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Viscous cosmology

Bulk viscosity in the cosmological context

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Preface

The work contained in this book is a Master's thesis in viscous cosmology, conducted at the physics department at the Norwegian University of Science and Technology (NTNU). The thesis was carried out within the international masters program of physics. The work was started the spring/summer of 2014 and ended mid mai 2015. The work has been carried out under the guidance of professor Iver Brevik at the department of Energy and Process Engineering at NTNU.

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B.D.N.

Summary and Conclusions

The main objective of this thesis has been to estimate the present value of the bulk viscosity of the cosmic fluid. The following list contains a summary of the main results found in the present work:

- The bulk viscosity is estimated for a two-component fluid consisting of pressure-less matter in mixture with the cosmological constant and also for a one-component fluid. Comparison with observations of the Hubble parameter as a function of redshift is performed. The investigation reveals that the magnitude of the viscosity is highly dependent on the model used.

- The formalism and results found in a recent paper by J. Wang and X. Meng is discussed and put on vitally more solid ground through more accurate solutions of the energy equation together with the Friedmann equations. This is seen as one of the major contributions given through the present work.

- For the one-fluid a degeneracy between the functional forms found in the literature for $\omega(\rho)$ and $\zeta(\rho)$ is pointed out. It is found instructive to use a constant equation of state parameter ω instead, and to include all inhomogeneity in $\zeta(\rho)$. In this case the viscosity is bound to be negative with the form assumed for $\zeta(\rho)$.

- For a constant equation of state parameter in the one-fluid case, it is shown that - with the functional form considered for $\zeta(\rho)$ - no other choice than $\omega = -1$ can explain observations well.

- In general for the one-fluid case, $\zeta \sim \rho$ is shown to give a rather good fit of the Hubble parameter measurements, whereas $\zeta \sim \sqrt{\rho}$ by eye sight gives a clearly less correct fit.

- When allowing for a varying equation of state parameter $\omega(\rho)$, however, both positive, negative and zero viscosity explains observations equally well. This is shown to be a consequence of having both a varying $\omega(\rho)$ and $\zeta(\rho)$.

- An estimate obtained from small perturbations of the overall cosmic pressure is shown to suggest an order of magnitude $\zeta_0 = 10^6$ Pa s for the present day viscosity. This is in good agreement with the amount of viscosity that is needed to make a visible change in the plotted curves of the Hubble parameter as a function of redshift. This is shown to be true for the two component fluid as well as for the one-fluid.

- Finally, through applying kinetic theory, as well as using Winberg's formalism, a non-negligible candidate is found as a cause for the viscosity; electromagnetic radiation slightly out of thermal equilibrium with matter. The present value of this viscosity is estimated to be $\zeta_P = 10^3$ Pa s.

Contents

Chapter 1

Introduction

1.1 Background

The so-called ΛCDM-model is often referred to as the standard cosmological model, providing relatively simple explanations for many different observations, such as the accelerated expansion of the universe and the relative lack of anisotropy in the Cold Microwave Background radiation (CMB). The theory invokes solving Einstein's field equations with the FRW-metric, a metric for a *homogeneous* and *isotropic* universe. Further on it assumes that the mass-energy content of the universe, the *cosmological fluid* as it may be called, can be described by the equation of state for an ideal fluid.

The theory is not exclusive, however, when it comes to possible fates of the universe as a whole. Possible scenarios include a so-called *Big Rip*, a *Little Rip*, *Pseudo Rip*, *Quasi Rip* or also *Bounce* cosmologies. Refer to (Brevik I. and Timoshkin, 2014) and further references therein for details. The ΛCDM-model contains a cosmological constant Λ. A constant is the simplest possible way to include what often is referred to as *dark energy*. There are now several theoretical candidates challenging the notion of a non-dynamical cosmological constant, but the most recent cosmological data (Planck Collab.XIV; Aghanim, 2013) finds no evidence for a dynamical cosmological constant. A dynamical cosmological constant should be thought to have decisive impact on the end time evolution of the universe, so its nature is an important missing factor for the overall picture. All in all, the Λ CDM- model will be the starting point of the discussion in the present work.

Problem formulation

In the present standard model of cosmology, the cosmic fluid is, as mentioned, assumed to be an ideal fluid. Although observations seem to suggest that this is a good approximation, it cannot be exact. Since (Misner, 1967), viscosity has been discussed in cosmological context. (Weinberg, 1971) continues the discussion of dissipation in cosmological context and argues that a vanishing bulk viscosity "*is the exception, rather than the rule, for a general imperfect fluid*". Concerns about the perfect fluid hypothesis are expressed also in late literature. In (Cardone et al., 2006) one finds; "*It is worth noting, however, that, in all the models considered so far (with the remarkable exception of UDE* [unified dark energy] *models), it has been aprioristically assumed that dark energy behaves as a perfect fluid so that its EoS* [equation of state] *is linear in the energy density. Actually, from elementary thermodynamics, we know* *that a real fluid is never perfect and, on the contrary, such an assumption is more and more inadequate as the fluid approaches its thermodynamical critical points or during phase transitions.*" Or, as (Brevik I., 1996) puts it; "*Based upon common experience in fluid mechanics we would however expect that the viscosity concept may be important in cosmology also.*" and "*From a physical point of view it would in our opinion be almost surprising if the viscosity concept were not of importance in cosmology.*" (Brevik and Heen, 1994).

Shear viscosity would not be in keeping with **the cosmological principle**, which states that the universe is homogeneous and isotropic on cosmological scale (100Mpc and beyond). Neither can heat conduction be compatible with such a principle, since this would mean that the universe is not in thermodynamic equilibrium. The only remaining candidate that is compatible with the cosmological principle is therefore the already mentioned **bulk viscosity**. This is the candidate that has been investigated in the present work.

Bulk viscous cosmology is also an alternative to gravity modifying theories (Nojiri and Odintsov, 2011) in that it alters the right hand side of Einstein's field equations instead of the left hand side.

Most earlier studies have concentrated their investigation of dissipation in cosmological context on the early universe, since the density and temperature were so high back then. But adding dissipation (bulk viscosity) to the energy-momentum tensor of the cosmic fluid can be of interest also for the late time evolution of the universe. Papers like (Brevik and Gorbunova, 2005), (Brevik, 2013) have developed equations for investigating this. However, in order for these formulae to have any decisive impact on predicting future evolution of the universe, the *magnitude* of the bulk viscosity must be determined as precisely as possible. **To find boundaries for the magnitude of the bulk viscosity is therefore the aim that is taken with this work.**

What is contained in this dissertation, is to great extent a phenomenological study, but also quite some room is found for discussing potential causes for the viscosity. The one approach might not exclude the other, but the former has been the major emphasis in the present work.

The idea is to develop a theoretical framework in which to study the inclusion of a bulk viscosity parameter in the energy-momentum tensor for the cosmological fluid. Most recent Hubble parameter measurements are used to constrain from the observational side.

The rest of the dissertation is structured as follows:

Chapter 2 uses the literature to develop the mathematical framework that is used in standard cosmology. General relativity, the Friedmann equations and the equation of state for a perfect fluid are discussed. Thereafter the formalism is sought extended to viscous cosmology, and specialized to include only bulk viscosity. The formalism found in (Weinberg, 1971), and the general argument given in (Zimdahl, 1996) are both discussed in particular detail.

Chapter 3 continues to develop the framework through existing literature. This time the classical side is sought for insight. After all viscosity has been dealt with in classical kinetic theory for a long time. (Hänel, 2004) and (Landau and Lifshitz, 1981) are investigated for formulae that could be applied in the present cosmological context.

Chapter 4 is per say the heart of the present work, and relies on no source in particular. An extensive amount of independent work is done in this chapter. The recent paper (Wang and Meng, 2014) is investigated and used as starting point for finding general solutions of the viscosity-extended Friedmann equations together with energy conservation. From these solutions, comparison with experiments (measurements of the Hubble parameter H as function of redshift z) is performed. Boundaries for the bulk viscosity are sought on these grounds.
A section investigating potential causes for the viscosity is also included.

Chapter 5 concludes the dissertation. The most important results and conclusions are summarized. Some suggestions as to what further work might contain are also given.

1.2 Objectives

In accordance with what is written above, the main objectives of this Master's thesis will be

1. to give an overview of viscous cosmology with particular emphasis on bulk viscosity,

2. to survey existing literature for useful results in viscous cosmology of the present and future universe.

3. to estimate the present day value of the bulk viscosity in the cosmological fluid.

Chapter 2

Establishing viscous cosmology

2.1 Standard cosmology

As mentioned in the introduction, there is a standard cosmological model, the so-called ΛCDM-model. The Λ denotes the cosmological constant and *CDM* is an abbreviation for *Cold Dark Matter*[1]. As also was mentioned in the introduction, there are three main ingredients in the ΛCDM-model; the *Einstein Equations*, the *Friedmann-Lemaitre-Robertson-Walker*-metric, and the equation of state of a *perfect fluid*. The following sections will deal with these topics separately. But first a word on notation and mathematics.

2.1.1 Notation and Convention

Writing equations in LATEX is already cumbersome enough, so the following standard shorthands will be used in the text:

- Einstein's summation convention is used, i.e.; repeated indices are being summed over. E.g. $v_\mu v^\mu = \sum_\mu v^\mu v^\mu$.
- Repeated Latin indices i, j, k, etc. run over spatial components only
- Repeated Greek indices (μ, ν, etc.) run over all four co-ordinate labels
- The Minkowski metric is $\eta = diag(-1, 1, 1, 1)$
- The speed c of light is set equal to unity ($c = 1$) throughout the work unless a) otherwise specified or b) unless c is found in the equation and c) except in numerical results.
- The four-velocity is normalized such that $U_\mu U^\mu = -1$.
- Both $\frac{\partial}{\partial x^\mu}$ and ∂_μ and $,\mu$ are all three used for the same: Partial derivative with respect to coordinate x^μ. Which one is used depends on what seems more tidy in the given context. In the same fashion, *covariant differentiation* (to be defined) is denoted by $;\mu$.

