{t(\alpha)} \arrow[d, "g_{t(\alpha)}"] \\ V_{s(\alpha)} \arrow[r, "M'_\alpha"] & V_{t(\alpha)} \end{tikzcd} \tag{13}$$

That is, $V \simeq V'$ if and only if $M'_\alpha = g_{t(\alpha)} M_\alpha g^{-1}_{s(\alpha)}$, which means M_α and M'_α are in the same orbit of G. Since this holds for every $\alpha \in Q_1$ that proves the claim.

By assumption the path algebra kQ is of finite representation type, which implies by Corollary 2.4.10 that there are only finitely many isomorphism classes of indecomposable representations in $\operatorname{rep}_k Q$. In particular there are only finitely many isomorphism classes of indecomposable representations V having dimension vector $\mathbf{dim}\, V = \mathbf{x}$. Hence we get by the above claim that the elements of M are divided into only a finite number of orbits in G.

Consider $G_0 \subset G$, where $G_0 = \{\lambda I_{x_1}, \ldots, \lambda I_{x_n} \mid \lambda \in k^*\}$. Observe that for $g \in G_0$ we get $M'_\alpha = g_{t(\alpha)} M_\alpha g^{-1}_{s(\alpha)} = \lambda\lambda^{-1} M_\alpha = M_\alpha$ for all $\alpha \in Q_1$. Hence, G_0 acts on M as the identity.

We get an onto morphism from G to each of the M_α's orbits in G, so $\dim M \leq \dim G$. Since G_0 acts on M as the identity we actually get $\dim M \leq \dim G - 1$. This argument requires the representations to be over an infinite field. The argument holds anyway, but requires further arguments. We have that $\dim G \leq \sum_{a \in Q_0} x_a^2$, while $\dim M = \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}$. Hence, by the above,

$$\sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)} \leq \dim G - 1 \leq \sum_{a \in Q_0} x_a^2 - 1.$$

This shows $q_Q(\mathbf{x}) \geq 1 > 0$ for $\mathbf{x} \neq 0$. Now, what remains is to show that $q_Q(\mathbf{x}) \geq q_Q(|\mathbf{x}|)$ to conclude that q_Q is positive definite. Observe that

$$q_Q(\mathbf{x}) \geq q_Q(|\mathbf{x}|) \Leftrightarrow \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)} \leq \sum_{\alpha \in Q_1} \left|x_{s(\alpha)}\right| \left|x_{t(\alpha)}\right|.$$

The latter clearly holds, so q_Q is positive definite. □

Lemma 4.4.5. *Let Q be a quiver whose underlying graph is a Dynkin diagram. Then the mapping $V \to$ **dim** V is a one-to-one correspondence between the set of isomorphism classes of indecomposable objects in* $\operatorname{rep}_k Q$ *and the positive roots of q_Q.*

Proof. Let $\overline{Q}$ be a Dynkin diagram, $\{a_1, \ldots, a_n\}$ be an admissible numbering of the vertices of Q and let V be an indecomposable object in $\operatorname{rep}_k Q$ such that $\mathbf{dim}\ V = \mathbf{x}$. We start by showing that $\mathbf{x}$ is a positive root of q_Q, before we show that the mapping is both injective and surjective.

By Theorem 4.3.11 the quadratic form q_Q is positive definite. Hence there exists a least integer s such that $c^{s-1}(\mathbf{x}) = (\zeta_{a_n} \cdots \zeta_{a_1})^{s-1}(\mathbf{x}) > 0$, but $c^s(\mathbf{x}) < 0$ by Lemma 4.3.15 (ii). This implies that there must exist some least $0 \leq t \leq n-1$ such that $\zeta_{a_t} \cdots \zeta_{a_1} c^{s-1}(\mathbf{x}) > 0$, but $\zeta_{a_{t+1}} \cdots \zeta_{a_1} c^{s-1}(\mathbf{x}) < 0$. Now, by consecutive use of Corollary 4.4.3 (ii) we get that $C^+(V)$, $(C^+)^2(V), \ldots, (C^+)^{s-1}(V)$ are indecomposable objects in $\operatorname{rep}_k Q$ and that

$$\mathbf{dim}\ (C^+)^j(V) = c^j(\mathbf{x})$$

for every $j \leq s-1$. Observe that $\zeta_{a_t} \cdots \zeta_{a_1} c^{s-1}(\mathbf{x}) = \zeta_{a_t} \cdots \zeta_{a_1} \mathbf{dim}\ (C^+)^{s-1}(V) > 0$. Hence $V' = \mathcal{S}^+_{a_t} \cdots \mathcal{S}^+_{a_1}(C^+)^{s-1}(V)$ is an indecomposable object of $\operatorname{rep}_k(\sigma_{a_t} \cdots \sigma_{a_1} Q)$ and

