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Waves Propagating on Fluids with Piecewise Constant Vorticity

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Abstract

The purpose of this study was to find the dispersion relation for waves on an arbitrary current velocity profile approximated by N linear layers.

Using a Stokes expansion to first order in the steepness parameter $\epsilon = ka$ the dispersion relation for systems with one, two and three linear layers was derived, a process revealing a general pattern for an increase in N. The pattern was then used to derive the dispersion relation for the N-layer system, which is shown valid even to second order theory. The dispersion relation was significantly simplified by a recursion formula which made it easy to solve numerically. To test the formula two possible current velocity profiles of water flowing down a river was investigated; a parabolic and a logarithmic profile.

The numerical analysis revealed that even a linear current velocity profile serve as a good approximation. The results also showed that we get $N + 1$ solutions. However, data supported a theory that only two of them are physically possible. The extra solutions probably originates from the unphysical bends in the velocity profile, and are not waves but layers of vortices drifting with the current.

Sammendrag (Norwegian)

Hensikten med dette arbeidet var å finne dispersjonsrelasjonen for bølger på en strømning med et tilfeldig hastighetsprofil ved å tilnærme hastighetsprofilet med N lineære lag.

Ved å bruke Stokes expansion til første orden i steilhetsparameteren $\epsilon = ka$ ble dispersionsrelasjonen for systemer med et, to og tre lag utleded, en prosess som avslørte et generelt mønster for en økning i N. Dette mønsteret ble så brukt for å utlede dispersjonsrelasjonen for et N-lags system, som også ble vist å være gyldig til andre ordens teori. Dispersjonsrelasjonen ble kraftig forenklet ved å bruke en rekursjonsformel og kunne enkelt løses numerisk. For å teste formelen ble to hastighetsprofiler for vann som strømmer ned en elv utforsket; et parabolsk og et logaritmisk profil.

Den numeriske analysen avslørte at selv et lineært profil er en god tilnærming. Resultatene viste også at vi får $N + 1$ løsninger. Imidlertid støtter dataene en teori om at kun to av løsningene er fysisk mulige. Resten av løsningen kommer antakeligvis fra den unaturlige knekken i hastighetsprofilet, og er ikke bølger men lag med virvler som driver med strmningen.

Preface

I was accepted as a student at NTNU (Norwegian university of science and technology) in 2010. After four years acquiring general physics knowledge I have spent the last year writing this master's thesis. From my stay at NTNU, I present to you, the jewel in the crown.

But first, I want to thank Simen Å. Ellingsen for the idea and Alex Hansen for letting me write this thesis outside the institute. Specially, I would like to thank my supervisors **Iver Brevik** and **Simen Å. Ellingsen** very much for all their help and patience during this year. I really can't thank you enough.

Highly appreciated was also the proofreading by Terje Toften and Iver Brevik.

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Ståle Toften

Look deep into nature and then you will understand everything better

- Albert Einstein

Symbols and Abbreviations

Numeral subscripts denote the layer number counting from 0 to N-1. The positional subscripts x and y represent partial derivatives.

A, B = Potential constants
C = Bernoulli constant
c = Wave velocity
c_p = Phase velocity
c_g = Group velocity
g = Constant of gravity
H = Finite water depth
h = Interface depth
k = Wavenumber
L = Characteristic channel length
N = Total number of layers
n = Specific layer
P = Pressure
R_e = Reynolds number
S = Shear
$\mathbf{S}$ = Vorticity vector
$S_{N-1,N} = S_{N-1} - S_N$
t = Time
U_0 = Surface current velocity
u = Velocity in x-direction
v = Velocity in y-direction
$\mathbf{V}$ = Velocity vector
x = Horizontal position parameter
y = Vertical position parameter
Z = Elevation constant or Dimensionless constant

Greek symbols

ϵ = Steepness. Small expansion parameter

η = Surface elevation

θ = Wave phase constant or river horizontal angle

λ = Wavelength

μ = Dynamic viscosity

ν = Kinematic viscosity

ρ = Fluid density

Φ = Velocity potential

ω = Wave angular velocity

Word explanations

bend: Point in the current velocity profile at which the derivative of u is discontinuous.

Contents

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1 Introduction

Approximately 70 % of the earths surface is covered with water [3] and as a consequence surface waves have engaged many scientists the last century. These scientists have produced a huge amount of research. Much of the research were incorporated in the famous "Surface Waves" review of Wehausen and Laitone's in 1960 [4]. Even thought the review is 55 years old, it is still very relevant and highly cited. Other classical works are Stoker's "Water Waves: The Mathematical Theory with Applications" from 1957 [5] and the fundamental book "Hydrodynamics" from Sir Lamb (1895) [6]. Among newer literature is LeBlond and Mysak's book "Waves in the Ocean" from 1981 [7].

One specific research area that has come into being is wave-current interactions, of which much is covered in the review of Jonsson [8]. An exciting application of these interactions are the stopping of linear gravity waves by opposing currents. Taylor [9] calculated in 1942 that an opposing current of constant velocity U, flowing over a sea of great depth which was at rest beneath, could stop all waves shorter than a certain critical length. Later in 1955 Evan's [10] made experiments where he stopped waves with pneumatic breakwater generated from bubbles. Evans also found the velocity profile of the breakwater to be almost linear with maximum velocity at the surface and still below some depth h. Taylor [11] then derived the mathematics behind Evan's experiments assuming the velocity profile was strictly linear. The constant and linear velocity profiles are sketched in Figure 1.

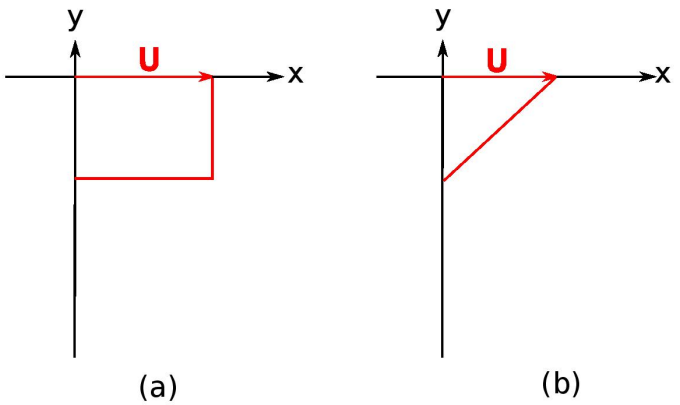

Figure 1: The rectangular profile with constant velocity to the left (a) and the triangular profile with constant shear/vorticity to the right (b).

Special for the triangular profile, in contrast of the rectangular profile, is the addition of rotation/vorticity. The dispersion equation for a rotational situation was derived by Thompson in 1949 [12] and Biesel in 1950 [13]. The influence of vorticity on current on waves has among others been investigated by Brink-Kjær [14].

In later years Betts [15] and Brevik [16] extended Taylors work also to include the effect of finite water depth for the rectangular and the triangle velocity profile respectively. Brevik [17] also studied the problem to third order in wave steepness using a Stokes expansion method, together with Brantenberg [18] he added the presence of surface tension and together with Sollie [19] he investigated the fluxes of mass, momentum and energy. The extension of Taylor's result was also done by Dalrymple [20] who had a bilinear profile[1]. In most recent years wave-current interactions have not been so heavily investigated, but some researchers like Ellingsen [21], Tyvand and Lepperød [22] are still active as of 2014.

[1]The author was first familiar with this reference at the last weeks before deadline

In this thesis the earlier work on breakwater will be extended to take into account any velocity profile. The idea is to divide the velocity profile into N layers with constant shear. When increasing the number of layers we should then be able to approximate any velocity profile. For an illustrative picture of this idea the reader is encouraged to take a look at Figures 4 and 5. Taylor and Goldstein [23] did something similar to this when they tried to approximate a heterogeneous fluid by layers of homogeneous fluids in 1931. Dalrymple [20] also suggested to use this approach but indicated that the mathematics would be tedious with more than two layers.

To clarify, there will be given a formula to calculate the possible waves on any given velocity profile (the dispersion relation). For simplicity this will be done with first order theory in wave steepness and effects like the fluid to be non-homogeneous and surface tension will be omitted. However, it will be shown that with respect to the phase velocity, first order theory is valid up to second order.

Hopefully this work can increase the knowledge and understanding of wave-current interactions. Taylors original idea was to use breakwater to stop waves approaching harbors, but today wave-current interactions have more areas of application. Sub-sea technology is rapidly developing and we harvest oil as never before. As a consequence it is important to better understand oil blowouts or other sub-sea leakages that can occur. A specific application regarding oil is in relation to cleaning spill. Here surface waves cause serious problems, and being able to better stop or damp them could really increase the cleaning success rate, and therefore save the environment when catastrophes occur. Other more usual applications is to use bubbles to prevent surface icing in for example harbors, or to use bubbles to keep water clean by circulation in water pits as in Figure 2.

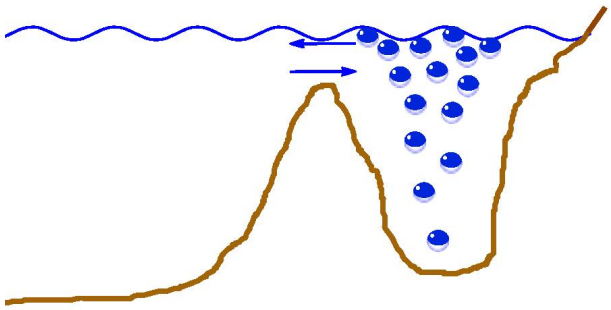

Figure 2: Circulation with pneumatic water current from bubbles. The current generated at the surface is largest allowing dirty water exit while fresh water enter from below. Understanding wave-current interactions help understand waves on such devices.

The above listed applications are only concerning surface waves in the ocean, but the knowledge can be transferred to other research areas where you have fluids, as for example in atmospheric physics.

1.1 Outline

First in Chapter 2 we will go through some of the necessary background theory on surface waves. We will also get to know the formalism and solving method used in the rest of the thesis.

Then in Chapter 3 we will derive the dispersion relation for the one-layer (linear) and the two-layer (bilinear) systems. These derivations have already been done by others in earlier research, but it will be necessary for us to go through them in detail before we take the next step. In the last part of Chapter 3 we will find the dispersion relation for a three-layer system.

In Chapter 4 there will be a quick look at second order theory. This is necessary to prove some of the assumptions made in Chapter 3 and it will also show that with respect to the phase velocity the results for the first order theory is valid to second order in wave steepness.

In Chapter 5 we will finally derive the dispersion relation for a N-layer system. This will be done by a recursion formula and will be viable for both finite and infinite water with some small adjustments.

Chapter 6 will be dedicated to investigate the N-layer dispersion relation numerically for two specific current velocity profiles: the parabolic and the logarithmic. These current profiles often occur in rivers and can easily be made in laboratories. We will look at the possible phase velocity, group velocity and amplitudes for waves on these profiles.

In Chapter 7 we will discuss the extra solutions that appear when we increase the number of layers.

In the end, in Chapter 8, we will summarize the main results in a short conclusion before we look at ideas for further work.

2 Background Theory and Solving Method

This chapter will describe the systems that are used in the thesis, derive the relevant equations and present the solving method.

2.1 Assumptions

In the text we will assume:
- homogeneous and incompressible fluids
- inviscid fluids
- stationary conditions, at least locally
- that surface tension can be neglected
- that the unperturbed current velocity profile is known

2.2 Water Waves

The results herein are applicable on any fluid waves, but we will concentrate on water waves. They can be divided in capillary waves and gravity waves. The first is due to surface tension. They are often too small for engineering purposes and will therefore be neglected in this study. The gravitational waves, on the other hand, are often not neglectable. They can be generated from for example tides, wind or earthquakes that make disturbances to the water equilibrium. Gravity sustain the waves, and thus the name gravitational waves. In this text will not care how the waves came into being. However, we will demand them to be small in height. That is, we want the relation (wave amplitude)/(wave length) to be small.

2.2.1 The Wave Equation

Mathematically a wave in one dimension has to satisfy the wave equation [24]

$$\frac{1}{v^2}\frac{\partial^2\eta(x,t)}{\partial t^2} = \frac{\partial^2\eta(x,t)}{\partial x^2} \tag{2.1}$$

where η is the wave, v is the wave velocity, t is time and x is the positional parameter. In our situation η will be a transversal wave and physically represent the water elevation. The wave equation has an infinite number of solutions. A solution that will meet all our later purposes is

$$\eta(x,t) = a\cos\left(kx - \omega t + \theta\right) = a\cos\left(k(x - \frac{\omega}{k}t) + \theta\right) \tag{2.2}$$

This solution is sometimes referred to as Airy waves. Here a is the wave amplitude, $k = (2\pi/\text{wavelength})$ the wavenumber, $\omega = (2\pi/\text{wave period})$ the wave angular frequency and θ the wave phase constant. Airy waves are linear in the steepness parameter $\epsilon = ka$ but can be extended to Stokes waves, which are correct up to third order for surface gravity waves [25].

2.2.2 Wave Phase Velocity and Group Velocity

If we insert the expression for the wave (2.2) into the wave equation (2.1) we get that $\omega^2/v^2 = k^2$. Or $v = \omega/k$. This v is what we call the phase velocity c_p. It describes how fast a wave with a certain wavenumber k travels with time.

Notice that ω can be a function of k. If ω is linear in k then all waves have the same phase velocity. If this is not true, we say that the waves are dispersive.

From the wave equation (2.1) we also see that if η_i and η_j solves the equation so does $\eta_i + \eta_j$ [2]. Let the two waves be almost identical with $k_i = k + \Delta k$, $k_j = k - \Delta k$, $\omega_i = \omega + \Delta\omega$ and $\omega_j = \omega - \Delta\omega$. Then we get by ignoring θ and a

$$\eta_i + \eta_j = \cos\Big((k + \Delta k)x - (\omega + \Delta\omega)t\Big) + \cos\Big((k - \Delta k)x - (\omega - \Delta\omega)t\Big)$$
$$\eta_i + \eta_j = 2\cos\left(\Delta k x - \Delta\omega t\right)\cos\left(kx - \omega t\right)$$

Here Δk and $\Delta\omega$ are much smaller than k and ω. Hence, the first cosine varies much slower than the second. We say that the first cosine modulates the amplitude. The end result is a wave moving with velocity ω/k but with amplitude varying as $\Delta\omega/\Delta k$ with respect to time. The first is the normal phase velocity introduced above, while the latter is the a new velocity called the group velocity. A band of waves Δk is called a wave packet and c_g describes how the wave packet is moving. The wave packet looks like pulse and all "real" waves are on this form. The monochromatic waves η_i and η_j are mathematical idealizations and cannot occur in real life because they extend to infinity in both x-directions.

When Δk is small we have $c_g = \partial\omega/\partial k$. Of course, if ω is linear in k, $c_g = c_p$ and the wave is non dispersive as it must be. That means that the wave packet keeps its form propagating forward. Since all the energy has to travel with the wave packet we have that c_g is related to energy transport. If $c_g = 0$ no energy can be transported.

$\omega(k)$ is often called the dispersion relation. However, because of the linear relationship between ω and c_p we will often call the equation involving c_p as the dispersion relation.

[2] The superposition principle works as long as we have a linear combination [26]

For waves on still water of finite depth H, the dispersion relation is known to be [27]

$$\omega^2 = gk \tanh(kH) \tag{2.3}$$

2.2.3 Shallow and Deep Water Waves

The depth of the ocean often comes in as an important parameter. The dispersion relation often take different form for deep water and for shallow water waves. We often use dimensionless constants to describe the system and normally kH is used to describe how deep the water is. Since k is inverse proportional to the wavelength λ, kH gives the relation between the water depth and the wavelength. If $kH \ll 1$ we call it shallow water waves, and if $kH \gg 1$ we call it deep water waves.

The dispersion relation for still water (2.3) is for shallow water waves $kH \ll 1 \implies \tanh(kH) = kH$

$$\omega^2 = gk^2H$$

and for deep water we get $kH \gg 1 \implies \tanh(kH) = 1$

$$\omega^2 = gk \tag{2.4}$$

2.3 Velocity and Vorticity

To describe the motion of the fluid, "Lagrangian" description will be used. This method describes the fluid motions by describing the velocity field in a region at a given moment. $\boldsymbol{V}$ will be used for the total velocity, while u will be used for velocity in the horizontal x-direction and v for the vertical y-direction. For simplicity the problem will be limited to two dimensions.

The velocity profile will be very important and it is necessary to be able to write it down mathematically. The undisturbed velocity profile is assumed already known, but the wave will also contribute to this profile. To make the mathematics easier we would like to write the wave perturbation as a scalar potential flow. However, potential flows normally require the velocity profile to be irrotational [28]. To understand why we can still use a potential flow in our systems, we will do as Ellingsen [29] and use the vorticiy equation. The vorticity equation can be found in for example "Viscous Fluid Flow" by White [30].

The vorticity equation is found by taking the curl of the Navier-Stokes equation. The Navier-Stokes equation will not be derived, but for a Newtonian fluid it is given as[3]

$$\rho \frac{d\boldsymbol{V}}{dt} = \rho \boldsymbol{g} - \nabla p + \mu \nabla^2 \boldsymbol{V}$$

Here $\mathbf{g}$ is the gravitational constant with direction, ρ is the density, μ the dynamic viscosity and p is the pressure in the fluid. To get the vorticity equation we take the curl and use the two vector identities $(\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} = \boldsymbol{\nabla}(V^2/2) - \boldsymbol{V} \times \boldsymbol{S}$ and $\nabla^2 \boldsymbol{V} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{V}) - \boldsymbol{\nabla} \times \boldsymbol{S}$ ($\boldsymbol{S} = \text{curl}\boldsymbol{V} = \boldsymbol{\nabla} \times \boldsymbol{V}$):

[3] The derivation is here done for a viscous fluid but is then of course also valid for inviscid fluids with $\mu = 0$.

$$\boldsymbol{\nabla} \times \rho(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}) = \boldsymbol{\nabla} \times \rho \boldsymbol{g} - \boldsymbol{\nabla} \times \boldsymbol{\nabla} p + \boldsymbol{\nabla} \times \mu \nabla^2 \boldsymbol{v}$$

We now use that the divergence of a curl is zero and that the curl of a gradient is zero. That means that the two first terms on the right hand side disappear. We divide by ρ and move terms so that we get

$$\begin{aligned} \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{V}}{\partial t} &= -\boldsymbol{\nabla} \times ((\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V}) + \boldsymbol{\nabla} \times \mu \nabla^2 \boldsymbol{V} \\ \frac{\partial \boldsymbol{S}}{\partial t} &= -\boldsymbol{\nabla} \times (\boldsymbol{\nabla}(V^2/2) - \boldsymbol{V} \times \boldsymbol{S}) + \boldsymbol{\nabla} \times \frac{\mu}{\rho}(\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{V}) - \boldsymbol{\nabla} \times \boldsymbol{S}) \end{aligned}$$

where we also used the vector identities. Using again that the curl of a gradient is zero together with vector cross product rules we get

$$\begin{aligned} \frac{\partial \boldsymbol{S}}{\partial t} &= -\boldsymbol{\nabla} \times (-\boldsymbol{V} \times \boldsymbol{S}) + \boldsymbol{\nabla} \times \frac{\mu}{\rho}(-\boldsymbol{\nabla} \times \boldsymbol{S}) \\ \frac{\partial \boldsymbol{S}}{\partial t} &= -(\boldsymbol{\nabla} \cdot \boldsymbol{S} + \boldsymbol{S} \cdot \boldsymbol{\nabla}) - \boldsymbol{v} - (\boldsymbol{\nabla} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{S} - \frac{\mu}{\rho}\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{S}) + \frac{\mu}{\rho}\nabla^2 \boldsymbol{S} \\ \frac{\partial \boldsymbol{S}}{\partial t} &= -(-\boldsymbol{S} \cdot \boldsymbol{\nabla})\boldsymbol{v} + (-\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{S} + \frac{\mu}{\rho}\nabla^2 \boldsymbol{S} \end{aligned}$$

where $\boldsymbol{\nabla} \cdot \boldsymbol{V} = 0$ for an incompressible fluid. The vorticity equation is then ultimately

$$\frac{\partial \boldsymbol{S}}{\partial t} = (\boldsymbol{S} \cdot \boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{S} + \frac{\mu}{\rho}\nabla^2 \boldsymbol{S}$$

If we now for every region assume that V is two dimensional with no z component and that $\boldsymbol{S} = S_0\hat{z}$ (hat denote direction), we see that all the terms on the left hand side is 0. This means that we have the desired property that the shear will remain constant $\partial S/\partial t = 0$. Since the assumption on $\boldsymbol{V}$ and $\boldsymbol{S}$ is true for the undisturbed velocity profile we only need to demand the same for the wave perturbation. Since the curl of a gradient is 0 we can use a two dimensional gradient flow. There will be added one potential per layer.