Also, refer to appendix A for a list of abbreviations.

[1] *Cold* as opposed to *hot* (i.e. relativistically moving) dark matter

2.1.2 Mathematical toolbox

Before laying out the foundation for the ΛCDM-model, a few mathematical concepts and tools will be reviewed (and only superficially so, as the reader is expected to have a fair grasp of the concepts already). This section is based on (Weinberg, 1972), a book found to be very instructive on general relativity and cosmology in general. Before giving away the mathematics, recall the following principle, which in fact is an alternative way of stating the *Principle of Equivalence of Gravitation and Inertia*, and from which all motivation for the following mathematical concepts as used in this context springs:

Principle of general covariance: Any physical eqution holds in a general gravitational field if the following two conditions are met:
1) The equation holds true in the absence of gravitation.
2)The equations are generally covariant, which is defined to mean that they are preserved under a general coordinate transformation $x \to x'$.

In order to make equations invariant under coordinate transformations, one needs information about how the physical quantities in the equations behave under coordinate transformations. The following definitions and properties should be well known to the reader. The notation will be such that a quantity Q is defined over unprimed coordinates, whereas Q' is the corresponding quantity defined over primed coordinates.

Now, make a coordinate transformation $x^\mu \to x'^\mu$. Then,

- **scalar** quantities s, which indeed are tensors of rank zero, remain invariant; $s = s'$.
- A **contravariant vector** is one that transforms such that

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V_\nu. \tag{2.1}$$

  As example, take differentials $dx^m u$, or scalar fields ϕ. Closely related, a

- **covariant vector** is defined to transform such that

$$U'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} U_\nu. \tag{2.2}$$

  For instance, if ϕ is a scalar field, then $\partial_\mu \phi$ will transform covariantly.

- A **tensor** is the higher dimensional extension of the vector introduced (a vector is a tensor of rank 1). Each of its indices must transform in a covariant or contravariant way. For a general tensor, with both contravariant and covariant components (upper and lower indices, respectively), one finds that it transforms as

$$T'^{\mu\lambda\ldots}{}_{\nu\ldots} = \frac{\partial x'^\mu}{\partial x^\kappa} \frac{\partial x'^\rho}{\partial x^\nu} \frac{\partial x'^\lambda}{\partial x^\sigma} \cdot \ldots \cdot T^{\kappa\sigma}{}_\rho. \tag{2.3}$$

  Beyond the above definitions there are three algebraic properties that make tensor equations easy to use:

  - *Linear combinations* of tensors give tensors;

$$T^\mu{}_\nu \equiv a A^\mu{}_\nu + b B^\mu{}_\nu.$$

– *Direct products* of two tensors give a new tensor with the same upper and lower indices;

$$T^{\mu}{}_{\nu}{}^{\rho} \equiv A^{\mu}{}_{\nu}B^{\rho}.$$

– *Contractions* of tensors is defined such that

$$T^{\mu}{}_{\nu}{}^{\rho\nu} \equiv T^{\mu\rho}.$$

- **The metric tensor** $g_{\mu\nu}$ is a covariant tensor of rank 2 and its inverse $g^{\mu\nu}$ is a contravariant tensor of rank 2. They must satisfy

$$g^{\lambda\mu}g_{\mu\nu} = \delta^{\lambda}{}_{\nu}. \tag{2.4}$$

Raising and lowering of indices by the metric tensor is then defined such that

$$g^{\alpha\nu}T^{\mu}{}_{\nu} \equiv T^{\mu\alpha}$$

and similarly for the inverse

$$g_{\alpha\nu}T^{\mu\nu} \equiv T^{\mu}{}_{\alpha}.$$

- **Tensor densities** are non-tensor quantities that transform like scalars except for powers of the Jacobian $J = |\partial x^{\rho}/x'^{\mu}|$. Here and throughout vertical bars denote determinant. An important tensor density of weight -2, is the determinant of the metric tensor,

$$g \equiv -|g_{\mu\nu}|. \tag{2.5}$$

It transforms as

$$g = \left|\frac{\partial x'}{\partial x}\right|^{-2} g'. \tag{2.6}$$

The importance of this tensor density arises in the context of integral calculus, as it happens that

$$\sqrt{g}d^4x \text{ is an invariant volume element.}$$

- The **affine connection** - here not distinguished from the **Christoffel symbol** - is another very important non-tensor, defined such that

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu}\partial x^{\nu}}. \tag{2.7}$$

Here $\xi^{\alpha}(x)$ is taken to be the locally inertial coordinate system. This entity can be shown to transform as

$$\Gamma'^{\lambda}{}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\rho}{}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} \tag{2.8}$$

The last term clearly makes it a non-tensor. The Christoffel symbols may be written in terms of the metric tensor $g_{\mu\nu}$ as

$$\Gamma^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\kappa}\left(g_{\kappa\nu,\mu} + g_{\kappa\nu,\mu} - g_{\mu\nu,\kappa}\right). \tag{2.9}$$

- **Covariant differentiation**, denoted by ; is defined such that, when acting upon a contravariant or covariant tensor, it gives back a tensor[2];

$$V^{\mu}{}_{;\lambda} \equiv V^{\mu}{}_{,\lambda} + \Gamma^{\mu}{}_{\lambda\kappa} V^{\kappa} \quad \text{and} \quad V_{\mu;\lambda} \equiv V_{\mu,\lambda} - \Gamma^{\kappa}{}_{\mu\lambda} V_{\kappa}. \tag{2.10}$$

Note that the covariant derivative of the metric tensor and its inverse is zero;

$$g^{\mu\nu}{}_{;\lambda} = 0 \quad \text{and} \quad g_{\mu\nu;\lambda} = 0 \tag{2.11}$$

- Finally stating three useful formulae;
  - the **gradient of a scalar** s
    $$s_{;\mu} = s_{,\mu}, \tag{2.12}$$
  - the **covariant curl** becomes
    $$V_{\mu;\nu} - V_{\nu;\mu} = V_{\mu,\nu} - V_{\nu,\mu}, \tag{2.13}$$
  - the **covariant divergence**
    $$V^{\mu}{}_{;\mu} = \frac{1}{\sqrt{g}} \left(\sqrt{g} V^{\mu}\right)_{,\mu}. \tag{2.14}$$

In ending, recall that the general procedure for finding the effect of gravitation on a system, is to write down the special-relativistic equations, with the Minkowski metric $\eta_{\mu\nu}$, and then let $\eta_{\mu\nu} \to g_{\mu\nu}$ and also exchange ordinary derivatives with covariant ones $(, \to ;)$ in order to account for the change of the basis vectors as well. These equations will obey the Principle of General Covariance, and thus hold true in the presence of gravitation.

2.1.3 Einstein's field equations

The ΛCDM-model builds on general relativity in explaining the universe's overall behaviour. General relativity (GR) as applied to cosmology, is contained in Einstein's field equations, which may be derived from Lagrangian mechanics, via Hamilton's variational principle. Starting with an action integral S_G for gravity, which according to GR is due to curvature of space-time, and with an action integral S_M for matter, one solves for $\delta(S_G + S_M) = 0$. With

$$\delta S_G = \frac{1}{2\kappa} \int \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \delta g^{\mu\nu} d^4 x \tag{2.15}$$

[2] That the generalized derivative transforms like a tensor must be so in order to formulate equations of motion in general coordinate independent form. The definition can be motivated as follows. Just like the partial derivative , of a vectorfield **A** with respect to a parameter λ can be defined as

$$\frac{d\mathbf{A}}{d\lambda} \equiv A^{\mu}{}_{,\nu} U^{\nu} \mathbf{e}_{\mu}$$

in a Cartesian coordinate system, the more general covariant derivative is defined in the same way

$$\frac{d\mathbf{A}}{d\lambda} \equiv A^{\mu}{}_{;\nu} U^{\nu} \mathbf{e}_{\mu}$$

over any coordinate system; Cartesian or not. Thus the change of the basis vectors $\mathbf{e}_{\mu}$ is incorporated. Here $U^{\nu} \equiv \partial_{\lambda} x^{\nu}$.

and

$$\delta S_M = \int \left(\frac{\partial[\sqrt{-g}L_M]}{g^{\mu\nu}} - \left\{ \frac{\partial[\sqrt{-g}L_M]}{\partial g^{\mu\nu}{}_{,\lambda}} \right\}_{,\lambda} \right) \delta g^{\mu\nu} d^4x. \tag{2.16}$$

(Grøn and Hervik, 2007) show that one obtains

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}} \tag{2.17}$$

which are the famous **Einstein's field equations**. It can be shown that $\kappa = \frac{8\pi G}{c^4}$. $T_{\mu\nu}$ is the **energy-momentum tensor**. It reads

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \left(\frac{\partial[\sqrt{-g}L_M]}{g^{\mu\nu}} - \left\{ \frac{\partial[\sqrt{-g}L_M]}{\partial g^{\mu\nu}{}_{,\lambda}} \right\}_{,\lambda} \right) \tag{2.18}$$

and will be of later use.