$$\mathbf{dim}\ V' = \mathbf{dim}\ (\mathcal{S}^+_{a_t} \cdots \mathcal{S}^+_{a_1}(C^+)^{s-1}(V)) = \zeta_{a_t} \cdots \zeta_{a_1} c^{s-1}(\mathbf{x})$$

by Corollary 4.4.3 (i). By the way t was chosen it is clear that $\zeta_{a_{t+1}}(\mathbf{dim}\, V') < 0$, and hence $V' \simeq S(a_{t+1})$ by Theorem 4.4.2 $(i)(a)$. Then clearly $\mathbf{dim}\, V' = \zeta_{a_t} \cdots \zeta_{a_1} c^{s-1}(\mathbf{x}) = \mathbf{e}_{a_{t+1}}$ and $\mathbf{x} = c^{-s+1}\zeta_{a_1} \cdots \zeta_{a_t}(\mathbf{e}_{a_{t+1}}) = c^{-s+1}\mathbf{p}_{a_{t+1}}$. Then by Proposition 4.3.17 the vector $\mathbf{x} = \mathbf{dim}\, V$ is a positive root of q_Q. Hence the mapping $V \to \mathbf{dim}\, V$ sends an indecomposable object of $\mathrm{rep}_k\, Q$ to a positive root of q_Q.

Next, let us show that the mapping is injective. We know that $V' = \mathcal{S}^+_{a_t} \cdots \mathcal{S}^+_{a_1}(C^+)^{s-1}(V)$ is indecomposable, and in particular $V' \simeq S(a_{t+1})$. Then Theorem 4.2.6 implies $V \simeq (C^-)^{-s+1}\mathcal{S}^-_{a_1} \cdots \mathcal{S}^-_{a_t}(S(a_{t+1}))$. Observe that the integers s, t only depend on the vector $\mathbf{x} = \mathbf{dim}\, V$. Then if V, W are two non-simple indecomposable representations having $\mathbf{dim}\, V = \mathbf{x} = \mathbf{dim}\, W$ we get that

$$\mathcal{S}^+_{a_t} \cdots \mathcal{S}^+_{a_1}(C^+)^{s-1}(V) \simeq S(a_{t+1}) \simeq \mathcal{S}^+_{a_t} \cdots \mathcal{S}^+_{a_1}(C^+)^{s-1}(W),$$

and hence

$$V \simeq (C^-)^{-s+1}\mathcal{S}^-_{a_1} \cdots \mathcal{S}^-_{a_t}(S(a_{t+1})) \simeq W.$$

If V, W are two simple representations having $\mathbf{dim}\, V = \mathbf{x} = \mathbf{dim}\, W$ it is obvious that $V \simeq W$. Thus the map is injective.

The last step is to show that the mapping is surjective. Let $\mathbf{x}$ be a positive root of q_Q. We then need to show that $\mathbf{x}$ is the dimension vector of some indecomposable representation V. By Proposition 4.3.17 the vector $\mathbf{x} = c^{-s}\mathbf{p}_{a_{i+1}} = c^{-s}\zeta_{a_1} \cdots \zeta_{a_i}(\mathbf{e}_{i+1})$ for some integers s, i. Then the indecomposable representation $V = (C^-)^s\mathcal{S}^-_{a_1} \cdots \mathcal{S}^-_{a_i}(S(a_{i+1}))$ satisfies $\mathbf{dim}\, V = \mathbf{x}$. □

The proof of Gabriel's Theorem will be closely connected to the previous results.

Theorem 4.4.6 (Gabriel's Theorem)**.** *Let Q be a quiver. The path algebra kQ is of finite representation type if and only if the underlying graph $\overline{Q}$ of Q is a Dynkin diagram.*

Proof. Let Q be a quiver whose underlying graph is a Dynkin diagram. Then q_Q has only finitely many roots by Corollary 4.3.12, which implies that there are only finitely many isomorphism classes of indecomposable objects in $\operatorname{rep}_k Q$ by Lemma 4.4.5. By Corollary 2.4.10 this implies that the path algebra kQ has only finitely many indecomposable finitely generated left kQ-modules, so kQ is of finite representation type.

Suppose kQ is of finite representation type. Then by Lemma 4.4.4 the quadratic form q_Q is positive definite. This implies by Theorem 4.3.11 that the underlying graph $\overline{Q}$ of Q is a Dynkin diagram. $\square$

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