For the total velocity in for example the upper part of the triangular velocity profile in Figure 1, we would have

$$u = U_0 + S_0 y - c + \Phi_{0x} \tag{2.5a}$$

$$v = \Phi_{0y} \tag{2.5b}$$

where U_0 is the velocity at the surface and Φ_0 the potential. The 0 subscript stands for it being the first layer and the x, y-subscript refers to the partial derivative with respect to x, y. Be careful with U_0. This is the surface velocity with respect to the bottom. Later u_0 will be used to describe the velocity with respect to the moving coordinate frame, which move with the wave phase velocity c. Then we have $u_0 = U_0 - c$ as the surface velocity. Brevik [16] includes the c in the potential, but it does not matter if we have it in the potential or in the unperturbed velocity profile because it is irrotational.

2.4 The System and Velocity Profiles

The reference frame is mostly set to follow the wave even though we sometimes will have to compare the wave with the still bottom. Following the wave the system become

time independent.

The system that will be used is sketched in Figure 3. The figure shows the system for N layers and an arbitrary velocity profile represented by the green curve. The formulas derived in this thesis will be general and work on all velocity profiles. However, to test the equations derived we will use two specific current profiles that often occur in open channel/river flows. This will be the parabolic and the logarithmic profile.

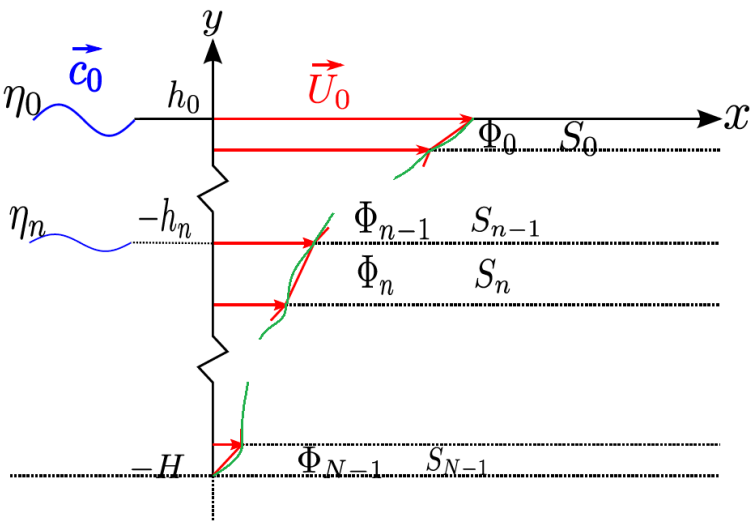

Figure 3: The system with interfaces, heights and velocity profile included. The green graph is an example of a real velocity profile that the system could approximate with N linear layers.

2.4.1 The Parabolic Velocity Profile

To describe a flow the Reynolds number R_e offer useful information. The Reynolds number tells us if the flow is viscously dominated or not. R_e is given as:

$$R_e = \frac{inertial forces}{viscous forces} = \frac{U_0 L}{\nu}$$

Where U_0 is the surface current velocity, L the characteristic channel length and ν the kinematic viscosity ($\nu = \mu/\rho$).

With low R_e we see that the viscous forces dominate and we call the flow laminar. Earlier we assumed inviscid fluids which means we actually cannot have a laminar flow, but we will still investigate this profile because it still gives useful information. From for example Michael Fowler's [31] derivation we can see that the velocity profile for a laminar river is

$$u(y) = \frac{\rho g \theta H^2}{2\nu}(1 - \frac{y^2}{H^2}) = U_0(1 - \frac{y^2}{H^2}) \tag{2.6}$$

where θ is the small downhill angle. In our examples we will use $U_0 = 0.5\,\text{m/s}$ and $H = 0.1\,\text{m}$. These values are chosen because they can easily be achieved in indoor laboratories. The velocity profile is given in Figure 4 which also shows the linear and bilinear approximation.

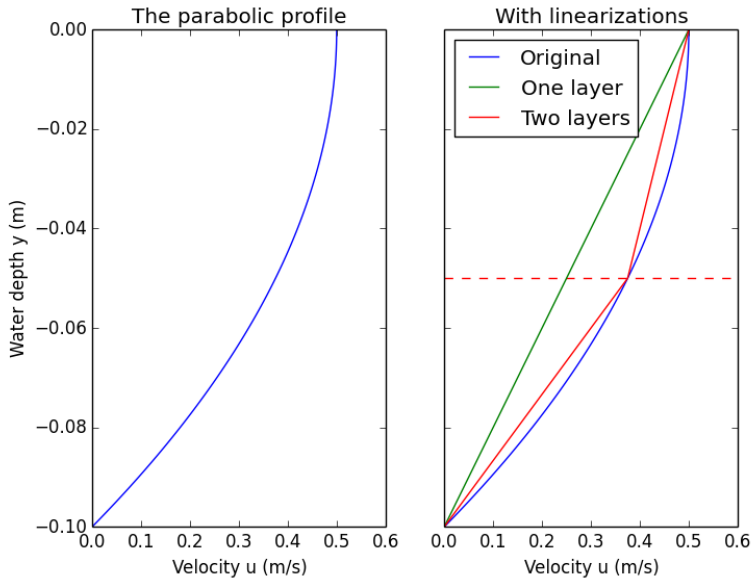

Figure 4: The parabolic velocity profile with linearizations for finite water depth with $U_0 = 0.5\,\text{m/s}$ and $H = 0.1\,\text{m}$.

2.4.2 The Logarithmic Velocity Profile

The other profile we will look at is the logarithmic. This profile also describes an open channel flow but here the Reynolds number is large.

Hickin's [32] calculates the Reynold number for a typical river of depth $H = 0.1\,\text{m}$ and with velocity $u = 0.5\,\text{m/s}$ at $10\,^{\circ}\text{C}$ to be $R_e = 7657$ which means viscous forces can almost be neglected. In practice such flows will have a very thin laminar layer at the bottom and be turbulent above.

Furthermore, Hickins derives a formula for the velocity profile of such a river, taking

also into account the characteristics of the riverbed. Without going into detail about the derivation, he ends up with the following formula for a rectangular open channel with depth $H = 2\,\text{m}$.

$$u = 0.253 \log_{10}(\frac{y + H}{0.005}) + 0.374 \tag{2.7}$$

This is the second velocity profile that will be used in this text and is sketched in Figure 5.Since it is not of importance for the problem the other variables such as viscosity, bed materials and density are omitted. They can however be found in the original text of Hickins. By setting $y = 0$ we get $U_0 = 1.03$.

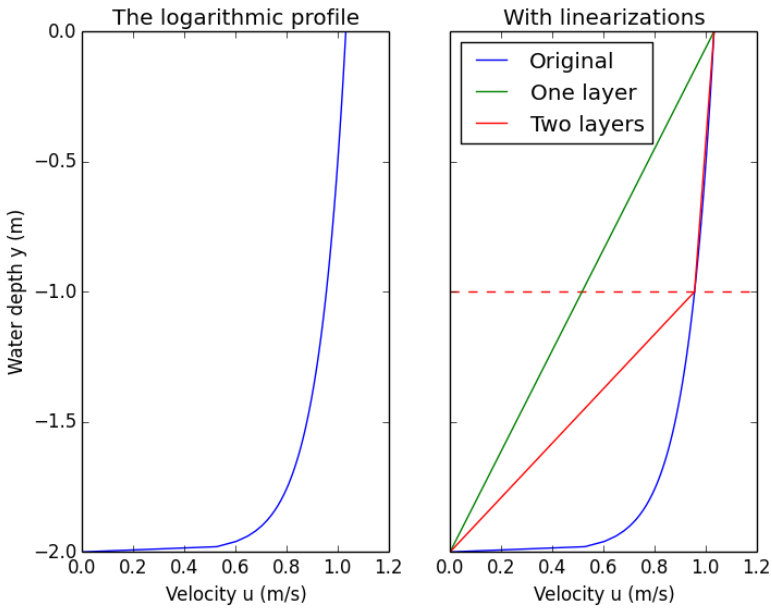

Figure 5: The logarithmic velocity profile with linearizations for finite water depth with $H = 2\,\text{m}$ and $U_0 = 1.03\,\text{m/s}$.

2.5 Boundary Conditions and Governing Equations

From Section 2.3 we now know that the added velocity due to the wave can be written as a potential flow. To determine these potentials we will need some equations.

2.5.1 The Continuity Equation

In classical physics mass cannot come from nothing or suddenly disappear [33] and for an incompressible fluid it is impossible for the mass to become more dense. If we consider a control volume anywhere in the fluid this means that the sum of the mass flux in and out of the volume must be zero. Making the volume infinite small and using that the mass flux is $\rho\mathbf{v}$ this give

$$\begin{aligned} \nabla(\rho\mathbf{v}) &= 0 \\ \nabla\mathbf{v} = \nabla^2\Phi &= 0 \end{aligned} \tag{2.8}$$

which is called the Laplace equation. Now we assume separation of variables, that is that $\Phi(x, y)$ can be written as $X(x)Y(y)$. Making this assumption might be wrong, but we will later see that this potential satisfies all the necessary condition and therefore is valid. The separation give

$$\begin{aligned} \nabla^2\Phi = \Phi_{xx} + \Phi_{yy} &= 0 \\ \frac{X_{xx}}{X} = -\frac{Y_{yy}}{Y} &(= -k^2) \end{aligned} \tag{2.9}$$

Here k is a constant. k has to be constant because the left hand side is only dependent on x and the right hand side only dependent on y. We call this constant k because it

later turns out to be the wave number we discussed previously that appeared in (2.2). The two differential equations are now solvable, however they have an infinite amount of solutions. The first order solution, also called the linear solution, which will be used in this text is [4]

$$\Phi_n = (A_n e^{ky} + B_n e^{-ky})(a_n \sin(kx) + b_n \cos(kx))$$

The n is added here to emphasize that there will be a unique potential for each layer and thus many constants. If we let the initial wave to first order theory be a cosine, Φ must be a sine because differentiating it with respect to x is supposed to give u. So $b_n = 0$. We can also include a_n in A_n and B_n, and divide the constants by c_0/k to make them dimensionless. We then end up with the following form for the potentials

$$\Phi_n = \frac{c_0}{k}(A_n e^{ky} + B_n e^{-ky}) \sin(kx) \tag{2.10}$$

2.5.2 Bottom Boundary Conditions

Now we look at the boundary conditions that needs to be satisfied. They will give us the governing equations, which will allow us to determine the constants in Φ.

The first boundary condition will be at the bottom. Either we have finite water depth H, or the water is said to be infinite deep. In the first case we have to demand that no water can go through the bottom. That is with N layers

[4] Here we could have used complex variables and included time dependence but it is not necessary or easier to do is this way.

$$
\begin{aligned}
v &= \Phi_{(N-1)y}\Big|_{y=-H} = 0 \\
&\frac{c_0}{k}(kA_{N-1}\mathrm{e}^{-kH} - kB_{N-1}\mathrm{e}^{kH})\sin(kx) = 0 \\
&A_{N-1}\mathrm{e}^{-kH} - B_{N-1}\mathrm{e}^{kH} = 0
\end{aligned}
$$

Where the x dependent terms in (2.10) are omitted because this have to be true for all x. This condition make it possible to redefine the potential flows in terms of a new constant D_{N-1} to

$$\Phi_{N-1} = \frac{c_0}{k} D_{N-1}\cosh(k(y+H))\sin(kx) \tag{2.11}$$

Otherwise, if we have infinite depth, we demand that the velocity cannot be infinite as $y \to -\infty$. For this to be true we must have $B_{N-1} = 0$ and

$$\Phi_{N-1} = \frac{c_0}{k} A_{N-1}\mathrm{e}^{ky}\sin(kx) \tag{2.12}$$

2.5.3 The Kinematic Boundary Condition

The second boundary condition is the kinematic. This condition has to be satisfied at all interfaces, where an interface is a curve separating two different layers. The interfaces will be called η_0, η_1 and so on until the last interface that will be η_{N-1}. So η_0 represent the free surface at the top, which in many occasions would be between air and water. In words the kinematic boundary condition state that no particles can go through these interfaces. That is, the derivative of the surface must be zero $F(x, y+h, t) = \eta - (y+h) = 0$ [4]. One can think of there being different liquids in each layer, and that these

liquids cannot mix.

$$\frac{dF}{dt} = \frac{d\eta}{dt} - \frac{dy}{dt} = 0$$

But $dy/dt = v = \Phi_y$ and $d\eta/dt = \eta_t + \boldsymbol{v} \cdot \nabla\eta = \eta_t + u\eta_x + v\eta_y$. However, from the choice of coordinate system $\eta_t = \eta_y = 0$. We are left with the kinematic boundary condition

$$u\eta_x = \Phi_y \tag{2.13}$$

2.5.4 The Bernoulli Equation

The last boundary condition is the dynamic or the Bernoulli equation. To derive this equation for the rotational case is very hard and often omitted, and we will also omit it here. Even Wehausen and Laitone [4] did not include this derivation. However, the equation states that along a streamline we have in two dimensions

$$\frac{1}{2}(u^2 + v^2) + gy = C \tag{2.14}$$

Here C is a constant called the Bernoulli constant[5]. In this text we will use the interfaces as streamlines and evaluate them at the region above and the region below. For both regions the streamline will be the same and C and gy are equal. By setting the equations for the two regions equal each other we can eliminate C and gy. However, at the free surface we can only evaluate the streamline from underneath.

[5] Here C includes the pressure term which is normally not included in the constant

2.6 The Stokes Expansion

To find the dispersion relation for water waves on the piecewise linear flow the Stokes expansion will be used. Stoke expansion uses a perturbation series approach with the conventional steepness parameter $\epsilon = ka$ as the small parameter. In other words, a solution will be on the form $Y = Y_0 + \epsilon^I Y_1 + \epsilon^{II} Y_2 + ...$ where ϵ is a small parameter. Since ϵ is small higher order terms can be neglected. For this method to be valid it is thus essential that the waves are small with respect to the total water depth, and that they are periodic. Roman subscripts are used to keep track of the order.

Lets consider the expansion scheme in detail. All potentials, elevations (interfaces), the phase velocity (in the laboratory frame) and the Bernoulli constant will be expanded. The scheme will be

$$\begin{aligned} \Phi(x,y) &= \epsilon\Phi_I(x,y) + \epsilon^2\Phi_{II}(x,y) + \epsilon^3\Phi_{III}(x,y) \\ \eta(x) &= \epsilon\eta_I(x) + \epsilon^2\eta_{II}(x) + \epsilon^3\eta_{III}(x) \\ c &= c_0 + \epsilon c_I + \epsilon^2 c_{II} + \epsilon^3 c_{III} \\ C &= C_0 + \epsilon C_I + \epsilon^2 C_{II} + \epsilon^3 C_{III} \end{aligned} \tag{2.15a}$$

Here Φ and η are used as a general potential and elevation, but of course we will have to put in Φ_n and η_n dependent on which layer we are considering. This is also true for the Bernoulli constant C which take different values at different streamlines/interfaces. c, on the other hand, is the same everywhere in the water.

What is done next now is to plug these expressions into the governing equations the Bernoulli equation(2.14) and the kinematic boundary condition (2.13). Then terms of equal order in ϵ are collected and used as separate equations.

For example would

$$A + B + \epsilon C = 2\epsilon D$$

give the two equations

$$A + B = 0 \qquad \text{and}$$
$$C = 2D$$

Since mainly first order terms will be used, and there are no zeroth order terms of Φ and η the roman subscript will be neglected. It is also shown in the Chapter 4 that $c_I = 0$ and C_I at the free surface is 0, hence we will not need roman subscripts here either. However, we will still use c_0 just to remind ourself that we work to first order.

The Stokes expansion was used by Wehausen and Laitonen [4] and later by Brevik ([16]- [19]). It is a fairly simple mathematical approach and it is easily generalized to more layers or to higher order theory. However, as done in this text, you loose the time evolution and have to assume stationary conditions.

There are also different ways do solve the problem. Ellingsen solved the one layer triangular problem with finite depth by using differential equation theory [29]. In this method he is able to keep time evolution as he also did when he solved the Cauchy-Poisson problem [21] where you do not have stationary conditions.

Another possibility is to do as is in LeBlond and Mysak [7] and use a matrix approach. The boundary equations are then written as a matrix equation and to find the dispersion relation they simply set the determinant to zero. This method is easily extended to take into account more layers by increasing the matrix size. Determinants can be quickly

determined by computers nowadays so this is perhaps a more efficient method.

Since the author is not familiar with any work on higher order theory with other than the Stokes expansion and because the idea was to generalize Brevik's work, the Stokes expansion was chosen.

3 One-, Two- and Three-Layer Systems

In this section the dispersion relation for the one-layer, two-layer and three-layer systems will be derived. The idea is to find a general formula or pattern for what happens when the number of layers increase, and in this way be able to find a formula for the general N-layer case. The calculations are done to first order theory. However, we will use some results from second order theory which will be proved in Chapter 4.

3.1 The One-layer System

We want to start with the easiest case, the one-layer system. The system is sketched in Figure 6. We will start with infinite water depth and then we will do the finite water depth situation. For infinite water depth the profile is not realistic as longs as $S_0 \neq 0$, because then $y \to -\infty \implies u \to -\infty$. However, $S_0 = 0$ means uninteresting still water. For finite water $u(-H)$ should be 0.

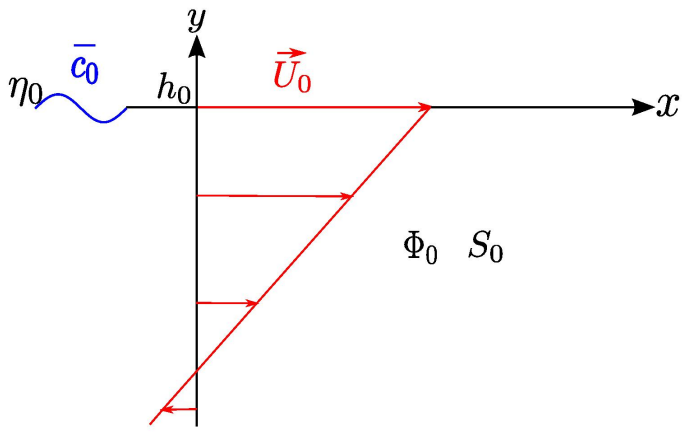

Figure 6: The velocity profile and elevation for an one-layer system with infinite water depth.

3.1.1 Infinite Water Depth

The velocity profile is given as

$$u = U_0 + S_0 y - c + \epsilon \Phi_{0x}$$
$$v = \epsilon \Phi_{0y}$$

We can see that as long as $S_0 \neq 0$ this profile will not be realistic as

Now to first order theory $c = c_0 + \epsilon c_I$. However, as earlier mentioned, it will be proved in Chapter 4 that $c_I = 0$ and we can safely use c_0 instead of c. Also remember that the 0 subscripts for c_0 represent it being the zeroth order phase velocity, while 0 subscript on the potential represent the first layer (there are no 0 in roman numbers). It might be confusing to use 0 subscript and not 1 to denote the first layer for the potentials and elevations, but this notation will make the formulas easier for the N-layer system.

The potential is then given from (2.12).

$$\Phi_0 = \frac{c_0}{k} A_0 \mathrm{e}^{ky} \sin(kx) \tag{3.1}$$

Remember $B_0 = 0$ because of the infinite water condition. We set η_0 to be a cosine in first order theory

$$\eta_0 = \frac{1}{k} \cos(kx) \tag{3.2}$$

so that the elevation from (2.15a) $\epsilon \eta_0 = a \cos(kx)$ where a is the amplitude. Remember

that the 0 subscript also here denote the layer we are in, and is really the same as η_I in (2.15a).

We now have two unknowns A_0 and c_0, and two equations (2.13) and (2.14). To solve the system we start by setting Φ_0 (3.1) and η_0 (3.2) into the kinematic boundary condition (2.13) at the free surface

$$u\epsilon\eta_{0x} - v = 0\Big|_{y=\epsilon\eta_0}$$

We then collect the first order terms. Since $\epsilon\eta_0$ is of first order we must use the zeroth order term of u for the product to be of first order. Hence $u = U_0 + S_0\eta - c_0 + \Phi_x = U_0 - c_0 + \Phi_x$. v must on the other hand be the first order term Φ_{0y}

$$\begin{aligned} u\epsilon\eta_{0x} - v = 0\Big|_{y=0} \\ (U_0 - c_0)(-\sin(kx)) = c_0 A_0 \sin(kx) \\ 1 - Y = A_0 \end{aligned} \tag{3.3}$$

where ϵ and $\sin(kx)$ is canceled out and $Y = U_0/c_0$. The expression is evaluated at $y = 0$ because $y = \epsilon\eta_0$ will only contribute with higher order terms in ϵ. Notice that we have a problem if the phase velocity equals the velocity at the interface U_0 and the left hand side is 0. This problem occurs at all interfaces and is a phenomenon called critical layer solutions. We will omit this problem until it is discussed in Section 7.1.