Energy conservation

The equation of continuity in relativistic formulation reads

$$\int_{\partial\Omega} T^{\mu\nu} n_\nu d\sigma = 0, \tag{2.19}$$

where $\partial\Omega$ denotes the surface with outward normal n^ν bounding a region Ω in space-time. This equation must be fulfilled for energy and momentum conservation to hold. By Gauss' integral theorem one has

$$\int_{\Omega} T^{\mu\nu}{}_{;\nu} \sqrt{-g} d^4x = 0, \tag{2.20}$$

or, on differential form

$$\boxed{T^{\mu\nu}{}_{;\nu} = 0.} \tag{2.21}$$

2.1.4 The FRW-metric

So far the Einstein field equations have been established, but any attempt at solving these equations requires a metric. In the case of a homogeneous and isotropic universe, one must have a metric on the form

$$ds^2 = -dt^2 + a(t)^2 (d\chi^2 + r(\chi)^2 [d\theta^2 + sin^2\theta d\phi^2]). \tag{2.22}$$

By requiring that the model be isotropic, it may be shown that

$$r(\chi) = R_0 S_k(\chi/R_0), \tag{2.23}$$

where

$$S_k(x) = \begin{cases} \sin y & \text{if} \quad k > 0 \\ y & \text{if} \quad k = 0 \\ \sinh y & \text{if} \quad k < 0 \end{cases} \tag{2.24}$$

One can now represent equation (2.22) as

$$ds^2 = -dt^2 + a(t)^2(d\chi^2 + R_0^2 S_k^2(\chi/R_0)[d\theta^2 + sin^2\theta d\phi^2]), \tag{2.25}$$

or, equivalently, on the form

$$\boxed{ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2[d\theta^2 + sin^2\theta d\phi^2]\right),} \tag{2.26}$$

which is the famous **Friedmann-Robertson-Walker** line element for a homogeneous, isotropic universe, from now on referred to as the FRW-line-element, and correspondingly as the FRW-metric. Refer to e.g. (Grøn and Hervik (2007),pp 269-271) for a detailed deduction of this line-element. All homogeneous, isotropic universe models may be represented by the line-element (2.26). It is important to note from equation (2.26) that there are three main options for the shape of the space described:

- **k>0** corresponds to spaces with constant positive curvature, and these universe models are referred to as **open** universe models.
- **k=0** corresponds to Eucledian spaces, hence these universe models are referred to as **flat**.
- **k<0** corresponds to spaces with constant negative curvature, and yields again what is denoted as **open** universe models.

The scale factor $a(t)$ may be determined by inserting the FRW-metric into Einstein's equations, (2.17) and solving. This will give the equations

$$3\frac{\dot{a}^2 + k}{a^2} = 8\pi G\rho \tag{2.27}$$

and

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} = 8\pi GP, \tag{2.28}$$

which are the famous **Friedmann equations**. Combining the two Friedmann equations, one obtains an often useful equation as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P), \tag{2.29}$$

which is the so-called **acceleration equation**. Although these last equations were not derived here, they will be in the more general viscous case.

Co-moving coordinates, expansion, redshift and other Useful Definitions

The scale factor appears as the freedom left for a space in which homogeneity and isotropy is required; the space may still expand equally much in all directions. It is customary to introduce **co-moving coordinates** in a reference frame. These coordinates are defined such that a reference point, say $p_r = (x_r, y_r, z_r)$ has constant spatial coordinates. The 3-velocity of a particle in such a co-moving reference frame must therefore be 0 ($dx_r/dt = dy_r/dt = dz_r/dt = 0$). For the 4-velocity one then finds

$$U = \gamma\left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = (1,0,0,0), \tag{2.30}$$

which will be of great use later. Recall that the convention in this work is to normalize according to $U_\mu U^\mu = -1$. Also

$$U_0 \equiv -1 \qquad \text{Co-moving coordinates.}$$

Now; even though a reference point in the co-moving frame of reference has constant coordinates, the space *itself* expands, and any continuum must expand with it. Light rays for example. To measure the expansion rate of the universe, one can measure how much the wave length of electromagnetic radiation is redshifted with time. Define now the **proper distance**

$$d_p = a(t)d_r,$$

where d_r here is taken to be the distance between two reference points in the co-moving frame. Then

$$v \equiv \frac{dd_p}{dt} = \frac{\dot{a}}{a}ad_r = Hd_p, \tag{2.31}$$

where

$$H \equiv \frac{\dot{a}}{a} \tag{2.32}$$

is defined as **the Hubble parameter**. and equation (2.31) is denoted **the Hubble law**. The Hubble law says that the velocity of a distant object is proportional to it's proper distance d_p. The present value of the Hubble parameter is denoted the **Hubble constant**, and is denoted H_0. It is $H_0 = 67,77$s/km Mpc. Refer to table 2.2 for details.

Consider now a light ray in an expanding universe. It is emitted at co-moving position r_e and observed at co-moving position r_o. Since light moves along null-geodesics, its line element

$$ds^2 = -dt^2 + a(t)^2 dx_i dx^i = 0.$$

Taking the direction of motion to be along the z-axis only gives $-a(t)dz = dt$. Denoting the spatial separation $r_e - r_o = z_e$, one should have

$$-\int_{z_e}^{0} dz_e = z_e = \int_{t_e}^{t_o} \frac{1}{a(t))}dt. \tag{2.33}$$

The separation distance z_e should be the same regardless of what times at which it is measured. Let us assume that the light-rays emitted have period T_e. The period as received is denoted T_o. Then

$$-\int_{z_e}^{o} dz_e = \int_{t_e+\Delta t_e}^{t_o+\Delta t_o} \frac{1}{a(t))}dt. \tag{2.34}$$

Equating equation (2.33) and (2.34) one may obtain

$$\int_{t_o}^{t_o+\Delta t_o} \frac{1}{a(t))}dt = \int_{t_e}^{t_e+\Delta t_e} \frac{1}{a(t))}dt. \tag{2.35}$$

The periods T_e and T_o are so small that the scale factor may be assumed constant with values $a(t_e)$ and $a(t_o)$, respectively, under the integration. Then

$$\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_o}{a(t_o)}, \tag{2.36}$$

which by use of $c = \lambda/T$ (where λ is the wavelength) may be rewritten to

$$z + 1 = \frac{a_0}{a(t)}, \tag{2.37}$$

where $a(t_o) \rightarrow a_0$, $a(t_e) \rightarrow a(t)$ and the **redshift z** defined as

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} \tag{2.38}$$

has been used. It is customary to set the present value of the scale factor $a_0 \equiv 1$, such that one obtains

$$\boxed{a = \frac{1}{1+z}.} \tag{2.39}$$

To measure the expansion of space, one can therefore measure the redshift of light emitted from distant celestial objects. Consult e.g. (Goldhaber) for the "Supernova Cosmology Project (SCP)" and futher references therein. Note that (2.39) was obtained on kinematical grounds, and is independent of the energy content of the universe.

The energy content of the universe will affect the curvature of the universe, however. Rewriting the first Friedmann equation (2.27) in terms of the Hubble parameter H (2.32) one has

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{3a^2}. \tag{2.40}$$

The specific value of the matter content ρ at which the universe has zero curvature ($k = 0$) is denoted ρ_c and is by the above equation determined to be

$$\rho_c = H^2 \frac{3}{8\pi G}. \tag{2.41}$$

It is here customary to introduce the **relative densities**

$$\Omega = \frac{\rho}{\rho_c} \quad \text{and} \quad \Omega_k = -\frac{k}{H^2 a^2} \tag{2.42}$$

which allow for rewriting the first Friedmann equation to

$$\boxed{\Omega + \Omega_k = 1.} \tag{2.43}$$

It is customary to define

$$E^2(z) = \frac{H^2}{H_0^2}. \tag{2.44}$$

For further reference, note in particular that $\dot{H}$ now may be rewritten in terms of $\partial_z E$.

$$\dot{H} = \frac{dH}{dz}\dot{z} \overset{(2.39)}{=} -(1+z)^2 \dot{a}\frac{\partial H}{\partial z} \overset{(2.44)}{=} -(1+z)H_0^2 E \frac{\partial E}{\partial z} \tag{2.45}$$

where also $H = \dot{a}/a$ was used in the last step. This result will be of later use.

2.1.5 Equation of state

The last ingredient in the ΛCDM-model is the equation of state for a perfect fluid, which is assumed to hold true for all essential components of the mass-energy composition in the universe. The equation

$$\boxed{P = \omega\rho} \tag{2.46}$$

is the **equation of state for a perfect fluid**.

With that the framework of the ΛCDM-model has been established, and it is time to extend the framework to include dissipation; i.e – it is time to move on to *Viscous Cosmology*. For the sake of future reference, however, a section on thermodynamics precedes the continuation.

2.2 Short intermezzo on thermodynamics

In this section, a few important relations from thermodynamics will be given. It might occur messy to the reader that the equations at times are expressed in terms of *densities*, and at times *per particle*. The reason for this, however, is the later use of the equations in the text. (Hemmer, 2009) is used as source.
In the following, and throughout the dissertation (unless otherwise explicitly specified), the notation given in this section is adopted (SI-like units given in brackets), so please refer back to the following list of symbols if variables are used without explanation in the rest of the dissertation:

- ρ = mass density [kg/m^3]
- M, m = mass [kg]
- N = number [−]
- n = number density [1/m^3]
- u = internal energy density [J/m^3]
- μ = chemical potential per particle [J/m^3]
- S = entropy [J/K], s = entropy per volume [J/Km^3] and σ = entropy per particle [J/K]
- V = volume [m^3], $\hat{V}$ = dimensionless volume [−]
- P = pressure [Pa=N/m^2]
- c_i = specific heat when variable i is held constant [J/kgK] and c_i' = specific heat per particle as i is held constant [J/K]

Using this notation, one defines

$$du = T ds - P d\hat{V}, \tag{2.47}$$

which, being a combination of the first and second law of thermodynamics is denoted the **thermodynamic identity**; valid for reversible, materially closed systems. Expressed in dimensionless form, with the entropy per particle, it reads

$$k_B T d\sigma = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right) = \frac{1}{n}\left[d\rho - \frac{\rho + P}{n} dn\right], \tag{2.48}$$

where k_B is the Boltzmann constant. Defining the **work-function** w as

$$w = \frac{\rho + P}{n}, \tag{2.49}$$

one might establish that the **chemical potential** per particle, μ, can be related to the work function per particle, denoted w', via

$$\mu = w' - T\sigma, \tag{2.50}$$

from which follows the relations

$$\left(\frac{\partial\mu}{\partial T}\right)_P = -\sigma \quad , \quad \left(\frac{\partial\mu}{\partial P}\right)_T = \frac{1}{n}. \tag{2.51}$$