The Bernoulli equation (2.14) give to first order

$$\begin{aligned}
\frac{1}{2}(u^2+v^2)+gy &= C_I\Big|_{y=\epsilon\eta_0} \\
\frac{1}{2}u^2+gy &= C_I\Big|_{y=\epsilon\eta_0} \\
u_0(S_0y+\epsilon\Phi_{0x})+gy &= C_I\Big|_{y=\epsilon\eta_0} \\
u_0\Phi_{0x}+\eta_0(g+S_0u_0) &= 0\Big|_{y=0}
\end{aligned} \tag{3.4}$$

Where y is evaluated at $\epsilon\eta_0$ as long as this contributes to first order terms. Otherwise y is evaluated at 0. If we take the mean over a wavelength we see that all terms except C_I disappear and thus $C_I = 0$. Noticing that u_0 can be written as $U_0 - c_0 = c_0(Y-1)$ we get by inserting for Φ_{0x}, η_0, A_0 and canceling the $\cos(kx)$-terms

$$\begin{aligned}
c_0(Y-1)c_0A_0+\frac{1}{k}(g+S_0c_0(Y-1)) &= 0 \\
\frac{g}{c_0^2k}-\frac{S_0}{c_0k}(1-Y) &= (1-Y)(1-Y) \\
(1-Y)^2+\frac{S_0}{U_0k}Y(1-Y)-\frac{g}{U_0^2k}Y^2 &= 0
\end{aligned} \tag{3.5}$$

This is the dispersion relation for infinite water with a linear velocity profile to first order theory. For still water ($U = 0 \implies Y = 0$ and $S = 0$) we see directly from the second line that $c_0 = \sqrt{g/k}$, which is a well known result and the same as we had in (2.4).

It is normal in literature to write the dispersion relation with dimensionless variables. We will later do as Brevik and Taylor and use $Z = F^{-2} = \frac{gh}{U_0^2} = \frac{g}{U_0^2k}kh$ where F is the Frode number. If we also substitute $S = U_0/h$ so that $\frac{S}{U_0k} = \frac{1}{kh}$ we can write the

equation only with the dimensionless variables Y, Z and kh. This makes the equation easier to analyze as we will do in Chapter 7. However, until then we will not use the dimensionless variable except for the Y. Introducing many variables might be more confusing when we increase the amount of layers and look for a pattern.

3.1.2 Finite Water Depth

For finite water the boundary condition states that water cannot go through the bottom and the potential is written as in (2.11)

$$\Phi_0 = D_0 \cosh\left(k(y+H)\right) \sin\left(kx\right)$$

Following exactly the same procedure as for infinite water we get for the kinematic boundary condition instead of (3.3)

$$\begin{aligned} u\epsilon\eta_{0x} - v &= 0\Big|_{y=\epsilon\eta_0} \\ (U_0 - c_0)(-\sin\left(kx\right)) &= c_0 D_0 \sin\left(kH\right)\sin\left(kx\right) \\ 1 - Y &= D_0 \sin\left(kH\right) \end{aligned}$$

and for the Bernoulli equation we get

$$\begin{aligned}
u_0\Phi_{0x} + \eta_0(g + S_0u_0) = 0\Big|_{y=0} \\
c_0(Y-1)c_0D_0\cosh(kH) + \frac{1}{k}(g + S_0c_0(Y-1)) = 0 \\
-c_0(Y-1)c_0(Y-1)\frac{1}{\tanh(kH)} + \frac{1}{k}(g + S_0c_0(Y-1)) = 0 \\
(1-Y)^2 + \frac{S_0}{U_0k}Y(1-Y)\tanh(kH) - \frac{g}{U_0^2k}Y^2\tanh(kH) = 0
\end{aligned}$$

If we multiply the last equation be c_0^2 and then solve the second order equation in $(1-Y)c_0 = c_0 - U_0$ with the abc-formula we get

$$c_0 - U_0 = -\frac{S}{2k} \pm \sqrt{(\frac{S}{2k}\tanh(kH))^2 + \frac{g}{k}\tanh(kH)}$$

This also a known result, see for example Ellingsen and Brevik [29]. They derive the same result keeping the time dependence and solving differential equations instead of collecting terms in the Stokes expansion. We also see that if the depth is large $\tanh(kH) \to 1$ and we reproduce the infinite water situation.

3.2 The Two-layer System

Lets now continue with a two-layer system. This system is drawn in Figure 7. We will only do this system for infinite deep water as it will be sufficient to see the pattern we are searching for when we increase the number of layers. .

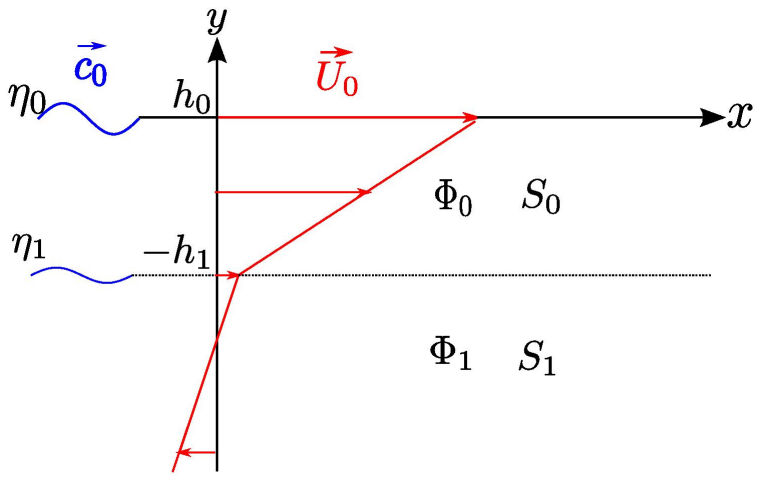

Figure 7: The velocity profile and elevations for a two-layer system with infinite water depth.

In this problem we have the following velocities for (2.5a) and (2.5b)

$$u = \begin{cases} U_0 + S_0 y - c_0 + \epsilon \Phi_{0x} & -h_1 < y \leq 0 \\ U_0 - S_0 h_1 + S_1(h_1 + y) - c_0 + \epsilon \Phi_{1x} & y \leq -h_1 \end{cases}$$

$$v = \begin{cases} \epsilon \Phi_{0y} & -h_1 < y \leq 0 \\ \epsilon \Phi_{1y} & y \leq -h_1 \end{cases}$$

Now we use the infinite depth condition (2.12) on the potentials from (2.10) to find the potentials

$$\Phi_0 = \frac{c_0}{k}(A_0 \mathrm{e}^{ky} + B_0 \mathrm{e}^{-ky}) \sin(kx) \tag{3.6}$$

$$\Phi_1 = \frac{c_0}{k}(A_1 \mathrm{e}^{ky} + B_1 \mathrm{e}^{-ky}) \sin(kx) = \frac{c_0}{k} A_1 \mathrm{e}^{ky} \sin(kx)$$

The elevations are set to be cosines and for the second interface we introduce the new dimensionless constant Z_1 (not the be mistaken with the previous introduced Z). The elevations have to be in phase and Z_1 gives the relative amplitude at the second interface with respect to the free surface

$$\eta_0 = \frac{1}{k} \cos(kx) \tag{3.7}$$

$$\eta_1 = \frac{Z_1}{k} \cos(kx)$$

To solve for the five unknowns, A_0, B_0, A_1, Z_1 and c_0, we need five equations. These equations come from the kinematic boundary condition and the Bernoulli equation at the two interfaces, in which we can insert $u, v, \Phi_0, \Phi_1, \eta_0$ and η_1.

The kinematic boundary condition (2.13) at the free surface

$$\begin{aligned} u\epsilon\eta_{0x} - v = (U_0 - c_0)\eta_{0x} - \Phi_{0y} = 0\Big|_{y=0} \\ (U_0 - c_0)(-\sin(kx)) = c_0 \sin(kx)(A_0 - B_0) \\ 1 - Y = A_0 - B_0 \end{aligned} \tag{3.8}$$

Remember $y = 0$ instead of η_0 to get terms to first order, and $Y = U_0/c_0$. The Bernoulli equation (2.14) analogous to (3.4) at the free surface yield

$$\begin{aligned} \frac{1}{2}(u^2 + v^2) + gy = C \\ (g + S_0(U_0 - c_0))\eta_0 = (c_0 - U_0)\Phi_{0x}\Big|_{y=0} \\ (\frac{gY^2}{U_0^2} + \frac{S_0}{U_0}Y(Y-1))\frac{1}{k}\cos(kx) = (1-Y)(A_0 + B_0)\cos(kx) \\ \frac{gY^2}{kU_0^2} + \frac{S_0}{kU_0}Y(Y-1)) = (1-Y)(A_0 + B_0) \end{aligned} \tag{3.9}$$

where the Bernoulli constant is set to zero by the same average argument as for the one-layer system.

The kinematic boundary condition at the second interface $y = \eta_1$ for region 0 (top layer) gives

$$\begin{aligned} (U_0 - S_0h_1 - c_0)\eta_{0x} - \Phi_{0y} = 0\Big|_{y=-h_1} \\ (U_0 - S_0h_1 - c_0)Z_1(-\sin(kx)) = c_0\sin(kx)(A_0\mathrm{e}^{-kh_1} - B_0\mathrm{e}^{kh_1}) \\ Z_1(1 + \frac{S_0h_1Y}{U_0} - Y) = A_0\mathrm{e}^{-kh_1} - B_0\mathrm{e}^{kh_1} \end{aligned} \tag{3.10}$$

and in region 1 we get

$$(U_0 - S_0 h_1 - c_0)\eta_{1x} - \Phi_{1y} = 0\Big|_{y=-h_1}$$

from which we see that to first order theory the kinematic boundary condition state that v is continuous across the interface

$$\begin{aligned} \Phi_{0y} &= \Phi_{1y}\Big|_{y=-h_1} \\ A_0 \mathrm{e}^{-kh_1} - B_0 \mathrm{e}^{kh_1} &= A_1 \mathrm{e}^{-kh_1} \end{aligned} \tag{3.11}$$

At the second interface the Bernoulli equation for the two regions give

$$\frac{1}{2}(u^2 + v^2) + gy = C$$

$$(U_0 + S_0 h_1 - c_0)(S_0 \eta_1 + \Phi_{1x}) = -g\eta_1 + \text{Bernoulli constant}\Big|_{y=-h_1}$$

$$(U_0 + S_0 h_1 - c_0)(S_1 \eta_1 + \phi_{1x}) = -g\eta_1 + \text{Bernoulli constant}\Big|_{y=-h_1}$$

which combined yield the following relation

$$\begin{aligned} S_0\eta_1 + \Phi_{0x} &= S_1\eta_1 + \Phi_{1x}\Big|_{y=-h} \\ \frac{Z_1}{k}\cos(kx)(S_0 - S_1) &= c_0 \cos(kx)(A_1 \mathrm{e}^{-kh_1} - A_0 \mathrm{e}^{-kh_1} - B_0 \mathrm{e}^{kh_1}) \\ Z_1(\frac{S_0 Y}{kU_0} - \frac{S_1 Y}{kU_0}) &= (A_1 \mathrm{e}^{-kh_1} - A_0 \mathrm{e}^{-kh_1} - B_0 \mathrm{e}^{kh_1}) \end{aligned} \tag{3.12}$$

Now we have five unknowns and the five equations from which they can be determined.

Combining (3.8) and (3.9) we get A_0 and B_0

$$\begin{aligned}
\frac{gY^2}{kU_0^2} + \frac{S_0}{kU_0}Y(Y-1)) &= (1-Y)(A_0+B_0) \\
\frac{gY^2}{kU_0^2} + \frac{S_0}{kU_0}Y(Y-1)) &= (1-Y)(A_0+A_0+Y-1) \\
A_0 &= \frac{1}{2}(1-Y-\frac{S_0Y}{kU_0}+\frac{gY^2}{kU_0^2(1-Y)}) \\
B_0 &= -\frac{1}{2}(1-Y+\frac{S_0Y}{kU_0}-\frac{gY^2}{kU_0^2(1-Y)})
\end{aligned} \tag{3.13}$$

then Z_1 and A_1 is given directly by (3.10) and (3.11)

$$Z_1(-Y+\frac{S_0h_1Y}{U_0}+1) = A_1\mathrm{e}^{-kh_1} = A_0\mathrm{e}^{-kh_1} - B_0\mathrm{e}^{kh_1}$$

In (3.12) we can now insert for A_0, B_0, A_1 and Z_1 to find the dispersion relation for Y

$$\begin{aligned}
Z_1(\frac{S_0Y}{kU_0}-\frac{S_1Y}{kU_0}) &= (A_1\mathrm{e}^{-kh_1} - A_0\mathrm{e}^{-kh_1} - B_0\mathrm{e}^{kh_1}) \\
(A_0\mathrm{e}^{-kh_1} - B_0\mathrm{e}^{kh_1})(\frac{S_0Y}{kU_0}-\frac{S_1Y}{kU_0}) &= -2B_0\mathrm{e}^{kh_1}(-Y+\frac{S_0h_1Y}{U_0}+1) \\
A_0\mathrm{e}^{-2kh_1}(\frac{S_0Y}{kU_0}-\frac{S_1Y}{kU_0}) &= B_0(-2+2Y-\frac{2S_0h_1Y}{U_0}+\frac{S_0Y}{kU_0}-\frac{S_1Y}{kU_0})
\end{aligned}$$

$$\begin{aligned} \mathrm{e}^{-2kh_1}((1-Y)^2 - \frac{S_0Y}{U_0k}(1-Y) + \frac{gY^2}{U_0^2k})(\frac{S_0Y}{U_0k} - \frac{S_1Y}{U_0k}) = \\ ((1-Y)^2 + \frac{S_0Y}{kU_0}(1-Y) - \frac{gY^2}{kU_0^2})(2 - 2Y + \frac{2S_0h_1Y}{U_0} - \frac{S_0Y}{U_0k} + \frac{S_1Y}{U_0k}) \end{aligned} \tag{3.14}$$

To check this answer we can for example compare it two the one-layer case and to Brevik's triangular profile result [16]. To simplify the system to Brevik's we set $S_0 = U_0/h_1$ and $S_1 = 0$. We get

$$\begin{aligned} &\mathrm{e}^{-2kh_1}((1-Y)^2 - \frac{Y}{kh_1}(1-Y) + \frac{gY^2}{U_0^2k})(\frac{Y}{kh_1}) = \\ &((1-Y)^2 + \frac{Y}{kh_1}(1-Y) - \frac{gY^2}{U_0^2k})(2 - \frac{Y}{kh_1}) \\ &\mathrm{e}^{-2kh}((1-Y)^2 - \frac{Y}{kh}(1-Y-ZY)(\frac{Y}{kh}) = \\ &((1-Y)^2 + \frac{Y}{kh}(1-Y-ZY)(2 - \frac{Y}{kh}) \end{aligned} \tag{3.15}$$

which reproduce Brevik's result [6]. Lets now see if we can reproduce the one-layer case. This can be done by setting $S_0 = S_1$ into (3.14) which will set the left hand side to 0. The last paranthesis on the right hand side is dependent on h_1 and can be chosen to be different from 0. Hence, the first paranthesis must be zero and as in the one-layer system we have

$$(1-Y)^2 + \frac{S_0Y}{kU_0}(1-Y) - \frac{gY^2}{kU_0^2} = 0$$

[6] In hes article U_0 is directed in the opposite direction so Y with $-Y$

This is exactly the dispersion relation found for one layer in (3.5). And is actually just the equation for $B_0 = 0$. Which is natural, when we go from two layers to one we must have $A_0 = A_1$ and $B_0 = B_1$. Also B_1 must be 0 because of the infinite water depth which give $B_0 = 0$.

3.3 The Three-layer System

The next system will be the three-layers system which is displayed in Figure 8. In this section there will be a lot of calculation which might be hard to follow. However, the calculations are not the most important thing here, but rather the similarities with the two-layer system. This is what we will use later for the N-layer system.

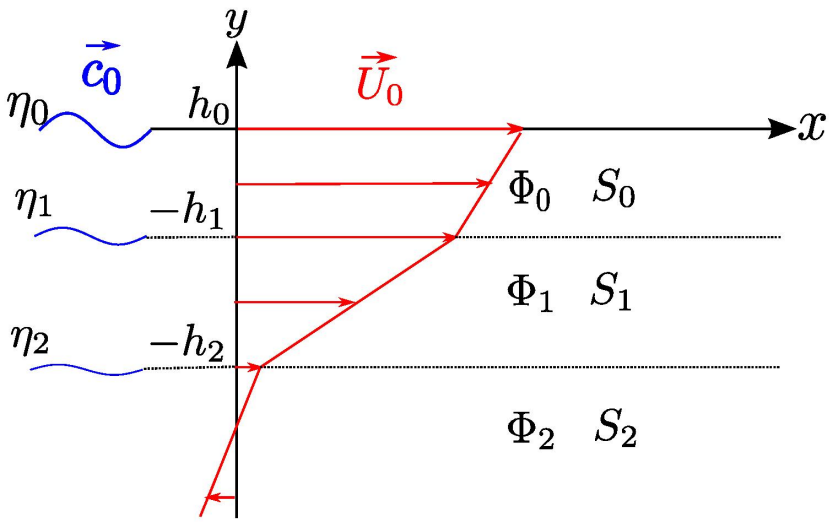

Figure 8: The velocity profile and elevations for a three-layer system with infinite water depth.

In this system the velocities will be

$$u = \begin{cases} U_0 + S_0 y - c_0 + \epsilon \Phi_{0x} & -h_1 < y \leq 0 \\ U_0 - S_0 h_1 + S_1(h_1 + y) - c_0 + \epsilon \Phi_{1x} & -h_2 < y \leq -h_1 \\ U_0 - S_0 h_1 + S_1(h_1 - h_2) + S_2(h_2 + y) - c_0 + \epsilon \Phi_{2x} & y \leq -h_2 \end{cases}$$

$$v = \begin{cases} \epsilon\Phi_{0y} & -h_1 < y \leq 0 \\ \epsilon\Phi_{1y} & -h_2 < y \leq -h_1 \\ \epsilon\Phi_{2y} & y \leq -h_2 \end{cases}$$

The potentials will be

$$\Phi_0 = \frac{c_0}{k}(A_0 \mathrm{e}^{ky} + B_0 \mathrm{e}^{-ky}) \sin(kx)$$

$$\Phi_1 = \frac{c_0}{k}(A_1 \mathrm{e}^{ky} + B_1 \mathrm{e}^{-ky}) \sin(kx)$$

$$\Phi_2 = \frac{c_0}{k} A_2 \mathrm{e}^{ky} \sin(kx)$$

where the A's and B's are dimensionless constants. The elevations with Z_1 and Z_2 as proportionality constants will be

$$\eta_0 = \frac{1}{k} \cos(kx)$$

$$\eta_1 = \frac{Z_1}{k} \cos(kx)$$

$$\eta_2 = \frac{Z_2}{k}\cos(kx)$$

Now the kinematic boundary condition (2.13) and the Bernoulli equation (2.14) at the free surface give

$$\begin{aligned} u\epsilon\eta_{0x} - v &= (U_0 - c_0) \\ epsiloneta_{0x} - \epsilon\Phi_{0y} &= 0\Big|_{y=0} \\ (U_0 - c_0)(-\sin(kx)) &= c_0\sin(kx)(A_0 - B_0) \\ 1 - Y &= A_0 - B_0 \end{aligned}$$

and analogous to (3.4)

$$\begin{aligned} (g + S_0(U_0 - c_0))\eta_0 &= (c_0 - U_0)\Phi_{0x}\Big|_{y=0} \\ (\frac{gY^2}{U_0^2} + \frac{S_0}{U_0}Y(Y-1))\frac{1}{k}\cos(kx) &= (1-Y)(A_0 + B_0)\cos(kx) \\ \frac{gY^2}{kU_0^2} + \frac{S_0}{kU_0}Y(Y-1)) &= (1-Y)(A_0 + B_0) \end{aligned}$$

Notice that these are the exact equations (3.8) **and** (3.9) **from the two-layer case**. This is a result of Φ_0 being calculated only by the conditions at the free surface. For Φ_0 it does not matter how many layers are underneath. Thus we do not need to calculate A_0 and B_0, because they are given in (3.13). Very importantly, however, notice that these expressions are functions of Y. There will be different values of Y solving the dispersion relation for the different systems (number of layers, velocity profile, finite/infinite). Hence, the equations are equal but their numerical values will be different from system

to system.