Entropy is a state function, and can be expressed in two of the three macroscopic variables P, V and T. Choosing T and P as independent variables, an infinitesimal change in the entropy per particle is

$$d\sigma = \left(\frac{\partial\sigma}{\partial T}\right)_P dT + \left(\frac{\partial\sigma}{\partial P}\right)_T dP. \tag{2.52}$$

It can be shown that

$$\left(\frac{\partial\sigma}{\partial T}\right)_P = \frac{c'_P}{T} \tag{2.53}$$

and also, when assuming an ideal gas, that

$$\left(\frac{\partial\sigma}{\partial P}\right)_T = -\frac{1}{P}. \tag{2.54}$$

The equation of state for an ideal gas is here written

$$P = Nk_B T. \tag{2.55}$$

Boltzmann's constant k_B can alternatively be expressed as $k_B = mR$, where m here is the mass of each particle and R is the specific gas constant ($[R]$ =m/kgK). Observe that (2.55) now can be expressed as

$$P = \frac{Nm}{V} C^2, \tag{2.56}$$

where $C^2 = RT$ One can think of $C = \sqrt{RT}$ as the average speed of the molecules in the gas. To make C^2 dimensionless, one may divide through by the speed of light squared, c^2, and denote the new constant ω. The equation becomes

$$P = \omega\rho, \tag{2.57}$$

where is the energy density defined to be $\rho \equiv Nmc^2/V$. The above equation is the same as (2.46). This equation is convenient when one deals with relativistic motion, so that speeds are comparable with c. For **pressure-less matter** ($P = 0$) one must necessarily have

$$\omega = 0 \tag{2.58}$$

and for **radiation** one finds

$$\omega = \frac{1}{3}, \tag{2.59}$$

which will be shown from the energy-momentum tensor later on. If choosing a fluid description of **the cosmological constant** one would have to choose

$$\omega = -1 \tag{2.60}$$

in order for the energy density to remain constant. These are not the only choices for equation of state parameters, but these are the ones needed in the present work.

2.3 Incorporating dissipation

The aim of this section is to go beyond the standard model by extending the formulae apparatus to the viscous case. This section will therefore include an overview, and to some extent derivations, of the equations still not given. The formulae is mostly relativistic.

2.3.1 Relativistic energy momentum tensor

It is natural to start by finding the general energy-momentum tensor for a *viscous*, *heat conducting* fluid. Starting with the perfect fluid case, the formulae will be extended to the more general case.

Perfect fluid

Following (Grøn and Hervik, 2007) again, it is here shown that the energy-momentum tensor of a perfect fluid may be derived from the general expression given as equation (2.18). The Lagrangian density for a perfect fluid is

$$L = -\rho, \tag{2.61}$$

where ρ is the proper energy density. To find $T^{\mu\nu}$, it is evident from equation (2.18) that one needs to calculate

$$\frac{\partial L_M}{\partial g^{\mu\nu}}$$

and

$$\frac{\partial L_M}{\partial g^{\mu\nu}{}_{,\lambda}},$$

of which the last beast vanishes, since the Lagrangian density L is independent of the derivatives of the metric. Using the thermodynamic relation

$$\left(\frac{\partial \rho}{\partial n}\right)_s = nw\partial n, \tag{2.62}$$

where, recall, nw is the work-function, together with equation (2.61) one finds that

$$\frac{\partial L_M}{\partial g^{\mu\nu}} = -nw\frac{\partial n}{\partial g^{\mu\nu}}. \tag{2.63}$$

To calculate δn the baryon number density will be defined in a co-moving orthonormal basis. Following the source, one defines a number flux vector n^μ such that

$$n^\mu = n\sqrt{-g}U^\mu, \tag{2.64}$$

with the number density n given by

$$n = \sqrt{\frac{g_{\mu\nu}n^\mu n^\nu}{g}}. \tag{2.65}$$

There are two constraints that have to be paid attention to. They are

$$\delta s = 0 \tag{2.66}$$

and

$$\delta n^\mu = 0. \tag{2.67}$$

Through straight forward, but rather technical manipulations, one may via (2.67), (2.65) (first equality) and (2.64) (second equality) now show that

$$\delta n = \frac{1}{2n}\left(\frac{n^\mu n^\nu}{g} - n^\mu n^\nu \frac{g_{\mu\nu}}{g^2}\delta g\right) = \frac{n}{2}\left(-U^\mu U^\nu \delta g_{\mu\nu} + \frac{U^\mu U^\nu}{g}\delta g\right) \tag{2.68}$$

which through further manipulations, and by $U^\mu U_\mu = -1$, becomes

$$\delta n = \frac{n}{2}(U_\mu U_\nu + g_{\mu\nu})\delta g^{\mu\nu}. \tag{2.69}$$

With $n\omega = \rho + P$ one therefore has

$$\frac{\partial L}{\partial g^{\mu\nu}} = -\frac{1}{2}(\rho + P)(U_\mu U_\nu + g_{\mu\nu}), \tag{2.70}$$

which through equation (2.18) gives

$$\boxed{T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu},} \tag{2.71}$$

the relativistic expression for the **energy-momentum tensor** of a **perfect fluid**.

Now; if the particles in the perfect fluid are moving relativistically, the fluid would behave like a gas of photons. Since the trace of the energy momentum tensor for an electromagnetic field vanishes, one has from equation (2.71) that

$$T^\mu{}_\mu = -(\rho + P)U_\mu U_\nu + 4P = 0 \rightarrow P = \frac{1}{3}\rho \tag{2.72}$$

which is the equation of state for radiation, as stated in the last section.

Lastly, it should be noted that in the following section, which is based on (Weinberg, 1971), the particle current is defined such that

$$N^\mu = nU^\mu. \tag{2.73}$$

This is not the same as the number flux vector that (Grøn and Hervik, 2007) defines (equation (2.64)). It is convenient, however, when working in co-moving coordinates, which will be the case. In these coordinates (2.73) gives

$$N^0 = n \qquad \text{(Co-moving coordinates)}.$$

Extending to non-ideal fluids

Having obtained formulae for the perfect fluid case, it is in this section sought to extend equations (2.71) and (2.73) with terms that incorporate dissipation. Formulae for the entropy and the entropy production are also included since they are important on thermodynamical grounds. However, they will not be of explicit use in the rest of the work. Following (Weinberg, 1971), one has

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu} + \Delta T^{\mu\nu} \qquad (2.74) \qquad\qquad N^\mu = nU^\mu + \Delta N^\mu. \qquad (2.75)$$

Dealing with the theory of relativity it needs be specified what is meant by the number density n, the energy density ρ and the four-velocity U^μ. Defining everything in a co-moving reference frame, one finds, in accordance with (2.73), that

$$n = -U_\mu N^\mu \quad (2.76) \qquad \rho = -U_\mu U_\nu T^{\mu\nu} \quad (2.77) \qquad U^\mu = (-N_\nu N^\nu)^{-\frac{1}{2}} N^\mu. \quad (2.78)$$

It follows that the terms due to dissipation must satisfy

$$U^\mu U^\nu \Delta T^{\mu\nu} = 0 \qquad (2.79) \qquad\qquad \Delta N^\mu = 0. \qquad (2.80)$$

What is more, the **equations of motion** are contained in the conservation laws

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0 \qquad (2.81) \qquad\qquad \frac{\partial N^\mu}{\partial x^\mu} = 0. \qquad (2.82)$$

So; thus far what has been found are general expressions for the energy-momentum tensor $T^{\mu\nu}$ of a dissipative fluid and for the particle current N^μ, equations (2.74) and (2.75) respectively, equations for the dissipative terms, given by equations (2.79) and (2.80), and finally also the equations of motion, as given above.

Another requirement that must be made is that the change in entropy always be positive. The *thermodynamical identity* gives the dimensionless change in entropy per particle, equation (2.48) as

$$kT d\sigma = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right) = \frac{1}{n}\left[d\rho - \frac{\rho + P}{n} dn\right]. \qquad (2.83)$$

Now, combining the three equations for the particle current N^μ, equations (2.75), (2.80) and (2.82), one readily finds

$$\frac{\partial U^\nu}{\partial x^\nu} = -\frac{U^\nu}{n}\frac{\partial n}{\partial x^\nu}. \qquad (2.84)$$

Turning to the next equation of motion; equation (2.81), it is rewritten by use of (2.74) together with the above expression and the fact that $U_\mu U^\mu = -1$. One obtains

$$U^\nu\left[\frac{\partial\rho}{\partial x^\nu} - \frac{(\rho + P)}{n}\frac{\partial n}{\partial x^\nu}\right] = U_\mu \frac{\partial}{\partial x^\nu}\Delta T^{\mu\nu}. \qquad (2.85)$$

Comparing this last expression with equation (2.83), one has

$$kTnU^{\nu}\frac{\partial\sigma}{\partial x^{\nu}} = U_{\mu}\frac{\partial}{\partial x^{\nu}}\Delta T^{\mu\nu}. \tag{2.86}$$

Expressing this in terms of space-time derivatives, one finds

$$\frac{\partial S^{\mu}}{\partial x^{\mu}} = -\frac{1}{T}\frac{\partial U_{\mu}}{\partial x^{\nu}}\Delta T^{\mu\nu} + \frac{1}{T^2}\frac{\partial T}{\partial x^{\nu}}U_{\mu}\Delta T^{\mu\nu}, \tag{2.87}$$

where S^{μ} now is the defined as the total entropy current four-vector

$$S^{\mu} = nk\sigma U^{\mu} - \frac{1}{T}U_{\nu}\Delta T^{\mu\nu}. \tag{2.88}$$

As mentioned; the requirement is now that *the change in entropy, as expressed by equation* (2.87) *is positive* for all possible fluid configurations.