At the second interface η_1 the kinematic boundary condition in region 0 will also be equal to the two-layer case (3.10)

$$\begin{aligned}(U_0 - S_0h_1 - c_0)\eta_{1x} - \Phi_{0y} = 0\Big|_{y=-h_1}\\ (U_0 - S_0h_1 - c_0)Z_1(-\sin(kx)) = c_0\sin(kx)(A_0\mathrm{e}^{-kh_1} - B_0\mathrm{e}^{kh_1})\\ Z_1(1 + \frac{S_0h_1Y}{U_0} - Y) = A_0\mathrm{e}^{-kh_1} - B_0\mathrm{e}^{kh_1}\end{aligned}$$

Which tell us that since the expressions for A_0 and B_0 is equal to the two-layer case, so is the expression for Z_1. In region 1 we get

$$(U_0 - S_0h_1 - c_0)\eta_{1x} - \Phi_{1y} = 0\Big|_{y=-h_1}$$

from which we see that

$$\begin{aligned}\Phi_{0y} = \Phi_{1y}\Big|_{y=-h_1}\\ A_0\mathrm{e}^{-kh_1} - B_0\mathrm{e}^{kh_1} = A_1\mathrm{e}^{-kh_1} - B_1\mathrm{e}^{kh_1}\end{aligned} \tag{3.16}$$

Now the Bernoulli equation at the second interface η_1 gives

$$\begin{aligned}\frac{1}{2}(u^2 + v^2) + gy = C\\ (U_0 + S_0h_1 - c_0)(S_0\eta_1 + \Phi_{0x}) = -g\eta_1 + \text{Bernoulli constant}\Big|_{y=-h}\end{aligned}$$

and

$$(U_0 + S_0 h_1 - c_0)(S_1\eta_1 + \Phi_{1x}) = -g\eta_1 + \text{Bernoulli constant}\Big|_{y=-h}$$

which combined gives the following relation

$$\begin{aligned} S_0\eta_1 + \Phi_{0x} &= S_1\eta_1 + \Phi_{1x}\Big|_{y=-h} \\ \frac{Z_1}{k}\cos(kx)(S_0 - S_1) &= c_0\cos(kx)(A_1 e^{-kh_1} + B_1 e^{+kh_1} - A_0 e^{-kh_1} - B_0 e^{kh_1}) \\ Z_1(\frac{S_0 Y}{kU_0} - \frac{S_1 Y}{kU_0}) &= (A_1 e^{-kh_1} + B_1 e^{kh_1} - A_0 e^{-kh_1} - B_0 e^{kh_1}) \end{aligned} \tag{3.17}$$

This last two equations (3.16) and (3.17) does now differ from those for the two-layer case because the bottom boundary condition have been shifted one layer down to region 2.

The kinematic condition at the third and last interface η_2 we get for region 1

$$\begin{aligned} (U_0 - S_0 h_1 - S_1(h_2 - h_1) - c_0)\eta_{2x} - \Phi_{1y} &= 0\Big|_{y=-h_2} \\ (U_0 - S_0 h_1 - S_1(h_2 - h_1) - c_0)Z_2(-\sin(kx)) &= c_0\sin(kx)(A_1 e^{-kh_2} - B_1 e^{kh_2}) \\ Z_2(1 + \frac{S_0 h_1 Y}{U_0} + \frac{S_1(h_2 - h_1)Y}{U_0} - Y) &= A_1 e^{-kh_2} - B_1 e^{kh_2} \end{aligned} \tag{3.18}$$

and for region 2 we have

$$(U_0 - S_0 h_1 - S_1(h_2 - h_1) - c_0)\eta_{2x} - \Phi_{2y} = 0\Big|_{y=-h_2}$$

from which we see that

$$\begin{aligned} \Phi_{1y} &= \Phi_{2y}\Big|_{y=-h_2} \\ A_1 \mathrm{e}^{-kh_2} - B_1 \mathrm{e}^{kh_2} &= A_2 \mathrm{e}^{-kh_2} \end{aligned} \tag{3.19}$$

The Bernoulli equation at the third elevation η_2 give

$$(U_0 + S_0 h_1 + S_1(h_2 - h_1) - c_0)(S_1\eta_2 + \Phi_{1x}) = -g\eta_2 + \text{Bernoulli constant}\Big|_{y=-h_2}$$

$$(U_0 + S_0 h_1 + S_1(h_2 - h_1) - c_0)(S_2\eta_2 + \Phi_{2x}) = -g\eta_2 + \text{Bernoulli constant}\Big|_{y=-h_2}$$

which combined gives the following relation

$$
\begin{aligned}
S_1\eta_2 + \Phi_{1x} &= S_2\eta_2 + \Phi_{2x}\Big|_{y=-h_2} \\
\frac{Z_2}{k}\cos(kx)(S_1 - S_2) &= c_0\cos(kx)(A_2\mathrm{e}^{-kh_2} - A_1\mathrm{e}^{-kh_2} - B_1\mathrm{e}^{kh_2}) \\
Z_2(\frac{S_1Y}{kU_0} - \frac{S_2Y}{kU_0}) &= (A_2\mathrm{e}^{-kh_2} - A_1\mathrm{e}^{-kh_2} - B_1\mathrm{e}^{kh_2})
\end{aligned}
\tag{3.20}
$$

These two equations (3.19) and (3.20) are almost the same as the expressions for the last interface in the two-layer system equation (3.11). This is natural because it is the same condition and the potentials above and below the last interface has the same form in both system. **This is the pattern we were searching for. Increasing the number of layers only add a potential determined by equations similar to** (3.16) **and** (3.17) **while the rest stay the same**.

Even though we have already found the pattern, we are so close to the dispersion relation for the three-layer system that we can just as well derive it. We started out with eight unknowns A_0, B_0, A_1, B_1, A_2, Z_1, Z_2 and c_0. However, we have already shown that the solutions for A_0, B_0 and Z_1 are equal to that derived from the two-layer case. We will therefore only need to solve for the five other.

From (3.17) we solve for A_1

$$
A_1\mathrm{e}^{-kh_1} = Z_1(\frac{S_0Y}{kU_0} - \frac{S_1Y}{kU_0}) - B_1\mathrm{e}^{kh_1} + A_0\mathrm{e}^{-kh_1} + B_0\mathrm{e}^{kh_1}
$$

Inserting this expression into (3.16) we get

$$\begin{aligned} A_0\mathrm{e}^{-kh_1} - B_0\mathrm{e}^{kh_1} &= Z_1(\frac{S_0Y}{kU_0} - \frac{S_1Y}{kU_0}) - B_1\mathrm{e}^{kh_1} + A_0\mathrm{e}^{-kh_1} + B_0\mathrm{e}^{kh_1} - B_1\mathrm{e}^{kh_1} \\ B_1\mathrm{e}^{kh_1} &= \frac{1}{2}Z_1(\frac{S_0Y}{kU_0} - \frac{S_1Y}{kU_0}) + B_0\mathrm{e}^{kh_1} \end{aligned} \tag{3.21}$$

and from (3.17) we see that A_1 must be

$$A_1\mathrm{e}^{-kh_1} = \frac{1}{2}Z_1(\frac{S_0Y}{kU_0} - \frac{S_1Y}{kU_0}) + A_0\mathrm{e}^{-kh_1} \tag{3.22}$$

Now we see that if A_1 and B_1 is known we can find Z_2 and A_2 directly from (3.18) and (3.19).

We need to use (3.20) to find the dispersion relation. We multiply both sides with $(1 + \frac{S_0h_1Y}{U_0} + \frac{S_1(h_2-h_1)Y}{U_0} - Y)$ to get

$$\begin{aligned} &Z_2(1 + \frac{S_0h_1Y}{U_0} + \frac{S_1(h_2-h_1)Y}{U_0} - Y)(\frac{S_1Y}{kU_0} - \frac{S_2Y}{kU_0}) = \\ &(A_2\mathrm{e}^{-kh_2} - A_1\mathrm{e}^{-kh_2} - B_1\mathrm{e}^{kh_2})(1 + \frac{S_0h_1Y}{U_0} + \frac{S_1(h_2-h_1)Y}{U_0} - Y) \end{aligned}$$

Now we can insert for Z_2 and A_2 from equations (3.18) and (3.19). Collecting terms of A_1 and B_1 we get

$$(A_1\mathrm{e}^{-kh_2} - B_1\mathrm{e}^{kh_2})(\frac{S_1Y}{kU_0} - \frac{S_2Y}{kU_0}) = -2B_1\mathrm{e}^{kh_2}(1 + \frac{S_0h_1Y}{U_0} + \frac{S_1(h_2-h_1)Y}{U_0} - Y)$$

$$A_1 \mathrm{e}^{-kh_2}(\frac{S_1 Y}{kU_0} - \frac{S_2 Y}{kU_0}) = -2B_1 \mathrm{e}^{kh_2}(1 + \frac{S_0 h_1 Y}{U_0} + \frac{S_1(h_2 - h_1)Y}{U_0} - Y - \frac{S_1 Y}{2kU_0} + \frac{S_2 Y}{2kU_0})$$

Now we need to insert for A_1 and B_1 from (3.22) and (3.21). The dispersion relation will then be

$$\begin{aligned}(\frac{1}{2}Z_1(\frac{S_0 Y}{kU_0} - \frac{S_1 Y}{kU_0})\mathrm{e}^{kh_1} + A_0)\mathrm{e}^{-kh_2}(\frac{S_1 Y}{kU_0} - \frac{S_2 Y}{kU_0}) = \\ - 2(\frac{1}{2}Z_1(\frac{S_0 Y}{kU_0} - \frac{S_1 Y}{kU_0})\mathrm{e}^{-kh_1} + B_0)\mathrm{e}^{kh_2} \\ (1 + \frac{S_0 h_1 Y}{U_0} + \frac{S_1(h_2 - h_1)Y}{U_0} - Y - \frac{S_1 Y}{2kU_0} + \frac{S_2 Y}{2kU_0})\end{aligned} \quad (3.23)$$

Let us settle with this expression since Z_1, A_0 and B_0 is equal as in the two-layer case. If we set $S_1 = S_2$ we should reproduce the two-layer system. In the dispersion relation the left hand side will be 0 because of the last factor. This means that the right hand side also has to be zero. The exponent can however not be zero so either the first or the second part must. Since the last parenthesis includes h_2 which can have any value and hence the paranthesis is not always 0, the first part must be 0. Thus we have

$$\begin{aligned}\frac{1}{2}Z_1(\frac{S_0 Y}{kU_0} - \frac{S_1 Y}{kU_0})\mathrm{e}^{-kh_1} + B_0 = 0 \\ Z_1(\frac{S_0 Y}{kU_0} - \frac{S_1 Y}{kU_0}) = -2B_0 \mathrm{e}^{kh_1}\end{aligned} \quad (3.24)$$

This is the dispersion relation for the two-layer case as given in equation (3.14) except that we have not inserted the expression for Z_1 and B_0 but they are equal in both

systems. This reduction support that (3.23) is the dispersion relation for the three-layer case and that all the steps deriving it are legal. We will not write the dispersion relation out any further. We will rather go to the N-layer system using the pattern we found. First, however, we will quickly look at second order theory to prove that $c_1 = 0$ which we took for granted in this chapter.

4 Determination of c_I With Second Order Theory

In the previous chapter we made the assumption that the first order term of the phase velocity $c_I = 0$, which simplified u. Also, at the free surface we took the mean over a wavelength to see that the first order term of the Bernoulli constant $C_{1,I}$ was 0. This was, however, only true because we had already set c_I to be 0. In this chapter we will prove that $c_I = 0$ in the same way that Brevik did [16].

Having subscripts for both the layer and the order can be confusing. Fortunately we will only need the top layer in this chapter and will thus omit the numeral subscripts to represent the layer. For only this chapter u_0 will be used as the zeroth order surface velocity and u_I as the first order surface velocity.

4.1 Relationship Between c_I and $C_{1,I}$

First we will quickly look at what $C_{1,I}$ would be if we did not assume $c_I = 0$. The Bernoulli equation at the free surface is from (2.14)

$$\frac{1}{2}(u^2 + v^2) + gy = C$$

which to first order in the Stokes expansion for small ϵ yield

$$\begin{aligned} u_I u_0 + g\epsilon\eta_I &= \epsilon C_1 \\ (S\eta_I - c_I)(U_0 - c_0) + g\eta_I &= C_1 \end{aligned}$$

The trick now is to take the mean or integrate over a wavelength as we did in equation (3.4). Because the wave is symmetric ($< \eta_1 >= 0$ where $<>$ represent the mean/integral over a wavelength)

$$\epsilon u_1 u_0 + g\eta_1 = C_1$$
$$C_1 = -c_1(U_0 - c_0)$$

So as we already knew, $c_I = 0 \implies C_I = 0$.

4.2 To Second Order in ϵ

From second order theory we can find an expression for c_I. The velocity from (2.5a) will to second order be (higher order terms are neglected as ϵ is small)

$$\begin{aligned} u &= u_0 + \epsilon u_I + \epsilon^2 u_{II} \\ u &= U_0 - c_0 + \epsilon(-c_1 - S\eta_I + \Phi_{Ix}) + \epsilon^2(-c_{II} - S\eta_{II} + \Phi_{IIx} + \eta_1 \phi_{Ixy}) \end{aligned} \tag{4.1}$$

where the layer subscript is omitted for simplicity as it is sufficient with only the top layer. Similar v from (2.5b) will be

$$v = \epsilon \Phi_{Iy} + \epsilon^2(\Phi_{IIy} + \eta_I \Phi_{Iyy}) \tag{4.2}$$

Following the same procedure as throughout the thesis we insert these expression into the governing equations, but collect terms to second order in epsilon. The kinematic

boundary condition (2.13) gives

$$\begin{aligned} u\eta_x &= \Phi_y \\ u_0\eta_{IIx} + u_1\eta_{Ix} &= \Phi_{IIy} \end{aligned} \tag{4.3}$$

but we already know the first order terms for η and Φ since these are the expression we used in Chapter 3. We have from (3.7) and (3.6)

$$\begin{aligned} \eta_I &= \frac{1}{k}\cos(kx) \\ \Phi_I &= \frac{c_0}{k}(A_I \mathrm{e}^{ky} + B_I \mathrm{e}^{-ky})\sin(kx) \end{aligned}$$

Inserting these expression into (4.3) together with (4.1) and (4.2) we get with $y = 0$ at the free surface

$$\begin{aligned} &u_0\eta_{IIx} + u_1\eta_{Ix} = \Phi_{IIy} \\ &\Phi_{IIy} = (U_0 - c_0)\eta_{IIx} + (-c_1 - S\eta_I + \Phi_{Ix})(-1\sin(kx)) \\ &= (U_0 - c_0)\eta_{IIx} + (-c_1 - S\frac{1}{k}\cos(kx) + c_0(A_I + B_I)\cos(kx))(-1\sin(kx)) \end{aligned} \tag{4.4}$$

The Bernoulli equation (2.14) yields

$$\begin{aligned} \frac{1}{2}(u^2 + v^2) + gy &= C \\ \frac{1}{2}(2u_0u_{II} + u_I^2 + \Phi_{Iy}^2) + g\eta_{II} &= C_{II} \end{aligned}$$

Since u_0 is independent of x, the derivative with respect to x of this equation is

$$\begin{aligned} g\eta_{IIx} &= -\frac{1}{2}(2u_0u_{IIx} + 2u_Iu_{Ix} + 2\Phi_{Iy}\Phi_{Iyx}) \\ g\eta_{IIx} &= -(U_0 - c_0)(-c_{II} - S\eta_{II} + \Phi_{IIx} + \eta_1\Phi_{Ixy})_x \\ &\quad -(-c_1 - S\eta_I + \Phi_{Ix})(-c_1 - S\eta_I + \Phi_{Ix})_x - \Phi_{Iy}\Phi_{Iyx} \\ g\eta_{IIx} &= -(U_0 - c_0)(-S\eta_{IIx} + \Phi_{IIxx} + \eta_{Ix}\Phi_{Ixy} + \eta_I\Phi_{Ixyx}) \\ &\quad -(-c_1 - S\eta_I + \Phi_{Ix})(-S\eta_{Ix} + \Phi_{Ixx}) - \Phi_{Iy}\Phi_{Iyx} \end{aligned} \tag{4.5}$$

We can now set in for the first order elevation η_I and potential Φ_I as we did in (4.4). However, this will make the expression very long and we will not do this. Notice in (4.4) that all the terms except those including η_{IIx}, Φ_{IIxx} and c_I have their x-dependence given as $\cos(kx)\sin(kx)$. In fact this will also be true for (4.5) if we insert for η_I and Φ_I. Even though we did not insert for η_I and Φ_I this should be easy to see. Now we can simplify the x-terms be rewriting $\cos(kx)\sin(kx)$ as $\frac{1}{2}\sin(2kx)$.

As for the potential Φ_I, also Φ_{II} has to satisfy the continuity equation (2.8). Again we use separation of variables and guess on one solution that satisfies the separation equation(2.9). We guess on the following for Φ_{II} and η_{II}

$$\begin{aligned} \Phi_{II} &= \frac{c_0}{k}(A_{II}\mathrm{e}^{2ky} + B_{II}\mathrm{e}^{-2ky})\sin(2kx) \\ \eta_{II} &= \frac{Z_{II}}{k}\cos(2kx) \end{aligned}$$

These expressions correspond to second order Stokes waves and are analogues to (3.7) and (3.6). We see that with this choice also η_{IIx}, Φ_{IIxx} in the governing equations (4.4) and (4.5) will have their x dependence as $\sin(2kx)$. If we set $c_I = 0$ all terms have the same x dependence and we can cancel the x-terms. The remaining are two equations

independent of both y and x with three unknowns (or free variables) A_{II}, B_{II} and Z_{II}. By the right chose of these variables the system is solvable in the same matter as for first order theory and the assumption that $c_I = 0$ was valid.

To solve for A_{II}, B_{II} and Z_{II} the reader is referred to Brevik [16]. We will not do these calculations, but rather be delighted that $c_p = c_0$ is valid to second order theory. Brevik [17] even went to third order to find c_{II} which will be the first correction to c_0.

5 The N-layer System

In this section the dispersion relation for a N-layer system will be derived using a recursion formula.

5.1 Introduction

The dispersion relation for a three-layer system (3.23) is much more complicated than that for the two-layer system (3.14). The exponential terms make it hard to simplify the solution further and trying to solve a system with many layers, N, by this brute force method will be messy as indicated by Dalrymple [20].

However, we made an important observation. Increasing the amount of layers does not change the expression for the layers near the surface or near the bottom, we just get another intermediate layer from which the constants in the potential can be found from (3.16) and (3.20).

So when we for example increased from two to three layers this did not affect the expressions for A_0 and B_0. From A_0 and B_0 we can find A_1 and B_1 regardless of how many layers are underneath because the equations involved only care about the two fluids at the interface. Therefore it should be possible to find a recursion formula which calculates A_n and B_n from A_{n-1} and B_{n-1} when n denote the nth layer and $0 < n < N$. Since A_0 and B_0 can be found by the free surface conditions this would solve our problem as long as the last layer with the bottom boundary condition contributes with the dispersion relation as it did in Chapter 3. Since the only difference from finite and infinite water is the bottom boundary condition, this method can be used for both finite and infinite water if we just use two different dispersion relations.

5.2 The Recursion Formula

Figure 9 shows the N-layer linearization. Let the potential and elevation at layer n be given by

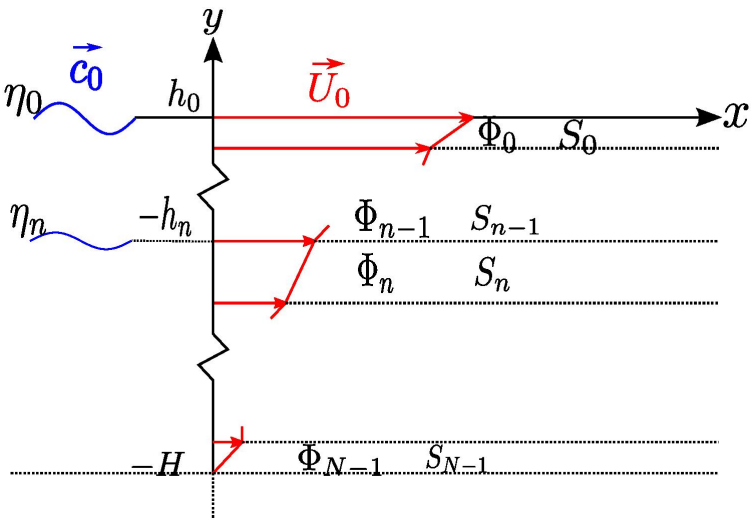

Figure 9: The velocity profile and elevations for a N-layer system with both finite and infinite water depth.

$$\eta_n = \epsilon \frac{Z_n}{k} \cos(kx) \tag{5.1}$$

$$\Phi_n = -c_0 x + \epsilon \frac{c_0}{k} (A_n \mathrm{e}^{ky} + B_n \mathrm{e}^{-ky}) \sin(kx) \Big|_{y=-h_n}$$

The unperturbed velocities at the interfaces, from now on called u_n, will be

$$u_n = U_0 - \sum_{i=0}^{n-1} S_i h_i - c_0$$

Then u and v, at layer n, can be written with terms up to first order as

$$u = u_n + \epsilon(\frac{c_0}{k}\cos(kx)(A_j \mathrm{e}^{-kh_n} + B_j \mathrm{e}^{kh_n}) + S_j \frac{Z_n}{k}\cos(kx)) \tag{5.2}$$

$$v = \epsilon c_0 (A_j \mathrm{e}^{-kh_n} - B_j \mathrm{e}^{kh_n})\sin(kx) \tag{5.3}$$

where j denotes if we are in the layer above or below the interface and take the values $n-1$ or n respectively. Inserting the expressions for η_n (5.1), u (5.2) and v (5.3) into the kinematic boundary condition (2.13) we get for the layer above the interface

$$\begin{aligned} u_n \eta_{nx} &= \Phi_{(n-1)y}\Big|_{y=-h_n} \\ u_n Z_n(-\sin(kx)) &= c_0 \sin(kx)(A_{n-1}\mathrm{e}^{-kh_n} - B_{n-1}\mathrm{e}^{kh_n}) \\ Z_n &= -\frac{c_0}{u_n}(A_{n-1}\mathrm{e}^{-kh_n} - B_{n-1}\mathrm{e}^{kh_n}) \end{aligned} \tag{5.4}$$

For the layer below it will be exactly the same except that we must replace A_{n-1} and B_{n-1} with A_n and B_n. Hence we can write

$$0 = A_{n-1}\mathrm{e}^{-kh_n} - B_{n-1}\mathrm{e}^{kh_n} - A_n \mathrm{e}^{-kh_n} + B_n \mathrm{e}^{kh_n} \tag{5.5}$$

Now from the Bernoulli equation (2.14) we have that u^2 is constant over the interface to first order in ϵ. u^2 to first order is the zeroth order term multiplied with the first order term. Since the unpurturbed terms are equal in both regions this simply means that the first order term of u must be constant across the interface. With the $\cos(kx)$ terms canceled out we get

$$\begin{aligned} c_0(A_n\mathrm{e}^{-kh_n} + B_n\mathrm{e}^{kh_n}) + S_n\frac{Z_n}{k} &= c_0(A_{n-1}\mathrm{e}^{-kh_n} + B_{n-1}\mathrm{e}^{kh_n}) + S_{n-1}\frac{Z_n}{k} \\ \frac{Z_n}{c_0k}(S_{n-1} - S_n) &= A_n\mathrm{e}^{-kh_n} + B_n\mathrm{e}^{kh_n} - A_{n-1}\mathrm{e}^{-kh_n} - B_{n-1}\mathrm{e}^{kh_n} \end{aligned} \tag{5.6}$$

We now have three equations (5.4), (5.5) and (5.6) and three unknowns A_n, B_n and Z_n assuming A_{n-1} and B_{n-1} are known. Z_n is of course found directly from (5.4). To solve for the rest we take (5.6) $\pm$ (5.5):

$$\begin{aligned} \frac{Z_n}{c_0k}(S_{n-1} - S_n) = {} & A_n\mathrm{e}^{-kh_n} + B_n\mathrm{e}^{kh_n} - A_{n-1}\mathrm{e}^{-kh_n} - B_{n-1}\mathrm{e}^{kh_n} \\ & \pm(-A_n\mathrm{e}^{-kh_n} + B_n\mathrm{e}^{kh_n} + A_{n-1}\mathrm{e}^{-kh_n} - B_{n-1}\mathrm{e}^{kh_n}) \end{aligned}$$

Either the A-terms or the B-terms vanish and we get

$$A_n = \frac{Z_n}{2c_0k}(S_{n-1} - S_n)\mathrm{e}^{kh_n} + A_{n-1} \tag{5.7}$$

$$B_n = \frac{Z_n}{2c_0k}(S_{n-1} - S_n)\mathrm{e}^{-kh_n} + B_{n-1} \tag{5.8}$$

which we can see correspond to the formulas for A_1 (3.22) and B_1 (3.21) in the three-layer system.

5.3 A_0 and B_0

If we now know A_0 and B_0 we can now find all the A_n and B_n until the last A_{n-} and B_{n-1}, for which we must be a little bit more careful because of the bottom boundary conditions. However, A_0 and B_0 was already found for the two-layer system and is given by (3.13)

$$A_0 = \frac{1}{2}(1 - Y - \frac{S_0 Y}{kU_0} + \frac{gY^2}{kU_0^2(1-Y)})$$
$$B_0 = -\frac{1}{2}(1 - Y + \frac{S_0 Y}{kU_0} - \frac{gY^2}{kU_0^2(1-Y)})$$

Nothing in the derivation for the two-layer system will change when we increase the number of layers because these expressions only care about the free surface.

Now we must use the bottom layer to find the dispersion relation. This will as previously mentioned be different for finite water depth and infinite water depth and we will look at them separately.

5.4 Finite Water Depth

For finite water depth, H, we have the condition that no water can go through the bottom. That is, v(-H)=0. This condition is in the Nth layer which from earlier notation will be on Φ_{N-1}. (5.3) give

$$v = 0 \implies A_{N-1}\mathrm{e}^{-kH} - B_{N-1}\mathrm{e}^{kH} = 0$$

where $-c_0 \sin(kx)$ has been omitted because it is not 0 for all x. This means

$$A_{N-1} = B_{N-1}\mathrm{e}^{2kH} \tag{5.9}$$

from the kinematic condition at the last interface we have from (5.5)

$$A_{N-1}\mathrm{e}^{-kh_{N-1}} - B_{N-1}\mathrm{e}^{kh_{N-1}} = A_{N-2}\mathrm{e}^{-kh_{N-1}} - B_{N-2}\mathrm{e}^{kh_{N-1}} \tag{5.10}$$

from which A_{N-1} and B_{N-1} can be found. Inserting (5.9) into (5.10) we get

$$\begin{aligned}
A_{N-1}\mathrm{e}^{-kh_{N-1}} - B_{N-1}\mathrm{e}^{kh_{N-1}} &= A_{N-2}\mathrm{e}^{-kh_{N-1}} - B_{N-2}\mathrm{e}^{kh_{N-1}} \\
B_{N-1}\mathrm{e}^{2kH} - B_{N-1}\mathrm{e}^{2kh_{N-1}} &= A_{N-2} - B_{N-2}\mathrm{e}^{2kh_{N-1}} \\
B_{N-1} &= \frac{A_{N-2} - B_{N-2}\mathrm{e}^{2kh_{N-1}}}{\mathrm{e}^{2kH} - \mathrm{e}^{2kh_{N-1}}} \\
A_{N-1} &= \frac{A_{N-2} - B_{N-2}\mathrm{e}^{2kh_{N-1}}}{\mathrm{e}^{2kH} - \mathrm{e}^{2kh_{N-1}}}\mathrm{e}^{2kH}
\end{aligned}$$

Remember also that Z_{N-1} can be found directly from (5.4). Using the last equation available, the Bernoulli equation (5.6), we can now solve for the last variable c_0. This is what we call the dispersion relation. Rearranging (5.6) by moving the left hand side to the right hand side we have:

$$A_{N-1}\mathrm{e}^{-kh_{N-1}} + B_{N-1}\mathrm{e}^{kh_{N-1}} - A_{N-2}\mathrm{e}^{-kh_{N-1}} - B_{N-2}\mathrm{e}^{kh_{N-1}} - \frac{Z_{N-1}}{c_0 k} S_{N-2,N-1} = 0 \tag{5.11}$$

Where $S_{N-2,N-1} = S_{N-2} - S_{N-1}$ to save space. Here only c_0 is unknown and can thus be determined.

5.5 Infinite Water Depth

For infinite water depth the boundary condition on the last potential Φ_{N-1} is that it cannot blow up when $y \to -\infty$. Mathematically $B_{N-1} = 0$. The kinematic boundary condition (5.5) is then

$$A_{N-1}\mathrm{e}^{-kh_{N-1}} = A_{N-2}\mathrm{e}^{-kh_{N-1}} - B_{N-2}\mathrm{e}^{kh_{N-1}}$$

and inserting this into the Bernoulli equation (5.6) gives

$$\begin{aligned} (A_{N-1}\mathrm{e}^{-kh_{N-1}} - A_{N-2}\mathrm{e}^{-kh_{N-1}} - B_{N-2}\mathrm{e}^{kh_{N-1}}) &= \frac{Z_{N-1}}{c_0 k}(S_{N-2} - S_{N-1}) \\ \frac{Z_{N-1}}{c_0 k}(S_{N-2} - S_{N-1}) + 2B_{N-2}\mathrm{e}^{kh_{N-1}} &= 0 \end{aligned} \tag{5.12}$$

This equation serve as the dispersion relation for infinite water.

5.6 Solving the Dispersion Relation

As mentioned earlier, writing out the dispersion relation in only the known constants S, h, U_0 and g will give very ugly expressions which are hard to simplify further because of the exponential terms. Naturally these expressions will also be very hard to solve analytically if we want to for example find c_0 as a function of k. Based on this we would rather solve the relation numerically.

Numerically the dispersion relation is very simple to solve by a guessing method. We guess on a value of Y (or c_0) and then we calculate A_0 and B_0 from (3.13). From A_0 and B_0 we can calculate all the constant until we have A_{N-1} and B_{N-1}. Inserting these values into the dispersion relation in equation (5.11) or (5.12) we get an error. That is, unless we guessed the exact right value of Y the left hand side in the dispersion relation is different from the right hand side which is 0. **The value of the left hand side is called the error, and will be used as a measure on how good the guessed Y was**. We then repeat this process for a rang of Y-values. If any y has an error that is almost 0 we say that this is an solution.

In the above we found Y for one given k. Now we do the same for a range of k-values. Finally we have c_p as a function of k. The derivative of this curve times k will be the group velocity $c_g(k)$.

We mentioned earlier that we have a problem when $u_n = 0$ and we loose the kinematic boundary condition. That is, when the phase velocity is the same as the stream velocity at an interface with respect to the frame of the bottom. Numerically we work around this problem by taking the limits on both sides. We do not plug in the c_0 values that cause problems, but rather a value little larger and a value a little smaller. Since these limit exists we oversee the problem for now until it is further discussed in Section 7.1.

The method require that the heights h_n, the shears S_n and the surface stream velocity

U_0 are given. If the shears and heights are not given but rather the velocity profile $u(y)$, as will be the case for the example in this text, we just pick some heights and calculate the velocity at these heights. From these velocities S is given as $S = \frac{\Delta u}{\Delta h}$. For simplicity the profiles in this text will be divided in **equally deep layers**. It would be more optimal to divide in many layers where the velocity is rapidly changing and fewer where the profile is almost constant. However, if we have enough layers this will not be important.

Why was this guessing method chosen? It was mainly chosen because it was the first method that came to mind which did not need the derivative of the equation. The derivative is not so trivial with N layers. Other numerical methods might probably be faster, but the chosen method was fast enough.

All numerical work by the author is given in the Appendix.

5.7 How Many Solutions Do We Analytically Have as a Function of N

Before we analyze two specific examples numerically we will do some analytic work to find out how many solutions should be present, so we know better what to expect.

We see that the dispersion relations (5.11) and (5.12) are to the first order in c_0 and should give one solution. However, also the A_n, B_n and Z_n include c_0. If we look more closely to for example (5.12), since it is the easiest mathematically (the physics should be almost the same), we can figure out how many solutions we have. This brute force process takes a little work and colors are added to make it less confusing. From the dispersion equation (5.12) inserting for Z_{N-1} we get:

$$\begin{aligned}
\frac{Z_{N-1}}{c_0 k} S_{N-2,N-1} + 2B_{N-2}\mathrm{e}^{kh_{N-1}} &= 0 \\
-\frac{c_0}{u_{N-1}}(A_{N-2}\mathrm{e}^{-kh_{N-1}} - B_{N-2}\mathrm{e}^{kh_{N-1}})\frac{1}{c_0 k} S_{N-2,N-1} + 2B_{N-2}\mathrm{e}^{kh_{N-1}} &= 0 \\
B_{N-2}\mathrm{e}^{kh_{N-1}}(2ku_{N-1} + S_{N-2,N-1}) - A_{N-2}\mathrm{e}^{-kh_{N-1}} S_{N-2,N-1} &= 0
\end{aligned}$$

Now we plug in for A_{N-2} and B_{N-2} from (5.7) and (5.8)

$$\begin{aligned}
&(\frac{Z_{N-2}}{2c_0 k} S_{N-3,N-2}\mathrm{e}^{-kh_{N-2}} + B_{N-3})\mathrm{e}^{kh_{N-1}}(2ku_{N-1} + S_{N-2,N-1}) - \\
&(\frac{Z_{N-2}}{2c_0 k} S_{N-3,N-2}\mathrm{e}^{kh_{N-2}} + A_{N-3})\mathrm{e}^{-kh_{N-1}} S_{N-2,N-1} = 0 \\
&(\frac{Z_{N-2}}{c_0 k} S_{N-3,N-2} + 2B_{N-3}\mathrm{e}^{kh_{N-2}})\frac{1}{2}\mathrm{e}^{-kh_{N-2}}\mathrm{e}^{kh_{N-1}}(2ku_{N-1} + S_{N-2,N-1}) - \\
&(\frac{Z_{N-2}}{c_0 k} S_{N-3,N-2} + 2A_{N-3}\mathrm{e}^{-kh_{N-2}})\frac{1}{2}\mathrm{e}^{kh_{N-2}}\mathrm{e}^{-kh_{N-1}} S_{N-2,N-1} = 0
\end{aligned} \tag{5.13}$$

We see from the blue expressions that a part of what we end up with is the same as what we started out with, except that we have gone one layer up from $N-1$ to $N-2$ and from $N-2$ to $N-3$. Since u_{N-1} is of first order in c_0 it now seems like we have a second order equation and that number of solutions increase with N. However, we have to make sure that the second term after the $-$ sign is not of higher order. Here only the red term can contain c_0. Inserting for Z_{N-2}, the red term is:

$$\frac{Z_{N-2}}{c_0 k} S_{N-3,N-2} + 2A_{N-3}\mathrm{e}^{-kh_{N-2}} =$$
$$-\frac{c_0}{u_{N-2}}(A_{N-3}\mathrm{e}^{-kh_{N-2}} - B_{N-3}\mathrm{e}^{kh_{N-2}})\frac{1}{c_0 k} S_{N-3,N-2} + 2A_{N-3}\mathrm{e}^{-kh_{N-2}} =$$
$$B_{N-3}\mathrm{e}^{kh_{N-2}} S_{N-3,N-2} - A_{N-3}\mathrm{e}^{-kh_{N-2}}(S_{N-3,N-2} - 2ku_{N-2}) =$$
$$(\frac{Z_{N-3}}{c_0 k} S_{N-4,N-3} + 2B_{N-4}\mathrm{e}^{kh_{N-3}})\frac{1}{2}\mathrm{e}^{-kh_{N-3}}\mathrm{e}^{kh_{N-2}} S_{N-3,N-2} -$$
$$(\frac{Z_{N-3}}{c_0 k} S_{N-4,N-3} + 2A_{N-4}\mathrm{e}^{-kh_{N-3}})\frac{1}{2}\mathrm{e}^{kh_{N-3}}\mathrm{e}^{-kh_{N-2}}(S_{N-3,N-2} - 2ku_{N-2})$$

Also the red term contribute with one blue term and one red term and we also go one layer up from $N-n$ to $N-n-1$. So if we in (5.13) insert for both the red term and the blue term we go from $N-2$ to $N-3$ and end up with similar terms as we started with plus some exponential and S terms. Continuing this process we will finally end up with A_0 and B_0, which we know are of the same order of c_0 from (3.13). Hence we know that the highest order term in (5.13) is the first term (everything before the minus sign). The order of the equation must thus be given by

$$\left(\frac{Z_1}{c_0 k}(S_0 - S_1) + 2B_0\mathrm{e}^{kh_1}\right) f(S_i, h_i, u_j)\Big|_{i=1,2....,N-1, j=2,3,...,N-1}$$

where f is a function that includes all the remaining parts after we have inserted for all the A_n and B_n terms. Since we have to do $N-2$ iterations to go from Z_{N-1} to Z_1 and we gain one c_0 term for each iteration the order of f must be c_0^{N-2}. If we look at the term inside the bracket this is exactly the same as the dispersion relation for the two-layer system (3.24). This is a third order equation. Hence the total order of the dispersion relation is $c_0^3 * c_0^{N-2} = c_0^{N+1}$. The dispersion relation is of order $N+1$ and should have $N+1$ solutions.

Another way to easily see that the number of solutions is increasing with N is to go backwards and set $S_{N-2} = S_{N-1}$ in the dispersion relations. This should reduce the system with one layer. And in fact the Z-term disappears and the order is reduced by 1.

However, that the equation is of order $N + 1$ does not mean we have to have $N + 1$ solutions, but only that it is possible. For example could some of the solutions be complex. It is also strange that by this argument we would have an infinite number of solutions for the best approximation when $N \to \infty$. This seem odd, and in fact we will later argue that only two of the solutions are physically possible, and the rest are probably errors due to the discontinuity in the derivative of u at the interfaces (hereafter called bends/breaks) in our linearized velocity profile.

5.8 Reproduce Earlier Results

To test if the forumla derived above is true it would be wise to test it against other results. However, to the author knowledge the dispersion relation has not been derived earlier for more than two layers. We could test it against the results for the two- and three-layer case, but since we have seen that we use the exact same formulas as in Chapte 3 this is omitted.

However, the python program[7] used in the next section was tested against the numerical results of Brevik and found identical.

[7] Python is a programming language

6 Numerical Analysis of Waves in Rivers

In this section we will work with the problem numerically. For this purpose we will use two current velocity profiles often used to describe water flows in open channels/rivers; a parabolic and a logarithmic velocity profile. These two profiles were introduced in Section 2.4.

We will first find the solutions for the phase velocity c_p for a given wavenumber k and see if there are $N+1$ solutions as we would guess from the analytic investigation in Section5.7. Then we will discuss the convergence of c_p when we increase the number of layers. Optimally we would like to know how many layers are necessary for a good approximation of the velocity profiles. The emphasis in this section will be on the upstream solution because this is the most exciting as it can be used to stop waves. We will, however, also look barely on the other solutions.

All python programs used to generate the results are found in the Appendix. Due to large running times associated with large N, only systems with up to six layers have been investigated. However, as we will see, even few layers are sufficient for good results.

6.1 The c_p Solutions

First we want to find the solutions for the recursion formulas 5.11 and 5.12 to see if they match the numbers of solutions $N+1$ found in section 5.7, and to find their values.

To do this we will use the method described in section 5.6, which we will repeat here. If we consider for example the finite water depth parabolic velocity profile, the dispersion relation was found by (5.11) to be:

$$A_{N-1}\mathrm{e}^{-kh_{N-1}} + B_{N-1}\mathrm{e}^{kh_{N-1}} - A_{N-2}\mathrm{e}^{-kh_{N-1}} - B_{N-2}\mathrm{e}^{kh_{N-1}} - \frac{Z_{N-1}}{c_0 k} S_{N-2,N-1} = 0$$

Now we guess on values of $Y = U/c_p$ (or c_0, they are equivalent) and calculate the left hand side of the equation. Since we most likely never guess on the exact right Y-value, the left hand side will not be zero. We call the value of the left hand side the **error** because this is how far off we got with a particular Y. We do this for a lot of Y-values and hope some of them have almost zero error.

But, the dispersion relations include the constants A_{N-1}, B_{N-1}, A_{N-2} and A_{N-2} which we do not know. However, from Y we can find A_0 and B_0 directly from (3.13) and from there we use the recursion formula (5.7) and (5.8) to find A_{N-1}, B_{N-1}, A_{N-2} and A_{N-2}.

To visualize the solutions we calculate the error for a range of Y-values for one particular k, and plot the errors as a function of Y. If we do the calculations for a one-, two-, three-, four-, five-, and six-layer system and plot them below each other in the same figure in that order, the results will look like Figure 10. Take notice, we plot the absolute value of the error.

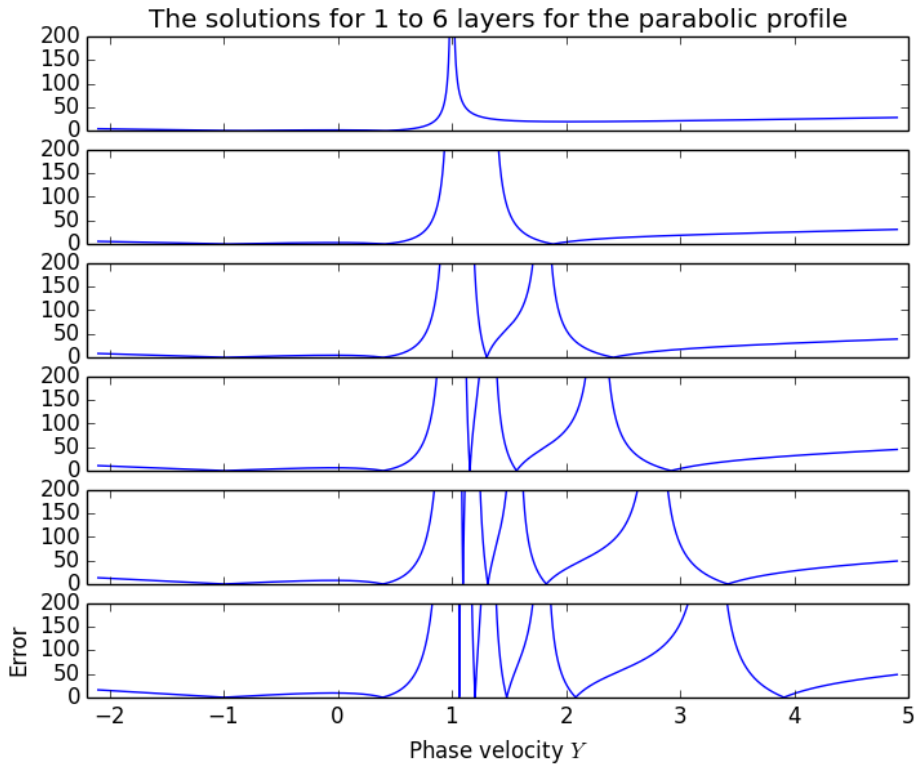

Figure 10: The absolute value error plot for the parabolic velocity profile with finite water depth for one to six layers with $U_0 = 0.5$ m/s, $H = 0.1$ m and k=10 m^{-1}. The one-layer system displayed at the top to the six-layer at the bottom. All layers in one system have the same height.