In meeting this requirement there are three conditions one can impose[3].

- Hydrodynamics assumes a continuous medium. To use such a description the variables involved are assumed to vary only slightly over the mean free path. This implies that $\Delta T^{\mu\nu}$ be linear in the space-time derivatives of the variables involved.
- In order for the entropy production to always be positive, there will appear no derivatives that cannot be expressed as the derivatives of T and U^{μ} in $\Delta T^{\mu\nu}$.
- Since the perturbations of $\Delta T^{\mu\nu}$ will be small, first order changes in $\Delta T^{\mu\nu}$ are the interesting ones. Thus the adiabatic equations of motion can be used to good approximation, and equation (2.86) now equated to zero yields

$$0 = nkTU^{\nu} = kT\left\{\left[\left(\frac{\partial\rho}{\partial n}\right)_T - \left(\frac{\rho+P}{n}\right)\right]U^{\nu}\frac{\partial n}{\partial x^{\nu}} + \left(\frac{\partial\rho}{\partial T}\right)_n U^{\nu}\frac{\partial T}{\partial x^{\nu}}\right\}, \tag{2.89}$$

which, by equation (2.84) reduces to

$$U^{\nu}\frac{\partial T}{\partial x^{\nu}} = \left(\frac{\partial\rho}{\partial T}\right)_n^{-1}\left[n\left(\frac{\partial\rho}{\partial n}\right)_T - \rho - P\right]\frac{\partial U^{\nu}}{\partial x^{\nu}}. \tag{2.90}$$

With the adiabatic equation, one may express $\partial_t T$ in terms of $\nabla\cdot\mathbf{U}$. Now constructing $\Delta T^{\mu\nu}$ in a locally co-moving coordinate system, it can be shown that the most general form $\Delta T^{\mu\nu}$ can take under the above requirements is contained in the three equations

$$\Delta T^{ij} = -\eta\left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x\hat{\imath}} - \frac{2}{3}\nabla\cdot\mathbf{U}\delta_{ij}\right) - \zeta\nabla\cdot\mathbf{U}\delta_{ij}, \tag{2.91a}$$

$$\Delta T^{i0} = -\chi\frac{\partial T}{\partial x^i} - \xi\frac{\partial U_i}{\partial t}, \tag{2.91b}$$

$$\Delta T^{00} = 0. \tag{2.91c}$$

The form of ΔT^{ij} resembles the terms in the classical Navier Stokes equations (to be discussed), which is the equation of momentum transfer in the general dissipative case in classical fluid mechanics. And indeed, η and ζ are coefficients of shear and bulk viscosity, respectively. The χ appearing in (2.91b) is recognized as the coefficient of *heat conduction*,

[3]See (Landau and Lifshitz (2009) p45) for similar arguments in the deduction of Navier Stokes equations

whereas ξ is a special relativistic effect with no non-relativistic counterpart.

Inserting for $\Delta T^{\mu\nu}$ in (2.87) one obtains the general expression

$$\left(\frac{\partial S^\mu}{\partial x^\mu}\right) = \frac{\eta}{2T}\left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x\hat{\imath}} - \frac{2}{3}\nabla\cdot\mathbf{U}\delta_{ij}\right)\left(\frac{\partial U_i}{\partial x^j} + \frac{\partial U_j}{\partial x\hat{\imath}} - \frac{2}{3}\nabla\cdot\mathbf{U}\delta_{ij}\right) + \frac{\zeta}{T}(\nabla\cdot\mathbf{U})^2 + (\chi\nabla T + \xi\dot{\mathbf{U}})\cdot\left(\frac{1}{T^2}\nabla T + \frac{1}{T}\dot{\mathbf{U}}\right). \tag{2.92}$$

One will now find that this expression will be positive for all possible fluid configurations if and only if

$$\xi = T\chi \qquad \text{and} \qquad \eta \geq 0, \qquad \zeta \geq 0, \qquad \chi \geq 0.$$

Since our expressions must be Lorentz invariant, equations (2.91) can easily be extended to a general inertial frame. Restating equation (2.74), therefore, one has

$$\boxed{\begin{aligned} T^{\mu\nu} &= (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu} + \Delta T^{\mu\nu}, \\ \Delta T^{\mu\nu} &= -\eta h^{\mu\lambda} h^{\nu\sigma}\left(\frac{\partial U_\lambda}{\partial x^\sigma} + \frac{\partial U_\sigma}{\partial x^\lambda} - \frac{2}{3}\eta_{\lambda\sigma}\frac{\partial U^\epsilon}{\partial x^\epsilon}\right) - \zeta h^{\mu\nu}\frac{\partial U^\lambda}{\partial x^\lambda} \\ &\quad - \chi(h^{\mu\lambda}U^\nu + h^{\nu\lambda}U^\nu)\left(\frac{\partial T}{\partial x^\lambda} + T\frac{\partial U_\lambda}{\partial x^\sigma}U^\sigma\right)\end{aligned}} \tag{2.93}$$

where $h^{\mu\nu} = \eta^{\mu\nu} + U^\mu U^\nu$ is the projection tensor[4]. This is the full expression for **the energy-momentum tensor of a non-ideal fluid in a general frame of reference**. Also extending the equation for entropy production, equation (2.92), one finds

$$\frac{\partial S^\mu}{\partial x^\mu} = \frac{\eta}{2T}h^{\nu\sigma}h^{\mu\lambda}\left(\frac{\partial U_\mu}{\partial x^\nu} + \frac{\partial U_\nu}{\partial x^\mu} - \frac{2}{3}\eta_{\mu\nu}\frac{\partial U^\epsilon}{\partial x^\epsilon}\right)\left(\frac{\partial U_\lambda}{\partial x^\sigma} + \frac{\partial U_\sigma}{\partial x^\lambda} - \frac{2}{3}\eta_{\lambda\sigma}\frac{\partial U^\epsilon}{\partial x^\epsilon}\right) + \frac{\zeta}{T}\left(\frac{\partial U^\mu}{\partial x^\mu}\right)^2 + \frac{\chi}{T^2}h^{\mu\nu}\left(\frac{\partial T}{\partial x^\mu} + T\frac{\partial U_\mu}{\partial x^\lambda}U^\lambda\right)\left(\frac{\partial T}{\partial x^\nu} + T\frac{\partial U_\nu}{\partial x^\sigma}U^\sigma\right) \tag{2.94}$$

for a general inertial system. The entropy four-current now reads

$$S^\mu = nk\sigma U^\mu + \frac{\chi}{T}h^{\mu\nu}\left(\frac{\partial T}{\partial x^\nu} + T\frac{\partial U_\nu}{\partial x^\lambda}U^\lambda\right). \tag{2.95}$$

2.3.2 Expressions for the viscosity coefficients

Now that the general formalism is established, some general expressions for the coefficients will be given. In this dissertation it is the bulk viscosity that is under investigation, and an expression for its coefficient will be derived. Along the ride, expressions for the other two energy dissipation coefficients will follow, added as they are for the sake of completeness. This section is again extensively based on (Weinberg, 1971).

It is worthwhile to note a special case in which the bulk viscosity vanishes. From (2.93) one finds the trace of the energy-momentum tensor as

$$T^\mu{}_\mu = 3P - \rho - 3\zeta\frac{\partial U^\mu}{\partial x^\mu}, \tag{2.96}$$

[4]The projection tensor projects down on the plane of simultaneity orthogonal to the four-velocity.

meaning the bulk viscosity enters along the diagonal as a contribution to the density and pressure. In the adiabatic limit, however, when all space-time derivatives are neglected, equation (2.97) becomes

$$T^{\mu}{}_{\mu} = 3p - \rho \rightarrow P = \frac{1}{3}\left[\rho + T^{\mu}{}_{\mu}\right]. \tag{2.97}$$

In the special case where $T^{\mu}{}_{\mu} \rightarrow f(\rho, n)$ one finds

$$P = \frac{1}{3}\left[\rho + f(\rho, n)\right].$$

Now, in general, the pressure is defined to be the same function of ρ and n in the general case as in the adiabatic case, so this last relation must hold true also in the non-adiabatic case. The conclusion must be that

$$T^{\mu}{}_{\mu} = f(\rho, n) \rightarrow \zeta = 0. \tag{2.98}$$

The source credits (Tisza, 1942) for this argument. Refer for an interesting discussion. As an example, the bulk viscosity for a simple gas of structureless point particles which interact only in a localized collision, will be negligible in the extreme-relativistic and non-relativistic limits.

In the following, it should be noted, however, that *the bulk viscosity should not be expected to vanish in the general case*, but rather as an exception.