Remember that all six plots approximate the same current velocity profile. That is they have the same current surface velocity U_0 and water depth H. However, the profile is divided in a different number of layers. In this text all layers in one system are equally deep, even though this is not necessary. This means that for example the layers in the two-layer system is twice as high as those in the four-layer system.

In Figure 10 we have cut the error axis (y-axis) at 200 for the plot to look better, because the error gets really large for some Y. However, in this plot it is still hard to see the solutions, so we will in the next figures limit the error-axis to 0.30. It will still be the same plot. The python script used to make these plots are found in the Appendix A.5.

6.1.1 Solutions for the Parabolic Velocity Profile

The parabolic velocity profile was given in (2.6) to be

$$u(y) = \frac{\rho g \theta H^2}{2\nu}(1 - \frac{y^2}{H^2}) = U_0(1 - \frac{y^2}{H^2})$$

For this profile k is set to $10\,\mathrm{m}^{-1}$, then 800 000 $Y = U/c_p$ values between -2.1 and 4.9 is used to calculate errors (no solutions was found outside this interval). Instead of plotting the error against Y as in Figure 10 we will make a variable shift and plot against $(c_p - U_0)/U_0$. This is also a dimensionless variable but if Y gives the solutions in the frame of the bottom, $(c_p - U_0)/U_0$ gives the solutions in the frame of the surface velocity. Therefore, we can easily see if the wave is faster or slower than the surface. We will later use the term downstream and upstream solutions which are relative to the solid bottom. Downstream then means in the same direction as the current $c_p > 0$ and upstream means against the current $c_p < 0$. Note, if the current is strong enough the upstream solution will become downstream.

Figure 11 show the plot for this particular case with one to six layers. Here the error axis go from 0 to 0.30 so that we can easily see the solutions, and as guessed there seem to be $N + 1$ solutions. Some of the solutions have an error that is not 0, however, this is probably due to the numerical approach. The best tested Y-value was probably not exactly the actual solution. Another explanation might be that the solutions are not

real, but have an imaginary part. Based on earlier research by Brevik and Sollie [2] we would expect some instable or imaginary solutions for some k-values. However, because $Z = \frac{hg}{U^2}$ is large this is probably not the case. We will discuss this matter more in Section 7.5.

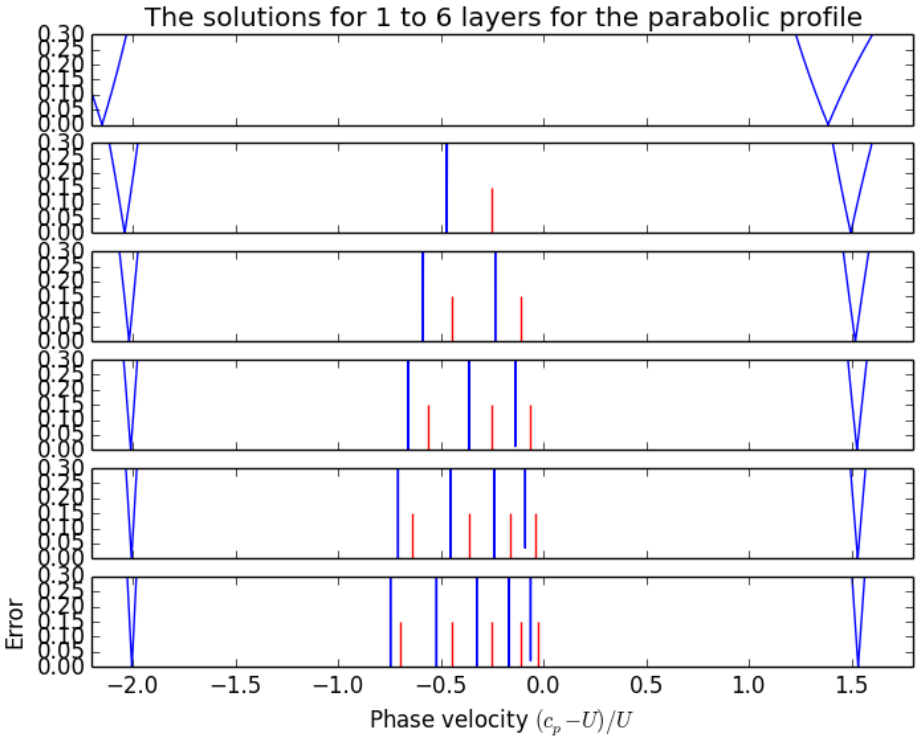

Figure 11: The solutions for the parabolic velocity profile with finite water depth for one to six layers with $U_0 = 0.5$ m/s, $H = 0.1$ m and k=10 m^{-1}. The one-layer system displayed at the top to the six-layer at the bottom. All layers in one system have the same height. Red lines denote the velocities at the interfaces.

Notice also the red lines. These lines denote the u_n or velocities at the interfaces. They

are included in the figure so that it is possible to compare the solutions to these velocities, something we will use in Chapter 7.

A popular approximation for shear flows is to use a constant profile $u(y) = U_0$. This should intuitively work fairly well in most cases because waves do not care about what happen at depths much larger then their wavelength. It would be interesting to compare this approximation to our linearizations. If we for the constant velocity profile change coordinate system to follow the current we get still water. From (2.3) we then have $(c_p - U_0)^2 = \frac{g}{k} tanh(kH)$ which with $g = 9.78$, $k = 10$, $U_0 = 0.5$ and $H = 0.1$ give $(c_p - U_0)/U_0 = \pm 1.72$. We see in Figure 11 that our leftmost and rightmost solutions are close to these but shifted a little to the negative x-direction. This is natural because the flow is stronger in the x-direction for the constant current velocity profile than for our linearizations.

Figure 12 shows almost the same system except that we have infinite still water below H. Here k is $25\,\mathrm{m}^{-1}$ and 200 000 Y-values between -4.1 and 13.9 have been tested. Again we have $N + 1$ solutions.

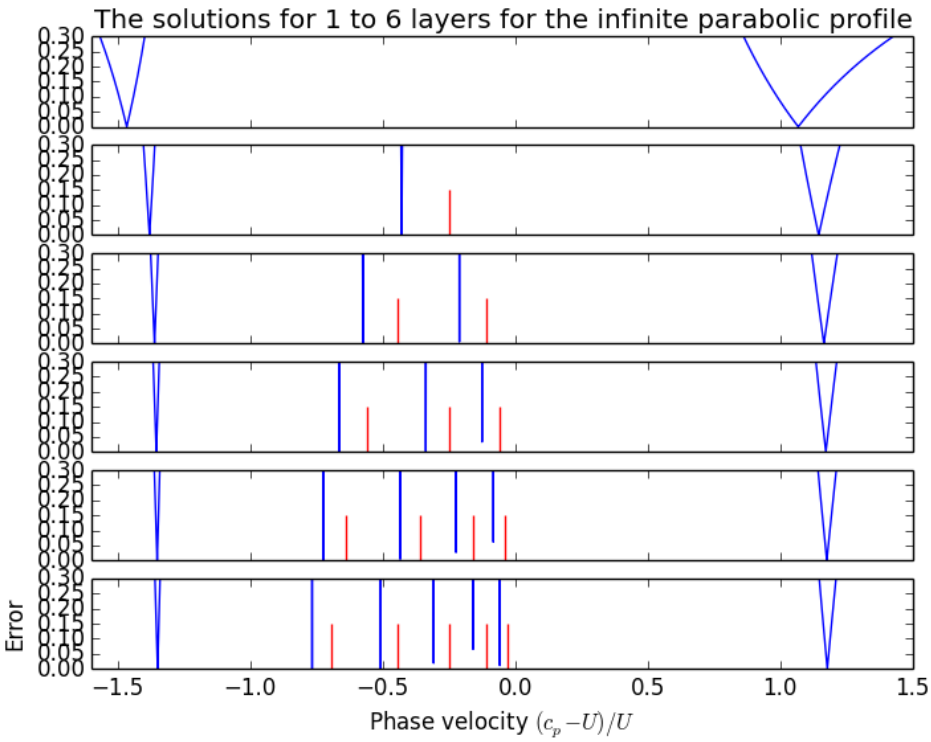

Figure 12: The solutions for the parabolic velocity profile with infinite water depth for one to six layers with $U_0 = 0.5$ m/s and k=25 m^{-1}. The one-layer system displayed at the top to the six-layer at the bottom. All layers in one system have the same height. Red lines denote the velocities at the interfaces.

6.1.2 Solutions for the Logarithmic Velocity Profile

The logarithmic velocity profile is from (2.7):

$$u = 0.253 \log_{10}(\frac{y+H}{0.005}) + 0.374$$

For this profile it was necessary to be more precise when testing different Y's to maintain low errors. As a consequence the computation was only done for four layers to keep running time and memory usage at reasonable levels. It was tested for $k = 5\,\mathrm{m}^{-1}$ with 8 200 000 Y-values between -2.8 and 1.5. The results are shown in Figure 13.

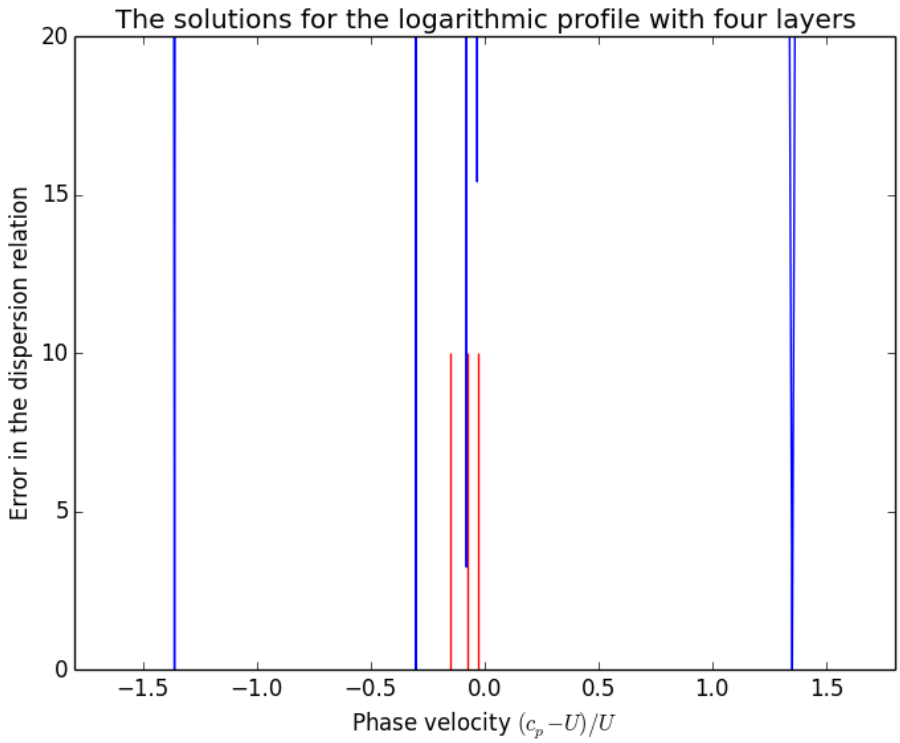

Figure 13: The solutions for the logarithmic velocity profile with finite water depth for four layers with $U_0 = 1.03\,\mathrm{m/s}$, $H = 2\,\mathrm{m}$ and k=15 m^{-1}. All layers have the same height. Red lines denote the velocities at the interfaces.

Since its not very relevant, there was not given much attention to why we had to be more precise for this profile. However, it could be because the bend in the velocity profile(or difference in shear) will be quite strong at the last interface as can be seen in Figure 5. However, this does not matter here. What do matter is that we again have $N + 1$ solutions.

6.2 The Phase Velocity, c_p

Now we want to look more into the phase velocity. We want to see how it converges with the number of layers and see its k-dependence. We can see the tendency of convergence in Figures 11 and 12. The solutions do not move much along the x-axis as the number of layers increases. Here we will concentrate on the leftmost solution (the upstream) and we will only investigate the finite depth version of the two profiles.

For the parabolic profile we tested 100 000 Y-values between -1000 and -0.4 to find the best solution for a given k. This was done for 20 k's between 0 and 40 m^{-1}. c_p was then plotted as a function of k. This was done for one to six layers and the result is given in Figure 14.

For the logarithmic profile we tested 100 000 Y-values between -100 and -0.1 and 20 k-values between 0.1 and 8 m^{-1}. The result is given in Figure 15.

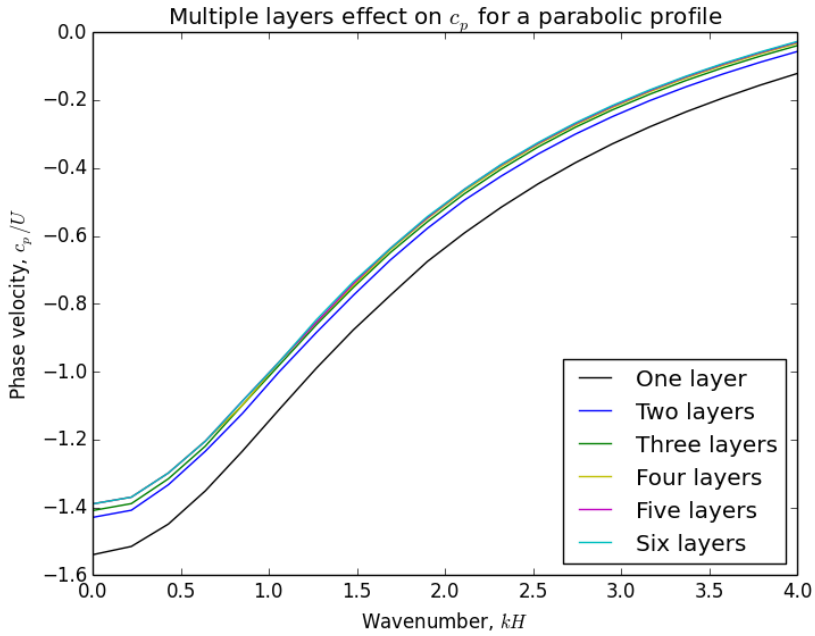

Figure 14: c_p as a function of k for the upstream solution on the finite depth parabolic velocity profile for one to six layers with $U_0 = 0.5\,\text{m/s}$ and $H = 0.1\,\text{m}$. All layers in one system have the same height.

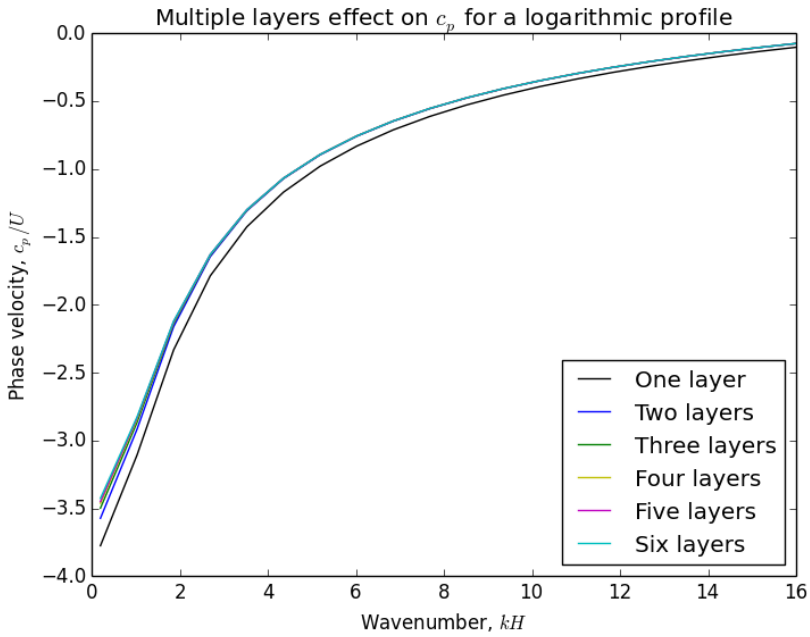

Figure 15: c_p as a function of k for the upstream solution on the finite depth logarithmic velocity profile for one to six layers with $U_0 = 1.032\,\text{m/s}$ and $H = 2.0\,\text{m}$. All layers in one system have the same height.

For both figures c_p seems to be quickly converging and the curves for $N > 2$ almost lay on top of each other. We can see that a one-layer system is a quite good approximation for both profiles, even for the logarithmic for which it did not look promising in Figure 5.

Earlier we compared our solutions to the approximation with a constant velocity profile $u(y) = U_0$. That result was only for one k-value and it could be interesting to see how the comparison would be for all k's. In Figure 16 the dispersion relation for the constant profile is plotted together with the two-layer linearization. We see that the constant

profile also is a fair approximation, even though the stronger current affect the velocity in positive x-direction.

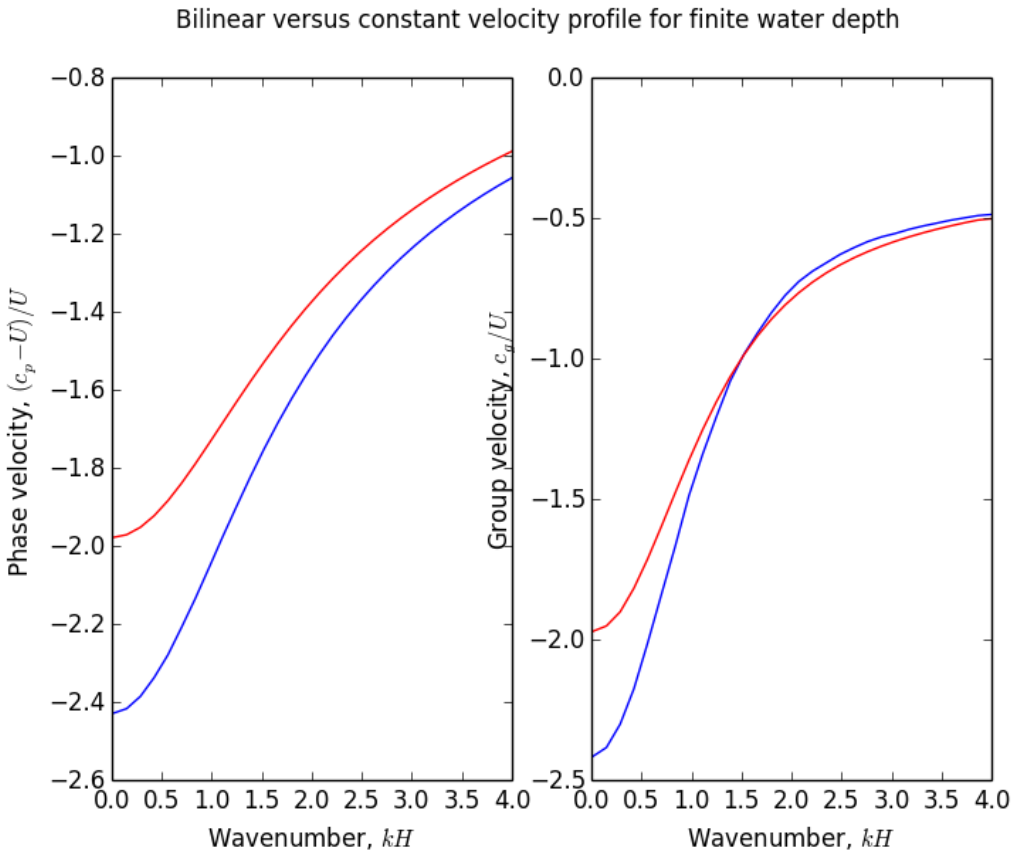

Figure 16: Comparison of c_p and c_g for the constant and the bilinear velocity profile approximations. The profile approximated is the parabolic finite depth profile with $U_0 = 0.5$ m/s and $H = 0.1$ m. Note, the y-axis is here $(c_p - U_0)/U_0)$ which explains why the curve for the bilinear profile looks different than in Figure 14 (they are of course the same). The constant profile is colored red and the bilinear blue. Both layers in the bilinear system have the same height.

6.3 Group Velocity, c_g

Investigating the group velocity is interesting due to the direct connection with energy transfer. Thinking back to Taylor's original pneumatic breakwater experiment it would be interesting to find when $c_g = 0$ because this is when the energy transfer is 0. When $c_g > 0$ the energy propagate with the current and away from the harbor from which we send the current. So we say that the current stop all waves with wavenumber larger than that for which $c_g = 0$.

From the theory chapter (2) we have the equations $c_g = \partial\omega/\partial k$ and $\omega = c_p k$. Thus we can easily find c_g from the c_p curves from the previous section in Figures 14 and 15. We only need to multiply them by k and differentiate. This is done numerically with the five-point stencil method which have an error of order Δk^4. The python script used for this differentiation is found in Appendix A.3.

The results for the parabolic profile is given in Figure 17 and the results for the logarithmic is found in Figure 18. The red lines mark $c_g = 0$, for which the energy transport is 0. Naturally also c_g converges quickly and even a one layer system is a fair approximation.