In the following, a fluid consisting of some material medium with very short mean free paths and mean free times and also some radiation quanta (photons, neutrinos or gravitons), with a finite mean free time τ will be considered. (Thomas, 1930) computes the energy-momentum tensor for such a fluid from solving the relativistic transport equation to first order in an expansion around local thermal equilibrium. It reads

$$\begin{aligned} T_{\mu\nu} &= p(T_M, n)\left(\eta_{\mu\nu} + U_\mu U_\nu\right) + \rho(T_M, n) U_\mu U_\nu \\ &\quad - 4b\tau T_M^3 \left\{\left(2U_\mu U_\nu U^\sigma + \frac{1}{3}\eta_{\mu\nu}U^\sigma + \frac{1}{3}\delta^\sigma{}_\mu U_\nu + \frac{1}{3}\delta^\sigma{}_\nu U_\mu\right)\frac{\partial T_M}{\partial x^\sigma}\right\} \\ &\quad - 4b\tau T_M^3 \left\{\frac{T_M}{15}\left(6\frac{\partial}{\partial x^\sigma}(U_\mu U_\nu U^\sigma) + \eta_{\mu\nu}\frac{\partial U^\sigma}{\partial x^\sigma} + \frac{\partial U_\mu}{\partial x^\nu} + \frac{\partial U_\nu}{\partial x^\mu}\right)\right\} + O(\tau^2), \end{aligned} \tag{2.99}$$

where T_M, n and U^μ are the temperature, number density and velocity of the material medium, respectively, and $p(T_M, n)$ and $\rho(T_M, n)$ are the total pressure and energy density that the matter and radiation would have if they were in thermal equilibrium at temperature T_M. The constant b is given such that

$$b = \begin{cases} a, & \text{for} \quad \text{photons or gravitons} \\ \frac{7}{8}, & \text{for} \quad \nu_e \text{ and } \bar{\nu}_e \text{ or } \nu_\mu \text{ and } \bar{\nu}_\mu \end{cases} \tag{2.100}$$

and $a = \frac{8}{15}\pi^5 k^4 h^{-3}$ is the Stefan-Boltzmann constant. The temperature T_M in this formula is defined such that

$$\left(\frac{\partial \rho}{\partial T}\right)_n (T - T_M) = -4bT^3\left(\frac{T}{3}\frac{\partial U^\gamma}{\partial x^\gamma} + U^\gamma \frac{\partial T_M}{\partial x^\gamma}\right). \tag{2.101}$$

With this equation one is in position to rewrite (2.99) to

$$\begin{aligned}
T_{\mu\nu} &= P(T,n)\left(\eta_{\mu\nu}+U_\mu U_\nu\right)+\rho(T,n)U_\mu U_\nu+\Delta T_{\mu\nu},\\
\Delta T_{\mu\nu} &= (T_M-T)\left\{\left(\frac{\partial P}{\partial T}\right)_n\left[\eta_\mu\nu+U_\mu U_\nu\right]+\left(\frac{\partial P}{\partial T}\right)_n U_\mu U_\nu\right\}\\
&\quad -4b\tau T^3\left\{\left(2U_\mu U_\nu U^\sigma+\frac{1}{3}\eta_{\mu\nu}U^\sigma+\frac{1}{3}\delta^\sigma{}_\mu U_\nu+\frac{1}{3}\delta^\sigma{}_\nu U_\mu\right)\frac{\partial T}{\partial x^\sigma}\right\}\\
&\quad +4b\tau T^3\left\{\frac{T_M}{15}\left(6\frac{\partial}{\partial x^\sigma}(U_\mu U_\nu U^\sigma)+\eta_{\mu\nu}\frac{\partial U^\sigma}{\partial x^\sigma}+\frac{\partial U_\mu}{\partial x^\nu}+\frac{\partial U_\nu}{\partial x^\mu}\right)\right\}+O(\tau^2),
\end{aligned} \tag{2.102}$$

which is starting to look comparable to equation (2.93). Before comparing, however, $U^\nu\partial T/\partial x^\nu$ has to be expressed in terms of $\partial U^\nu/x^\nu$. From (2.83) one deduces that

$$\left[\frac{\partial}{\partial T}\left\{\frac{1}{nT}\left(\left(\frac{\partial\rho}{\partial n}\right)_T-\left(\frac{\rho+P}{n}\right)\right)\right\}\right]_n=\left\{\frac{\partial}{\partial n}\left[\frac{1}{nT}\left(\frac{\partial\rho}{\partial T}\right)_n\right]\right\}_T, \tag{2.103}$$

which may be written as

$$T\left(\frac{\partial\rho}{\partial T}\right)_n=\rho+P-n\left(\frac{\partial\rho}{\partial n}\right)_T. \tag{2.104}$$

This result was a consequence of the entropy being a state function, or exact differential. The final step is to rewrite equation (2.90) to

$$U^\nu\frac{\partial T}{\partial x^\nu}=-\frac{T(\partial P/\partial T)_n}{\partial\rho/(\partial T)_n}\frac{\partial U^\nu}{\partial x^\nu}+O(\tau). \tag{2.105}$$

Now, for a locally comoving Lorentz frame, this equation becomes

$$\frac{\partial T}{\partial t}=-\frac{T(\partial P/\partial T)_n}{(\partial\rho/\partial T)_n}\nabla\cdot\mathbf{U}+O(\tau). \tag{2.106}$$

Inserting this into equation (2.101), one finds

$$\begin{aligned}
T-T_M &= -4bT^3\left(\frac{\partial\rho}{\partial T}\right)_n^{-1}\left(\frac{\partial T}{\partial t}+\frac{T}{3}\nabla\cdot\mathbf{U}\right)+O(\tau^2)\\
&\quad -4bT^4\left(\frac{\partial\rho}{\partial T}\right)_n^{-1}\left(\frac{1}{3}-\frac{(\partial P/\partial T)_n}{\partial\rho/(\partial T)_n}\right)\nabla\cdot\mathbf{U}+O(\tau^2).
\end{aligned} \tag{2.107}$$

By use of equations (2.106) and (2.107), the expression for $\Delta T_{\mu\nu}$ in a comoving Lorentz frame becomes

$$\begin{aligned}
\Delta T_{ij} &= -4bT^4\tau\left[\frac{1}{3}\left(\frac{\partial P}{\partial\rho}\right)_n-\right]^2\nabla\cdot\mathbf{U}-\frac{4}{15}bT^4\tau\left[\frac{\partial U_i}{\partial x^j}+\frac{\partial U_j}{\partial x\hat{\imath}}-\frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{U}\right]+O(\tau^2),\\
\Delta T_{i0} &= \frac{4}{3}bT^3\tau\left[\frac{\partial T}{\partial x\hat{\imath}}+T\frac{\partial U_i}{\partial t}\right]+O(\tau^2),\\
\Delta T_{00} &= O(\tau^2).
\end{aligned} \tag{2.108}$$

Finally, upon comparing the result with equations (2.91) (a-c), expressions for the energy dissipation coefficients are found as

$$\zeta = 4bT^4\tau\left[\frac{1}{3} - \left(\frac{\partial P}{\partial \rho}\right)_n\right]^2, \tag{2.109a}$$

$$\eta = \frac{4}{15}bT^4\tau, \tag{2.109b}$$

$$\chi = \frac{4}{3}bT^3\tau. \tag{2.109c}$$

As initially mentioned, these coefficients are valid for a fluid consisting of some material medium with very short mean free paths and mean free times together with radiation quanta (photons, neutrinos or gravitons). Note that equation (2.109a) reduces to $\zeta = 0$ if the medium with which the radiation interacts is highly relativistic ($P \simeq \frac{1}{3}\rho$). This is just as it should be, in accordance with equation (2.98); the bulk viscosity vanishes for a fluid consisting solely of particles with negligible mass. Also, refer to (Straumann, 1940), where the energy-momentum tensor found in (Thomas, 1930) is extended such as to take scattering processes into account. This modifies the transport coefficients, since these depend on the specific model.

2.3.3 Expanding Weinberg's formalism

Before proceeding into laying out the general formulation that Weinberg continues with in his article (Weinberg, 1971), a short intermezzo on an article by Zimdahl (Zimdahl, 1996) is commented on. In this article, Weinbergs formula for the bulk viscosity is generalized, as seen below.

Zimdahl

Starting from the statement that "*In the simplest cosmological models there is no way to study entropy producing processes except through bulk viscosity.*", Zimdahl continues with saying that although this is obvious on formal grounds, bulk viscosity is in a cosmological context not by far understood as well as the other transport phenomena. The aim of the article is to, by what is denoted as a "*heuristic mean free time argument*", show that "*different cooling rates for two perfect fluids are sufficient for the existence of a nonvanishing bulk viscosity of the system as a whole.* The paper also shows that, unlike what was done in earlier works, it is not necessary to introduce heat fluxes in order to show this.

The starting point is to assume an energy-momentum tensor that is the sum of two perfect fluid components;

$$T^{\mu\nu} = T_1{}^{\mu\nu} + T_2{}^{\mu\nu}. \tag{2.110}$$

That is to say; each of the components are given by equation (2.71), and one has

$$T_A^{\mu\nu} = \rho_A U^\mu U^\nu + P_A h^{\mu\nu}, \tag{2.111}$$

and $A = 1,2$. Also, it is demanded that energy-momentum conservation hold for each component. Equation (2.21) gives

$$T_A{}^{\mu\nu}{}_{;\nu} = 0. \tag{2.112}$$

As will be shown later in the general entropy producing case, this gives (with no entropy production), that

$$\dot{\rho}_A = -\theta(\rho_A + P_A), \tag{2.113}$$

where $\theta \equiv U^\mu{}_{;\mu}$. Further taking the number density n_A and the temperature T_A as thermodynamical variables, it is assumed that

$$P_A = P_A(n_A, T_A) \quad \text{and} \quad \rho_A = \rho_A(n_A, T_A). \tag{2.114}$$

By equation (2.113) and by use of $N_A{}^\mu$ one finds

$$\dot{T}_A = -T_A\theta \frac{\partial P_A/\partial T_A}{\partial \rho_A/\partial T_A} \tag{2.115}$$

where also the general relation

$$\frac{\partial \rho_A}{\partial n_A} = \frac{\rho_A + P_A}{n_A} - \frac{T_A}{n_A}\frac{\partial P_A}{\partial T_A} \tag{2.116}$$

following from the requirement that the entropy be a state function (same as equation (2.104)), has been utilized.

Note that inserting $\theta = 3\dot{a}/a$ into equation (2.115), reproduce the well known relations

$$T \sim \frac{1}{a} \quad \textbf{(radiation)} \quad , \quad T \sim \frac{1}{a^2} \quad \textbf{(matter)} \quad \text{and} \quad T = \text{const} \quad \textbf{(dust)} \tag{2.117}$$

for radiation, ordinary matter and for dust, respectively. To see this, the equation of state $P_r = n_r k T_r = (1/3)\rho_r$ has to be used for radiation, $P_m = n_m k T_m = (2/3)(\rho_m + n_m mc^2)$ for ordinary matter, and $P_d = 0 \cdot \rho = 0$ for pressure-less matter (dust).

The important step now, is to let the two fluids interact, and to see them as an effective one-fluid, instead of two single fluids. With a new particle number density $n = n_1 + n_2$ and with an equilibrium temerature T, the overall equations of state are

$$P = P(n, T) \quad \text{and} \quad \rho = \rho(n, T), \tag{2.118}$$

where P is the equilibrium pressure, and the equilibrium temperature T is defined through

$$\rho_1(n_1, T_1) + \rho_2(n_2, T_2) = \rho(n, T). \tag{2.119}$$

It is then shown in the article that this implies

$$P_1(n_1, T_1) + P_2(n_2, T_2) \neq P(n, T). \tag{2.120}$$

Defining the viscous pressure π as

$$\pi = P_1(n_1, T_1) + P_2(n_2, T_2) - P(n, T), \tag{2.121}$$

and by equating it with $\pi = -\zeta\theta$ (ignoring heat fluxes and shear viscosity), it is then shown in the article, through a mean free time argument, that

$$\boxed{\zeta = -\tau T \frac{\partial \rho}{\partial T}\left(\frac{\partial P_1}{\partial \rho_1} - \frac{\partial P}{\partial \rho}\right)\left(\frac{\partial P_2}{\partial \rho_2} - \frac{\partial P}{\partial \rho}\right),} \tag{2.122}$$

where
$$\frac{\partial P_i}{\partial \rho_i} \equiv \frac{\partial P_i/\partial T}{\partial \rho_i/\partial T}.$$
As is shown in the article, the non-vanishing viscous pressure is a consequence of the different temperature evolution laws of the two subsystems. Further on it is shown that this expression reduces to the Weinberg formula, equation (2.109a), when one of the two components is taken to be radiation ($\omega = 1/3$) and the other component is matter with kinetic energy ($\omega = 2/3$)
$$\zeta = \frac{\tau}{3} n_r k T \frac{n_m}{2n_r + n_m}, \tag{2.123}$$
which reduces to approximately the Weinberg formula (2.109a) whenever $\rho_m >> \rho_r$.[5]

Equation (2.122), then, should be taken as a generalized formula for the bulk viscosity coefficient of two interacting fluids phenomenologically described as one fluid. The result is a rather general result, since general equations of state (2.114) and (2.118) have been used in the derivation.

2.3.4 Extending with general relativity

So far a formulation of the energy-momentum tensor for a special-relativistic, non-ideal fluid has been derived. It is now time to generalize to a *general-relativistic* formulation. The extension is, as (Weinberg, 1971) points out, rather trivial. All that is needed, is to exchange the Minkowski metric $\eta_{\mu\nu}$ with a general metric $g_{\mu\nu}$, and to exchange the derivative with covariant derivatives, here denoted by ;. From equations (2.93) for $T^{\mu\nu}$, (2.75) for the particle current N^μ, (2.80) for it's four-divergence $N^\mu{}_{;\mu}$, equations (2.81) and (2.82) for the conservation laws, (2.94) for the entropy and (2.95) for the production of it per unit volume and with (2.105), a summary of the most useful general relativistic formulae is

$$T^{\mu\nu} = P g^{\mu\nu} + (\rho + P) U^\mu U^\nu - g h^{\mu\kappa} h^{\nu\lambda} \left(U_{\kappa;\lambda} + U_{\lambda;\kappa} - \frac{2}{3} g_{\lambda\kappa} U^\sigma_{;\sigma} \right), \\ -\zeta h^{\mu\nu} U^\lambda_{;\lambda} - \chi (h^{\mu\lambda} U^\nu + h^{\nu\lambda} U^\mu) \left(T_{;\lambda} + T U_{\lambda;\kappa} U^\kappa \right) \tag{2.124a}$$
$$h^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu, \tag{2.124b}$$
$$N^\mu = n U^\mu, \tag{2.124c}$$
$$T^{\mu\nu}{}_{;\nu} = 0, \tag{2.124d}$$
$$N^\mu{}_{;\mu} = 0, \tag{2.124e}$$
$$S^\mu{}_{;\mu} = \left(\frac{\eta}{2T} \right) h^{\mu\kappa} h^{\nu\lambda} \left(U_{\mu;\nu} + U_{\nu;\mu} - \frac{2}{3} g_{\mu\nu} U^\sigma{}_{;\sigma} \right) \tag{2.124f}$$
$$\left(U_{\kappa;\lambda} + U_{\lambda;\kappa} - \frac{2}{3} g_{\kappa\lambda} U^\sigma{}_{;\sigma} \right) + \frac{\zeta}{T} (U^\sigma{}_{;\sigma})^2 \tag{2.124g}$$
$$+ \frac{\chi}{T^2} h^{\mu\nu} \left(T_{;\mu} + T U_{\mu;\kappa} U^\kappa \right) \left(T_{;\nu} + T U_{\nu;\lambda} U^\lambda \right), \tag{2.124h}$$
$$S^\mu = n k \sigma U^\mu + \frac{\chi}{T} h^{\mu\nu} \left(T_{;\nu} + T U_{\nu;\lambda} U^\lambda \right), \tag{2.124i}$$
$$U^\mu T_{;\mu} = -T \left(\frac{\partial P}{\partial \rho} \right)_n U^\mu{}_{;\mu} + O(\tau) \tag{2.124j}$$

[5] Note in passing that the Weinberg formalism does not distinguish between the case $\omega = 2/3$ and $\omega = 0$. It seems strange if this is exact. Could the reason be that scattering processes are not accounted for?

where, again $U_\mu U^\mu = -1$. These expressions appear as rather cumbersome and messy, however, so before going on a few often useful tensors will be introduced, and the above general formulae rewritten to an easier format. The projection tensor was introduced among the general formulation above and is for consistency listed again underneath. Following (Brevik and Heen, 1994) one has
the **projection tensor**

$$h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \tag{2.125}$$

from which one defines the **rotation tensor** as

$$\omega_{\mu\nu} = h^\alpha{}_\mu h^\beta{}_\nu U_{[\alpha;\beta]} = \frac{1}{2}(U_{\mu;\alpha} h^\alpha{}_\nu - U_{\nu;\alpha} h^\alpha{}_\mu) \tag{2.126}$$

and the **expansion tensor** as

$$\theta_{\mu\nu} = h^\alpha{}_\mu h^\beta{}_\nu U_{(\alpha;\beta)} = \frac{1}{2}(U_{\mu;\alpha} h^\alpha{}_\nu + U_{\nu;\alpha} h^\alpha{}_\mu), \tag{2.127}$$

from which the **scalar expansion** follows as

$$\theta = \theta^\mu{}_\mu = U^\mu{}_{;\mu}. \tag{2.128}$$

From the above, one defines the **shear tensor** as

$$\sigma_{\mu\nu} = \theta_{\mu\nu} - \frac{1}{3} h_{\mu\nu}\theta, \tag{2.129}$$

which is traceless, such that $\sigma^\mu{}_\mu = 0$.

From these tensors one is in position to rewrite the covariant derivative of U_μ as

$$U_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3} h_{\mu\nu}\theta - A_\mu U_\nu, \tag{2.130}$$

where $A_\mu = \dot{U}_\mu = U^\nu U_{\mu;\nu}$ is the **four-acceleration** of the fluid. Refer to appendix C for classical counterparts.
A bit to the side of the above defined tensors, one also defines the **space-like heat flux density four-vector** as

$$Q^\mu = -\chi h^{\mu\nu}(T_{,\nu} + T A_\nu). \tag{2.131}$$

With these tensors established, the most important among equations (2.124) become

$$T_{\mu\nu} = \rho U_\mu U_\nu + (P - \zeta\theta) h_{\mu\nu} - 2\eta\sigma_{\mu\nu} + Q_\mu U_\nu + Q_\nu U_\mu, \tag{2.132a}$$
$$N^\mu = n U^\mu, \tag{2.132b}$$
$$T^{\mu\nu}{}_{;\nu} = 0, \tag{2.132c}$$
$$N^\mu_{;\mu} = 0, \tag{2.132d}$$
$$S^\mu{}_{;\mu} = \frac{2\eta}{T}\sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{\zeta}{T}\theta^2 + \frac{1}{\kappa T^2} Q_\mu Q^\mu, \tag{2.132e}$$
$$S^\mu = n k_B \sigma U^\mu + \frac{1}{T} Q^\mu, \tag{2.132f}$$

which is a much more convenient notation.

2.4 Viscous cosmology

In the start of this chapter, the ΛCDM-model was given with its constituents; Einstein's field equations, the FRW-metric and the perfect fluid equation of state. In the viscous cosmology discussed in this work, the FRW metric and the Einstein equations will, quite naturally, be kept. The change, however, will be in the energy-momentum tensor, which now is describing an imperfect fluid and therefore involves dissipation terms as well. The new energy-momentum tensor is naturally equation (2.132a).