The reason for the different kH scales is that $kH > 1$ correspond to deep water wave and this is more appropriate for the turbulent logarithmic profile.

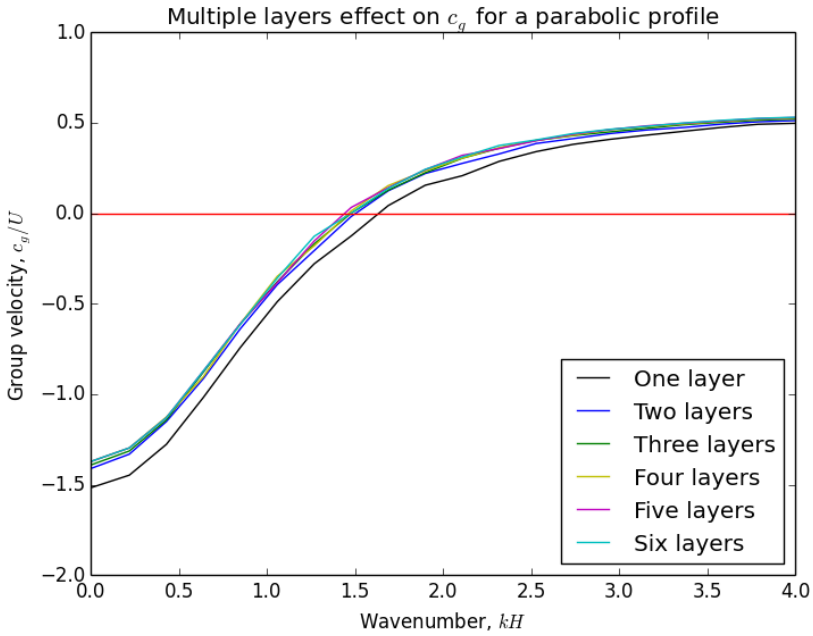

Figure 17: Show the upstream solution as a function of k for the parabolic velocity profile for one to six layers with $U_0 = 0.5\,\text{m/s}$ and $H = 0.1\,\text{m}$. All layers in one system has the same height. Red line denote $c_g = 0$.

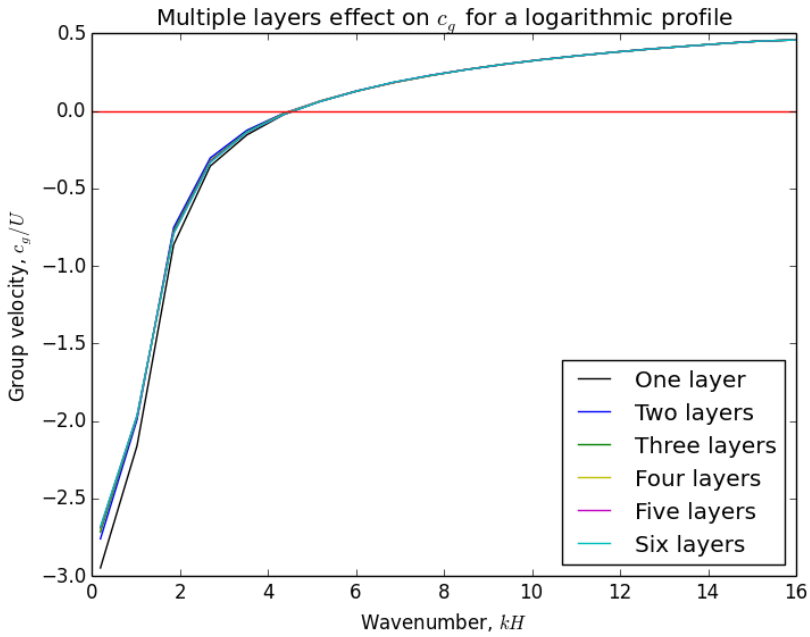

Figure 18: Show the upstream solution as a function of k for the logarithmic velocity profile for one to six layers. $U_0 = 1.032\,\text{m/s}$ and $H = 2.0\,\text{m}$. All layers in one system has the same height. Red line denote $c_g = 0$.

For specially interested readers the results for an example of the rightmost(largest) downstream solution for the parabolic profile is given in Figure 19. This figure also includes the phase velocity c_p. As we can see the group velocity cannot be zero for this mode. Which is natural since we cannot stop a wave be sending a current in its direction.

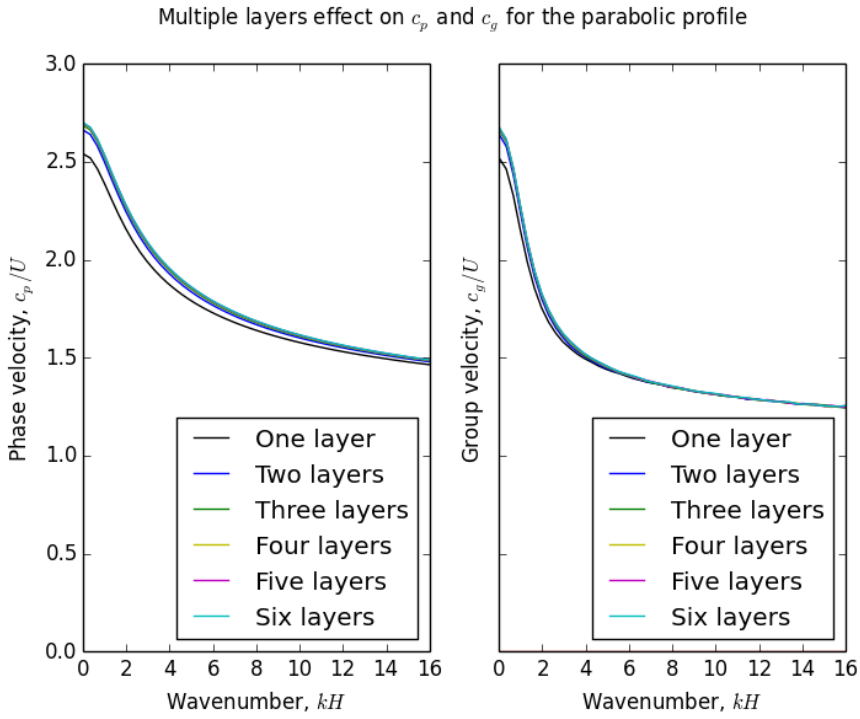

Figure 19: c_p and c_g for the downstream solution as a function of k for the parabolic velocity profile with finite water depth for one to six layers with $U_0 = 0.5$ m/s and $H = 0.1$ m. All layers in one system has the same height.

6.4 Amplitudes, Z_n

It can be interesting to see how the wave amplitude decreases with depth, and here we will find the amplitudes at the interfaces. This also contribute with some information we will use in the next chapter.

The recursion formula actually finds the Z_n's as an immediate step. This means that

from (5.1) the amplitudes can be found directly. By inserting the solutions into the recursion formula we can find the amplitude relative to the surface at all the interfaces for all the solutions. Notice however, that we cannot say anything about the wave amplitude at other heights. The bottom part of the code in Appendix A.5 writes out the solutions from the solutions plot and the code in Appendix A.4 calculates the amplitudes.

In table 1 the amplitudes for the parabolic profile with four layers are tabled for all the five solutions. Here $U_0 = 0.5\,\text{m/s}$, $H = 0.1\,\text{m}$ and $k = 10\,\text{m}^{-1}$.

Solution$((c_p - U_0)/U)$	Interface (n)		
(m/s)	1	2	3
-2.00887	0.77928	0.58078	0.35752
-0.65739	-1.4940	-5.6508	-33.158
-0.36153	-8.1074	-38.900	8.0222
-0.13603	-97.766	22.969	2.6068
+1.52714	0.62340	0.33528	0.13422

Table 1: The relative amplitudes Z_n for the different solutions appearing in the four layer description of the parabolic velocity profile. $U_0 = 0.5\,\text{m/s}$, $H = 0.1\,\text{m}$ and $k = 10\,\text{m}^{-1}$. Blue colors mark realistic amplitudes.

We see that the two standard solutions, those appearing for all N [8], here marked in blue, have realistic decaying amplitudes. The extra solutions, which we will call the rest of the solutions, have on the other hand unrealistic and strange amplitudes. We notice that the amplitudes increase in negative value until a maximum negative value before they all of sudden turns positive and start decaying more normally.

For the logarithmic profile the results for four layers are shown in table 2 with $H = 2\,\text{m}$ and $k = 5\,\text{m}^{-1}$.

[8] These are the leftmost and rightmost solutions in Figure 11

Solution($c_p - U_0$)	Interface (n)		
(m/s)	1	2	3
-1.36111	0.08414	0.00719	0.00082
-0.30226	-128.73	-1861.2	-32595
-0.07987	-2815.0	-264896	609.97
-0.03311	-134766	601.64	81.245
+1.34883	0.08008	0.00595	-0.00420

Table 2: The relative amplitudes Z_n for the different solutions appearing in the four layer description of the logarithmic velocity profile. $H = 2\,\mathrm{m}$ and $k = 5\,\mathrm{m}^{-1}$. Blue color marks realistic amplitudes. The red amplitude is negative but should have been positive.

We see again that the standard solutions marked in blue have realistic amplitudes while the extra solutions follow the same pattern as for the parabolic profile. The amplitude at the last interface for the downstream solution is marked in red because it is negative, which would be strange. However, this is probably just due to the solution not being accurate enough. Inserting the exact solution would probable give a very small but positive amplitude.

7 The Extra Solutions

In the previous chapter we did numerical analysis on two river profiles and found that for these two profiles even a linear or a bilinear velocity profile is a good approximation. This is together with the recursion formula the main result of the thesis. However, we cannot just ignore all the extra solutions that appear when we increase N. In this section we will argue that these extra solutions are unphysical and permissible to neglect.

We will start by discussing critical layer solutions as these solutions, in addition to the extra solutions, also are present mathematically. As well they share many properties with the extra solutions.

7.1 Critical Layers

Critical layer solutions have earlier mostly been discussed for internal gravity waves, but they are also present in laterally shear flows as we have in this text [7]. A critical layer can occur at singularities such as those we encountered in Chapter 3 and 5 when we had $u_n = 0$.

According to Miles [34] we have a critical layer at depth $y = y_c$ if $u(y_c) = c$ and $u'(y_c) \neq 0$. All these conditions are satisfied at our interfaces and we should expect to have critical layer solutions also in our systems.

Critical layer solutions have been investigated by many and are a well known phenomenon. They are not a regular waves but more like a layer of disturbances that drift with the current. Lord Kelvin showed that vortices looking as ”cat’s eye” formed near such layers [35], see Figure 20. Based on this, we can start to think that maybe the extra solutions are not real waves either.

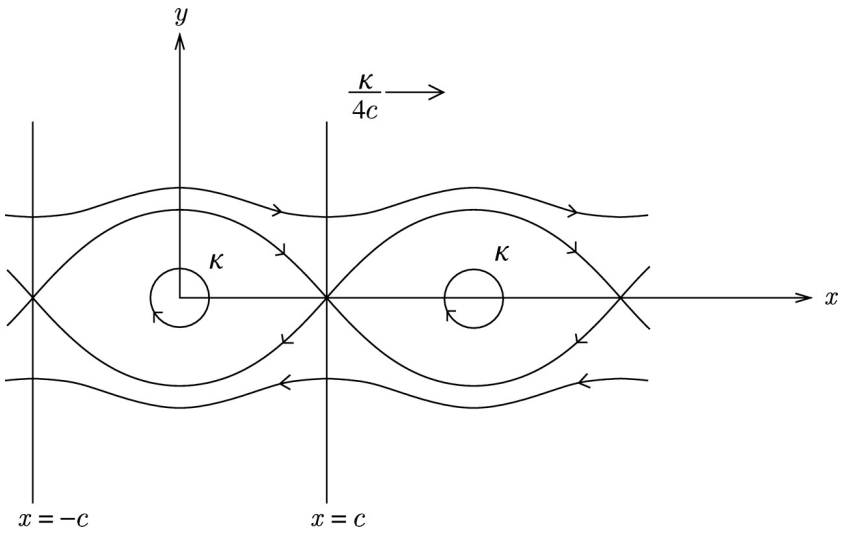

Figure 20: Kelvin-Stuart vortices. Figure taken from Fraenkel [1]

7.2 Drifting Solutions?

First we look at the extra solutions velocities. For the two river profiles the velocities are given in Figures 11, 12 and 13. We see that all the extra solutions lay in the interval $-1 < (c_p - U)/U < 0$ or that $0 < c_p < U$. So c_p is always positive with respect to the bottom, and always less than the surface velocity. This observation strengthen the hypothesis that also the extra solutions are disturbances drifting with the current.

Figures 11, 12 and 13 only shows the extra solutions for one specific value of k and we might just have been lucky with this k. Thus Figure 21 are added. It shows both c_p and c_g for the extra solution in the two-layer finite depth parabolic profile as a function of k. We see that the solution is in the drifting range for all k, that is $c_p > 0$ and $c_p < U_0$.

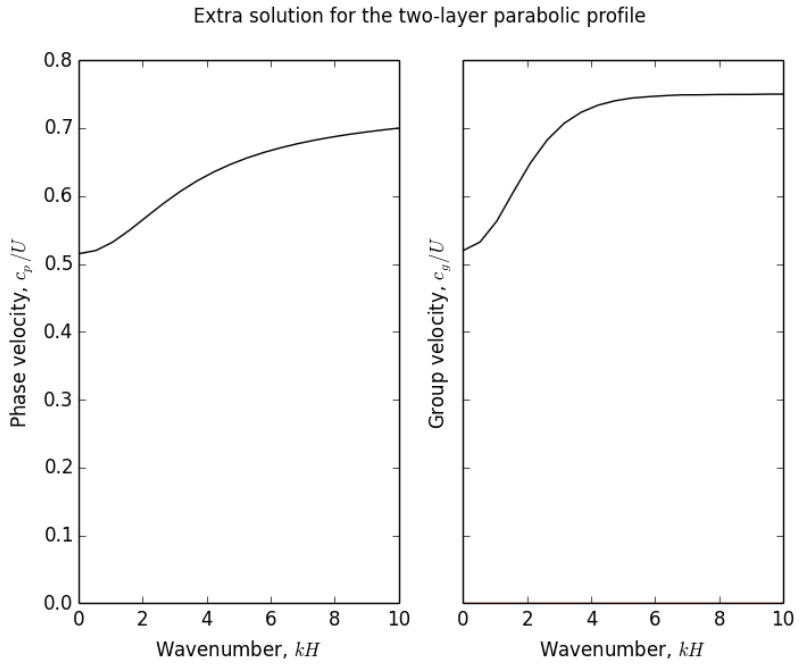

Figure 21: The extra solution for the two-layer finite water depth parabolic velocity profile with $U_0 = 0.5\,\text{m/s}$, $H = 0.1\,\text{m}$ and $k = 20\,\text{m}^{-1}$.

7.3 3D Generalization

Another observation worth mentioning is that the extra solutions do not come in pairs. Thinking of a generalization to three dimension, waves often appear as spherical waves. Think for example if you throw a rock in the ocean spherical waves will propagate outwards. This would represent the two original solutions in two dimensions. If the extra solutions were drifting disturbances this would fit well with them coming in odd numbers.

7.4 Strange Amplitudes for the Extra Solutions

The amplitudes also give some valuable information. First of all we notice in tables 2 and 1 that the amplitudes are rather large, even up to 134766 times the surface amplitude. This number could of course be very different if we had plugged in the exact solutions and not the numerical guess, but they are obviously way too large for natural waves. How can they physically get so large? Remember that we demand the velocity profile to be as given, and this might require large energy input.

Further we noticed that the amplitudes have a sign change. This sign change happens at different interfaces for the different solutions. The sign change could be explained if the solutions were drifting vortices like the ”cat’s eye” from Figure 20. Such a vortex would obviously have a sign change for the amplitude across it and the amplitude would decrease away from it. This fits the results from tables 2 and 1 very well if we assume that the solution at the second row have a vortex below the third interface, that the solution at the third row have a vortex between the second and the third interface, and that the solution at the forth row have a vortex between the first and second interface.

Are the extra solutions in the regions just described? Well if they are drifting disturbances they would have to drift at a height where their ”phase”[9] velocity is equal to the current velocity. Since the current velocity in both river profiles are strictly decreasing with depth, each velocity is connected to a height. So to see if an solutions is in a specific layer we can just check if its velocity is between the velocities at the interfaces below and above the layer. We will use Figures 11, 13 and 12 in which the red lines denote the velocity at the interfaces. The red line furthest to the right represent the first interface since its velocity is closest to U_0 and the red line furthest to the left denote the last interface. From the tables 2 and 1 we see that the second row is the rightmost extra solution and thus lie below the third interface. Also the third and forth row fit with our hypothesis in the end of last paragraph. This observation really indicates that

[9] Phase velocity is not a expression normally used for drifting disturbances

the extra solutions are drifting vortices.

7.5 Extra Solutions Couple and Become Unstable

For the standard upstream solution the phase velocity is increasing with respect to the positive x-axis as kH increase as can be seen in Figure 14. It will be fair to assume that it at some point might interfere with the extra solutions. No figure in this text confirms this, because the python programs have problems when $c_p \to 0$ because this implies that $Y \to \infty$. We could have rewritten the program to work around this problem (use c_o instead of Y) but we will rather use the results of Brevik and Sollie [2]. In their article about instability they looked at the triangular profile in Figure 1 with infinite water depth. They sketched all the three solutions in the same plot for different values of $Z = hg/U_0^2$. Their plots are given in Figures 22 and 23.

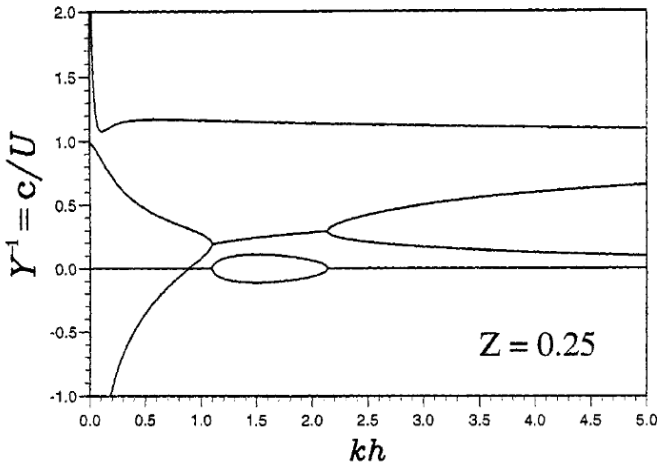

Figure 22: Large instability bubble for the two-layer system with $Z = 0.25$. Figure from Brevik and Sollie [2]. From the top the downstream solution, then the extra solution and last the upstream solution. The line at $Y^{-1} = 0$ including the stability bubble shows the imaginary part of the two solutions that couple.

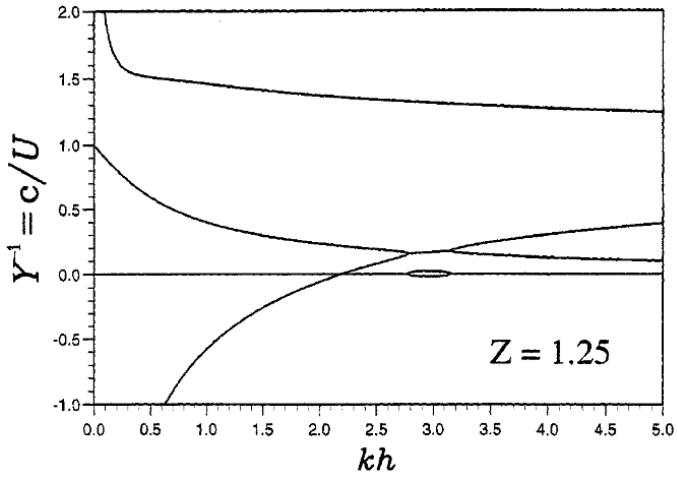

Figure 23: Small instability bubble for the two-layer system with $Z = 0.25$. Figure from Brevik and Sollie [2]. From the top the downstream solution, the extra solution and last the upstream solution. The line at $Y^{-1} = 0$ including the bubble shows the imaginary part of the two solutions that couple.

Here we see the convenience of the dimensionless variables. Since the dispersion relation (3.15) is written in only the three dimensionless variables Z, Y and kh we can set one of them constant and plot the two others. Here Z is set to 0.25 and 1.25 respectively and Y is plotted as a function of kh. We see that we have a larger instability bubble for smaller Z. The bubble show the imaginary part of the solutions. All their work is done analytically. For reference, in our earlier studies Z was approximately 4 for the parabolic profile and 20 for the logarithmic. We would therefor not expect instability bubbles there and only real solutions.

What happens physically is that the upstream solutions and the extra solution approach the same value. When they get close enough they will couple at the interfaces because the equation is global. All the solutions are present at the same time and we cannot look at them individually. This coupling make the solutions complex and in

fact the two solutions will be complex conjugates resulting in one of them being stable and the other unstable. Mathematically we can see the unstableness if we make the complex generalization of the wave given by (2.2)

$$\eta(x,t) = a\cos\left(k(x-\frac{\omega}{k}t)+\theta\right) = \Re[a\mathrm{e}^{(ik(x-\frac{\omega}{k}t)+\theta)}]$$

Then a negative imaginary part in ω would make the wave grow with time, while a positive imaginary part would not.

If the N-layer system acts the same way the upstream solution would probably couple with all this extra solutions. This coupling would affect the downstream solution so drastically it seems unlikely. It would be easier if the extra solutions were unphysical and could be neglected. In addition also the critical layers can couple with the downstream solutions as shown by Fabrikant [36].

7.6 Extra Solutions Origin From Velocity Bends

Since we do not have any extra solutions for the one-layer system it is reasonable to believe that the each extra solution originates from an interface and not from a layer, and hence $N-1$ interfaces give $N-1$ extra solutions. These solutions do most likely come from the unphysical bends/breaks in the velocity profile at the interfaces. If this is true they would not be present for a real physical velocity profile.

The rectangular profile in Figure 1 does not just have bends in u, but it is also discontinuous. Solving this two-layer problem give four solutions [7] where two are complex conjugates. It appears like a bend gives one extra solution, and a discontinuity give two extra solutions. Both cases are unphysical and cannot happen in nature.

8 Conclusion

Wave-current interactions have several areas of application as for example stopping water gravity waves by pneumatic breakwater. Its therefore of great interest to understand these interactions as much as possible. Over the years many researchers have investigated wave-current interactions, but until this point it has been mostly on linear or bilinear flows. The purpose of this project has been to extend this work to take into account any arbitrary current velocity profile by approximating the velocity profile with N linear layers.

First we reproduced earlier work with finding the dispersion relation for water waves on a linear and a bilinear current with the Stokes expansion method. Then we solved the three-layer system for which the dispersion relation got very complex. However, a comparison of the intermediate solving steps for the one-, two- and three-layer revealed a pattern for an increase in N. The governing equations which are evaluated at the interfaces do only care about the two layers they divide. Since the perturbation from the wave on the velocity profile can be written as a potential flow, and this potential flow takes the same form for all layers except the bottom, the governing equations also take the same form at all the interfaces except the last. Thus it was possible to make a recursion formula. Starting from the free surface, for which we can find the potential, we can work our way down and find all the potentials. The governing equations at the last interface then ultimately gives us the dispersion relation. By changing the bottom boundary condition we can find the dispersion relation for both finite and infinite water depth.

The dispersion relation for systems with many layers will be very hard to solve analytically. However, with the recursion formula derived herein the relation is easy to solve numerically. To test the recursion formula two possible current velocity profiles of water flowing down a river were investigated; a parabolic and a logarithmic pro-

file. For possible waves on these profiles the phase velocity, the group velocity and the amplitudes at the interfaces was studied for different N.

The numerical analysis revealed that for these two profiles even a linear profile was a good approximation. This observation strengthens results of earlier research for which a linear profile was used as an approximation. For a bilinear system the results were really good, and it seems unnecessary to divide into more layers. If its desired to approximate the velocity profile with a constant profile $u(y) = U_0$, the results show that also this works fairly well.

The results also showed that for N layers the dispersion relation have $N + 1$ solutions. Since we only have two solutions for the still water case, the constant velocity profile and the one-layer system, we would expect only two solutions for all N, and hence we have $N - 1$ extra solutions. Most likely the extra solutions come from the unphysical bends in the velocity profile, and is therefore a weakness in the method of approximating the current velocity profile by linear layers. The phase velocity of these extra solution indicate that they can be disturbances drifting with the current, instead of waves, much similar to critical layer solutions, which appear when the phase velocity is equal the current velocity at an interface. The investigation of the amplitudes of the extra solutions support a hypothesis that the extra solutions, like the critcal layer solutions, are drifting layers of vortices (for example Kelvin-Stuart vortices).

8.1 Further Work

There are numerous possibilities for further work on the subject. For example extending the work to account for higher order theory, heterogeneous fluids or surface tension.

Another idea is to solve the problem in three dimensions. In three dimensions physics sometimes get more understandable. An example of this, is when the upstream and downstream solution in two dimensions become a spherical solution in three dimensions. Perhaps this could help better understand the extra solutions.

Perhaps more difficult, is to test the results herein experimentally, or to use the recursion formula to find the current profile that stops wave most energy efficiently.

Appendices

A Python Code for Numerical Analysis

This appendix will give the python codes used by the author. The codes are build up by four main functions that do the numerical calculations. Then for each figure a separate script is written that uses these functions to get data vectors that are later plotted.

For the plotting scripts to work it is thus important to import the following:

```
1 from __future__ import division
2 import numpy as np
3 from matplotlib import pyplot as plt
4 import sys
5 from functions import infinite, finite, derivative, amplitude
6 import pickle
7 from scipy.signal import argrelextrema
```

Here the functions in line five are the four main functions which will be given first, before we look at the plotting scripts used in the thesis.

A.1 Infinite

This function takes in the velocity profile, a range of k-values and a range of Y-values. For each k, all the Y-values are tested in the dispersion relation (5.12). Each Y-value will have an error connected to it, and the program picks the Y-value which contribute with the least error. Then we have one Y-value for all k and an error connected to this Y-value. The Y-values and errors are then returned.

```
def infinite(U, S, h, kvec ,Yvec): #Returns cp and cg for a water
    profile with infinite depth
    n=len(S)                    #Number of layers
    nk=len(kvec)                #Number of k's
    nY=len(Yvec)                #Number of Y's
    cp=np.zeros(nk)             #To store the cp that are correct for
        each k
    u=[0]*n             #Make u_n vector
    error=[0]*nY        #To store the error for each Y-value
    errors=[0]*nk       #To store the error for each k-value
    g=9.7826087         #Constant of gravity

    for ik in range(nk): # Need to find one value cp for each k
        k=kvec[ik]      #Pick k-value

        Ytrue=0 #To know what Y is, Y=U/cp

        for j in range(nY):
            Y=Yvec[j] #Pick a Y-value
            c0=U/Y      #Program use c values instead of Y to calculate
            u[0]=U-c0
            for i in range((n-1)):
                u[i+1]=u[i]-(h[i+1]-h[i])*S[i]
            A=0.5*(-(g+S[0]*u[0])/(k*u[0]*c0)-u[0]/c0)
            B=A+u[0]/c0
            for i in range((n-2)): #Use the recursion function to find
                the constants A & B
                Zn=-c0/u[i+1]*(A*np.exp(-k*h[i+1])-B*np.exp(k*h[i+1]))
```

```
26                  A=0.5*Zn/(c0*k)*(S[i]-S[i+1])*np.exp(k*h[i+1])+A
27                  B=0.5*Zn/(c0*k)*(S[i]-S[i+1])*np.exp(-k*h[i+1])+B
28
29              #Testing with the dispersion relation to see how correct
                    this value of Y is
30              Zn=-c0/u[n-1]*(A*np.exp(-k*h[n-1])-B*np.exp(k*h[n-1]))
31              error[j]=2*B*np.exp(k*h[n-1])+Zn/(c0*k)*(S[n-2]-S[n-1])
32
33          err=sys.float_info.max #Variable to store the lowest error
34          for i in range(len(error)): #Find the Y-value with least error
35              if abs(error[i])<err:
36                      err=abs(error[i])
37                      Ytrue=Yvec[i]
38          cp[ik]=U/Ytrue
39          errors[ik]=err
40      return (cp, errors)
```

A.2 Finite

This function does exactly the same as the ”Infinite” above except it uses the dispersion relation for finite water depth (5.11).

```
def finite(U, S, h, H, kvec ,Yvec): #Returns cp and cg for a water profile with infinite depth
    n=len(S)                    #Number of layers
    nk=len(kvec)                #Number of k-values
    errors=[0]*nk               #To store the error for each k-value
    nY=len(Yvec)                #Number of Y-values
    cp=np.zeros(nk)             #To store the cp that are correct for each k
    u=[0]*n                     #Make u_n vector
    error=np.zeros(nY)          #To store the error for each Y-value
    g=9.7826087                 #Constant of gravity

    for ik in range(nk): # Need to find one value of cp for each k
        k=kvec[ik]

        Ytrue=0 #To know what Y is, Y=U/cp

        for j in range(nY):
            Y=Yvec[j] #Pick a Y-value
            c0=U/Y      #Program use cp-values isntead of Y to calculate
            u[0]=U-c0
            for i in range((n-1)):
                u[i+1]=u[i]-(h[i+1]-h[i])*S[i]
            A=0.5*(-(g+S[0]*u[0])/(k*u[0]*c0)-u[0]/c0)
            B=A+u[0]/c0
            for i in range((n-2)): #Use the recursion function to find the constants A & B
                Zn=-c0/u[i+1]*(A*np.exp(-k*h[i+1])-B*np.exp(k*h[i+1]))
                A=0.5*Zn/(c0*k)*(S[i]-S[i+1])*np.exp(k*h[i+1])+A
                B=0.5*Zn/(c0*k)*(S[i]-S[i+1])*np.exp(-k*h[i+1])+B

```

```
29              #Testing with the dispersion relation to see how correct
                    this value of Y is
30              if n==1:
31                  error[j]=A*np.exp(-k*H)-B*np.exp(k*H)
32              else:
33                  Zn=-c0/u[n-1]*(A*np.exp(-k*h[n-1])-B*np.exp(k*h[n-1]))
34                  tempA=A
35                  B=(A-B*np.exp(2*k*h[n-1]))/(np.exp(2*k*H)-np.exp(2*k*h[
                        n-1]))
36                  A=B*np.exp(2*k*H)
37                  error[j]=2*(A-tempA)*np.exp(-k*h[n-1])-Zn/(c0*k)*(S[n
                        -2]-S[n-1])
38
39          err=sys.float_info.max #Variable to store the lowest error
40          for i in range(len(error)): #Find the Y-value with least error
41              if abs(error[i])<err:
42                      err=abs(error[i])
43                      Ytrue=Yvec[i]
44          errors[ik]=err
45          cp[ik]=U/Ytrue
46      return (cp, errors)
```

A.3 Derivative

Derivative is a simple program that takes the derivative using the five-point stencil. This is a familiar formula so there are probably several better programs available that do exactly the same.

Its inputs are a vector y, which in this context will be $c_p k$, and a small length Δx or her Δk. The output is the derivative of y which here will be the group velocity c_g. The error is in the order of Δk^4.

```
def derivative(y,dx): #Calculate derivative of a list of points with five-point stencil
    n=len(y)     #Number of y-values
    yder=[0]*n  #New vector for the derivatives
    for i in range(n-4):
        i=i+2    #We need to start on the 3rd value because we need to know two values before and two values after to use the five-point stencil
        yder[i]=(-y[i+2]+8*y[i+1]-8*y[i-1]+y[i-2])/(12*dx)
    yder[0]=(y[1]-y[0])/dx #Use first order formula for the first point and the last
    yder[n-1]=(y[n-1]-y[n-2])/dx
    yder[1]=(y[2]-y[0])/(2*dx) #Use second order formula for the second and second last point
    yder[n-2]=(y[n-1]-y[n-3])/(2*dx)
    return yder
```

A.4 Amplitudes

This function calculates the amplitudes at all the interfaces for a given solution Y. So if we already know the solution, we can plug it in together with k and the velocity profile to generate the amplitudes. Remember the amplitude is found directly from (5.4).

```
def amplitude(U, S, h, H, k ,Y): #Returns cp and cg for a water profile
     with infinite depth
    g=9.7826087  #Constant of gravity
    n=len(S)    #Number of layers
    u=[0]*n     #Vector for u_n
    Z=[0]*(n-1)     #Vector for the amplitudes
    c0=U/Y       #The program uses c_0 instead of Y to calculate
    u[0]=U-c0
    for i in range((n-1)):  #Calculating the velocities at the
        interfaces
        u[i+1]=u[i]-(h[i+1]-h[i])*S[i]
    A=0.5*(-(g+S[0]*u[0])/(k*u[0]*c0)-u[0]/c0) #A_0
    B=A+u[0]/c0         #B_0
    for i in range((n-2)): #Use the recursion function to find the
        constants A & B
        Z[i]=-c0/u[i+1]*(A*np.exp(-k*h[i+1])-B*np.exp(k*h[i+1]))
        A=0.5*Z[i]/(c0*k)*(S[i]-S[i+1])*np.exp(k*h[i+1])+A
        B=0.5*Z[i]/(c0*k)*(S[i]-S[i+1])*np.exp(-k*h[i+1])+B
    Z[n-2]=-c0/u[n-1]*(A*np.exp(-k*h[n-1])-B*np.exp(k*h[n-1]))    #
        Calculates the amplitudes directly for each layer
    return Z
```

A.5 Number of Solutions

This script explains how the figures that show the solutions in Section 6.1.1 and 6.1.2 were made. First the "Finite" function is used for one k value and one Y-value. This will return the error for this particular Y. The process is repeated for a range of Ys, still one at a time. All the errors are collected in a vector which is then plotted versus the Y vector itself. From this we can see what Y-values are closest to the actual solutions. As done in the bottom we can also find the minimums of the error vector instead of using eyes.

When the calculations are done for several system with one to six layers they are all plotted in the same plot. Such that it should be easier to see the solutions, the plot have a y-max value.

```
1
2  from __future__ import division
3  import numpy as np
4  from matplotlib import pyplot as plt
5  import sys
6  from functions import infinite, finite, derivative, amplitude
7  import pickle
8  from scipy.signal import argrelextrema
9
10 #Finding solutions for a spesific k-value
11 Yn=800000
12 Ymin=-2.1
13 Ymax=4.9
14
15 k=[10]
16 U=0.5
17 H=0.1
18
19 Y=[0]
20 Yvec=np.linspace(Ymin, Ymax, num=Yn)
21
```

```
#One layer
h=[0.0]
S=[U*10]
error1=[0]*Yn
for i in range(Yn):
    Y[0]=Yvec[i]
    cp, error1[i]=finite(U, S, h, H, k, Y)

#Two layers
h=[0.0, 1/20]
S=[0.25*20*U, 0.75*20*U]
error2=[0]*Yn
for i in range(Yn):
    Y[0]=Yvec[i]
    cp, error2[i]=finite(U, S, h, H, k, Y)

#Three layers
h=[0.0, 1/30, 2/30]
S=[1*10/3*U, 3*10/3*U, 5*10/3*U]
error3=[0]*Yn
for i in range(Yn):
    Y[0]=Yvec[i]
    cp, error3[i]=finite(U, S, h, H, k, Y)

#Four layers
h=[0.0, 1/40, 2/40, 3/40]
S=[1*10/4*U, 3*10/4*U, 5*10/4*U, 7*10/4*U]
error4=[0]*Yn
for i in range(Yn):
    Y[0]=Yvec[i]
    cp, error4[i]=finite(U, S, h, H, k, Y)

#Five layers
h=[0.0, 1/50, 2/50, 3/50, 4/50]
S=[1*10/5*U, 3*10/5*U, 5*10/5*U, 7*10/5*U, 9*10/5*U]
error5=[0]*Yn
for i in range(Yn):
```

```
    Y[0]=Yvec[i]
    cp, error5[i]=finite(U, S, h, H, k, Y)

#Six layers - should give 7 solutions
error6=[0]*Yn
h=[0.0, 1/60, 2/60, 3/60, 4/60, 5/60]
S=[1*10/6*U, 3*10/6*U, 5*10/6*U, 7*10/6*U, 9*10/6*U, 11*10/6*U]
for i in range(Yn):
    Y[0]=Yvec[i]
    cp, error6[i]=finite(U, S, h, H, k, Y)

cpvec=[0]*Yn
for i in range(Yn):  #To plot versus (U/Yvec[i]-U)/U instead of Y
    cpvec[i]=(U/Yvec[i]-U)/U

#Plotting
f, axarr = plt.subplots(6, sharex=True, sharey=True)
axarr[0].plot(cpvec, error1)
axarr[0].set_title('The solutions for 1 to 6 layers for the parabolic
    profile')
axarr[1].plot(cpvec, error2)
axarr[1].vlines((-U*1/4)/U,0.0,0.15,'r') #Example on how the red lines
    are made
axarr[2].plot(cpvec, error3)
axarr[3].plot(cpvec, error4)
axarr[4].plot(cpvec, error5)
axarr[5].plot(cpvec, error6)

plt.ylabel('Error')
plt.xlabel('Phase velocity $(c_p-U)/U$')
plt.ylim(ymax=50)
plt.xlim(xmin=-2.2,xmax=1.8)

#Want to find the solutions for the two-layer system
error=error2
ne=3
sol=[0]*ne #Should have 4 solutions
```

```
j=0
for i in range(len(error)-2):
    i=i+1
    if (error[i]<error[i+1]) and (error[i]<error[i-1]) and (error[i
        ]<[0.3]):
        sol[j]=cpvec[i+1] #Picks as solution if it is less that the
            value before, the value after and less than 0.3.
        j=j+1
print[sol]
```

A.6 c_p and c_g

This script plots c_p and c_g as a function of k. c_p is basically found directly with the "Finite" or the "Infinite" function. Multiplying c_p by k and using "Derivative" gives c_g.

```
1
2  from __future__ import division
3  import numpy as np
4  from matplotlib import pyplot as plt
5  import sys
6  from functions import infinite, finite, derivative
7  import pickle
8
9  #The importance of multiple layers simulations
10 YN=10000
11 kN=50
12
13 U=0.5
14 H=0.1
15 k=np.linspace(0.1,40,num=kN)
16 Y=np.linspace(-1000 ,-0.4, num=YN)
17
18 #One layer
19 h=[0.0]
20 S=[U*10]
21 cp1, err1=finite(U, S, h, H, k, Y)
22 cg1=derivative((cp1*k),(k[1]-k[0]))
23
24 #Two layers
25 h=[0.0, 1/20]
26 S=[0.25*20*U, 0.75*20*U]
27 cp2, err2=finite(U, S, h, H, k, Y)
28 cg2=derivative((cp2*k),(k[1]-k[0]))
29
30 #Three layers
31 h=[0.0, 1/30, 2/30]
32 S=[1*10/3*U, 3*10/3*U, 5*10/3*U]
```

```
cp3, err3=finite(U, S, h, H, k, Y)
cg3=derivative((cp3*k),(k[1]-k[0]))

#Four layers
h=[0.0, 1/40, 2/40, 3/40]
S=[1*10/4*U, 3*10/4*U, 5*10/4*U, 7*10/4*U]
cp4, err4=finite(U, S, h, H, k, Y)
cg4=derivative((cp4*k),(k[1]-k[0]))


#Five layers
h=[0.0, 1/50, 2/50, 3/50, 4/50]
S=[1*10/5*U, 3*10/5*U, 5*10/5*U, 7*10/5*U, 9*10/5*U]
cp5, err5=finite(U, S, h, H, k, Y)
cg5=derivative((cp5*k),(k[1]-k[0]))

#Six layers
h=[0.0, 1/60, 2/60, 3/60, 4/60, 5/60]
S=[1*10/6*U, 3*10/6*U, 5*10/6*U, 7*10/6*U, 9*10/6*U, 11*10/6*U]
cp6,err6=finite(U, S, h, H, k, Y)
cg6=derivative((cp6*k),(k[1]-k[0]))


for i in range(len(k)): #Making variable dimensionless for the plotting
    cp1[i]=cp1[i]/U
    cg1[i]=cg1[i]/U
    cp2[i]=cp2[i]/U
    cg2[i]=cg2[i]/U
    cp3[i]=cp3[i]/U
    cg3[i]=cg3[i]/U
    cp4[i]=cp4[i]/U
    cg4[i]=cg4[i]/U
    cp5[i]=cp5[i]/U
    cg5[i]=cg5[i]/U
    cp6[i]=cp6[i]/U
    cg6[i]=cg6[i]/U
    k[i]=k[i]*H
```

```
70
71 #Plotting
72 plt.figure()
73 plt.ylabel('Phase velocity, $c_p/U$')
74 plt.xlabel('Wavenumber, $kH$')
75 plt.title("Multiple layers effect on $c_p$ for a parabolic profile")
76 plt.plot(k,cp1,'k')
77 plt.plot(k,cp2,'b')
78 plt.plot(k,cp3,'g')
79 plt.plot(k,cp4,'y')
80 plt.plot(k,cp5,'m')
81 plt.plot(k,cp6,'c')
82 plt.legend(('One layer','Two layers','Three layers','Four layers','Five
       layers','Six layers'), loc=4)
83 plt.show()
84
85 plt.figure()
86 plt.ylabel('Group velocity, $c_g/U$')
87 plt.xlabel('Wavenumber, $kH$')
88 plt.title("Multiple layers effect on $c_g$ for a parabolic profile")
89 plt.plot(k,cg1,'k')
90 plt.plot(k,cg2,'b')
91 plt.plot(k,cg3,'g')
92 plt.plot(k,cg4,'y')
93 plt.plot(k,cg5,'m')
94 plt.plot(k,cg6,'c')
95 plt.axhline(y=0,xmin=0,xmax=35, color='r')
96 plt.legend(('One layer','Two layers','Three layers','Four layers','Five
       layers','Six layers'), loc=4)
97 plt.show()
```

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