With the general formalism in order from previous sections, it is time to calculate the formulae listed in (2.124) for the FRW-metric, which reads

$$ds^2 = g_{\mu\nu}x^\mu x^\nu = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2[d\theta^2 + sin^2\theta d\phi^2]\right), \tag{2.133}$$

previously given as equation (2.26). So, in the following section, the equations listed in (2.124) will be reduced by endowing them with the FRW-metric. All the important equations will be summarized again in the end of the section. In co-moving coordinates one has (2.30)

$$U^r = U^\theta = U^\phi = 0 \;\; ; \qquad U^0 = 1. \tag{2.134}$$

In this case the energy-momentum tensor simplifies extensively. Imposing (2.134) on the projection tensor (2.124b), one finds

$$h^{\mu\nu} = g^{\mu\nu} + \delta^\mu{}_0\delta^\nu{}_0 U^0 U^0 \begin{cases} 0, & \text{if} \quad \mu = \nu = 0 \\ g^{ij}, & \text{else.} \end{cases} \tag{2.135}$$

With (2.134) and (2.135) the energy-momentum tensor (2.124a) reads

$$\begin{aligned} T^{\mu\nu} = {} & Pg^{\mu\nu} + (\rho+P)\delta^\mu{}_0\delta^\nu{}_0 - \eta(g^{\mu\kappa} + \delta^\mu{}_0\delta^\kappa{}_0)(g^{\nu\lambda} + \delta^\nu{}_0\delta^\lambda{}_0)\left[U_{\kappa;\lambda} + U_{\lambda;\kappa} - \frac{2}{3}g_{\lambda\kappa}U^\sigma_{;\sigma}\right] \\ & -\zeta(g^{\mu\nu} + \delta^\mu{}_0\delta^\nu{}_0)U^\lambda{}_{;\lambda}, \end{aligned} \tag{2.136}$$

and one can observe that the heat conduction coefficient χ is already gone (the first bracket in the χ-term in (2.124a) vanishes). From equation (2.133) it is evident that the only non-zero entries in the FRW-metric are the diagonal ones. Then, from (2.136) it is evident that $T^{\mu\nu}$ will be diagonal too, so in the following the diagonal terms will be evaluated. Starting with T^{00} one immediately finds

$$T^{00} = \rho, \tag{2.137}$$

and by rather easy calculations one obtains as expected

$$T^{i0} = 0. \tag{2.138}$$

Now, for the spatial parts one has

$$
\begin{aligned}
T^{ij} &= Pg^{ij} - \eta g^{i\kappa} g^{j\lambda} \left[U_{\kappa;\lambda} + U_{\lambda;\kappa} - \frac{2}{3} g_{\lambda\kappa} U^{\sigma}{}_{;\sigma} \right] - \zeta g^{ij} U^{\lambda}{}_{;\lambda} \\
&= Pg^{ij} - \eta g^{i\kappa} g^{j\lambda} \left[-\Gamma^{\sigma}{}_{\lambda\kappa} U_{\sigma} - \Gamma^{\sigma}{}_{\kappa\lambda} U_{\sigma} - \frac{2}{3} g_{\lambda\kappa} \Gamma^{\sigma}{}_{\tau\sigma} U^{\tau} \right] - \zeta g^{ij} \Gamma^{\lambda}{}_{\tau\lambda} U^{\tau} \\
&= Pg^{ij} + \eta g^{ii} g^{jj} \left[2\Gamma^{0}{}_{ij} + \frac{2}{3} g_{ji} \Gamma^{\sigma}{}_{0\sigma} \right] - \zeta g^{ij} \Gamma^{\lambda}{}_{0\lambda} \\
&= \left(P - 3\zeta \frac{\dot{a}}{a} \right) g^{ij},
\end{aligned}
\tag{2.139}
$$

where it has been used that $\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta}$, that $g^{\mu\nu}$ is diagonal and that U^0 is the only non-zero velocity component. The last equality might not be obvious, but follows from the Christoffel symbols calculated and listed in appendix B.
So; it is seen that the diagonal character of the FRW-metric is enough to exclude the shear viscosity η – which indeed is just how it should be. These results could also have been found more easily by noting that the rotation and shear tensors, equations (2.126) and (2.129) respectively, vanish; $\omega_{\mu\nu} = \sigma_{\mu\nu} = 0$.

Gathering all the terms, the overall tensor is given as

$$
\begin{aligned}
T_{ij} &= \left(P - 3\zeta \frac{\dot{a}}{a} \right) g_{ij}, \\
T_{i0} &= 0, \\
T_{00} &= \rho.
\end{aligned}
\tag{2.140}
$$

Or, more compactly, with $\theta = U^{\mu}{}_{;\mu} = 3\frac{\dot{a}}{a}$;

$$
\boxed{T_{\mu\nu} = \rho U_{\mu} U_{\nu} + (P - \zeta\theta)\, h_{\mu\nu}.}
\tag{2.141}
$$

This is the **general-relativistic energy-momentum tensor for an imperfect fluid with the FRW-metric**. As indicated above; only the bulk viscosity, which does not violate the cosmological principle, does show up in this energy-momentum tensor. Rewriting equation (2.141) to

$$
T_{\mu\nu} = \left(\rho + P - 3\zeta \frac{\dot{a}}{a} \right) U_{\mu} U_{\nu} + \left(P - 3\zeta \frac{\dot{a}}{a} \right) g_{\mu\nu}
\tag{2.142}
$$

and comparing with equation (2.71), one sees that the only change compared to the perfect fluid case is the additional ζ-term. The pressure is modified due to the viscosity.
In the start of this chapter the famous Friedmann equations, equations (2.27) and (2.28), were given as a result of inserting the FRW-metric (2.26) and a perfect fluid energy-momentum tensor (2.71) into the Einstein equations (2.17). The Friedmann equations determined how the scale factor evolve with time as a function of the energy content in the universe. It should be instructive, therefore, to now obtain the corresponding equations valid in the viscous case. Restating the Einstein equations, they read (2.17)

$$
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},
\tag{2.143}
$$

where the Γ's are the Christoffel symbols. To obtain expressions for the scale factor $a(t)$ by direct calculations, one therefore must calculate the components of the Ricci curvature tensor $R_{\mu\nu}$. The Ricci tensor is defined as the contraction of the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$, (see e.g. Grøn and Hervik (2007), p159) such that, in a coordinate basis one finds

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} = \Gamma^{\alpha}{}_{\nu\beta,\alpha} - \Gamma^{\alpha}{}_{\nu\alpha,\beta} + \Gamma^{\rho}{}_{\nu\beta}\Gamma^{\alpha}{}_{\rho\alpha} - \Gamma^{\rho}{}_{\nu\alpha}\Gamma^{\alpha}{}_{\rho\beta}. \tag{2.144}$$

Also,by contraction of equation (2.143) with $g^{\mu\nu}$ one finds the Ricci scalar as

$$R = -\kappa T + 4\Lambda, \tag{2.145}$$

where $R = R^{\mu}{}_{\mu}$ and $T = T^{\mu}{}_{\mu}$. Inserting this back into equation (2.143) the result is

$$R_{\mu\nu} = \Lambda g_{\mu\nu} + \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}{}_{\lambda}\right). \tag{2.146}$$

As a side comment it is worthwhile noting that the Ricci curvature tensor vanishes for an empty universe; $T_{\mu\nu} = \Lambda = 0$ gives $R_{\mu\nu} = 0$! Now, using (2.141) to calculate the brackets on the RHS of (2.146) one finds

$$T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}{}_{\lambda} = \rho U_{\mu}U_{\nu} + (P - \zeta\theta)\, h_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\left(\rho - 3(P - \zeta\theta)\right). \tag{2.147}$$

Now designating the RHS of (2.146) as the *source term*, $S_{\mu\nu}$, equation (2.147) gives

$$\begin{aligned} S_{00} &= \frac{\kappa}{2}\left(\rho + 3(P - \zeta\theta)\right) - \Lambda \\ S_{0i} &= 0, \\ S_{ij} &= \frac{\kappa}{2}\left(\rho - (P - \zeta\theta)\right) a^2\gamma_{ij} + \Lambda g_{ij}. \end{aligned} \tag{2.148}$$

By use of the Christoffel symbols calculated in appendix **??** one finds (from (2.144)) the Ricci tensor components for the FRW metric to become

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{0i} &= 0 \\ R_{ij} &= \tilde{R}_{ij} + (a\ddot{a} + 2\dot{a}^2)\gamma_{ij}, \end{aligned} \tag{2.149}$$

where $\tilde{R}_{ij}$ is the spatial Ricci tensor, calculated from the metric γ_{ij}.

Also, since γ_{ij} is the metric of a maximally symmetric space, it must follow (Weinberg (1972), pp 383 and 471) that[6]

$$\tilde{R}_{ij} = 2k\gamma_{ij} \tag{2.150}$$

and thus that

$$R_{ij} = (a\ddot{a} + 2\dot{a}^2 + 2k)\gamma_{ij}. \tag{2.151}$$

The next step is to compare equations (2.148) with equations (2.149). Using equation (2.151), this give the two equations

[6]Note that (Weinberg, 1972) uses a different sign-convention for the Ricci tensor. The signs of equations 15.1.6 and 15.1.8 in the source have to be switched.

$$-3\frac{\ddot{a}}{a} = 4\pi G\big(\rho + 3(P - \zeta\theta)\big) - \Lambda, \tag{2.152a}$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G\big(\rho - (P - \zeta\theta)\big) + \Lambda, \tag{2.152b}$$

when natural units are used, and $\kappa = 8\pi G$. By straight forward algebraic manipulations, one may rewrite these two equations to the more familiar forms

$$\boxed{\begin{aligned} 3\frac{\dot{a}^2 + k}{a^2} &= 8\pi G\rho + \Lambda \\ -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} &= 8\pi G\,[P - \zeta\theta] - \Lambda. \end{aligned}} \tag{2.153a, 2.153b}$$

Except for the cosmological constant-term, which this time was included, the first of these equations is the exact same result as obtained in equation (2.27). The first Friedmann equation is left unchanged. The second equation resembles equation (2.28), but this time with an altered pressure; $P \to P - \zeta\theta$. Again the cosmological constant, which in the perfect fluid case was chosen to be omitted, has been included.

So far so good. Now turning to the rest of equations (2.124). The entropy four-current S^μ, given by equation (2.132f), and it's four-divergence ${S^\mu}_{;\mu}$, equation (2.132e) become

$${S^\mu}_{;\mu} = \frac{\zeta}{T}\theta^2 \qquad , \qquad S^0 = nk_B\sigma \qquad \text{and} \qquad S^i = 0 \tag{2.154}$$

when invoked with the FRW-metric. From this it follows, since ${\Gamma^0}_{00} = 0$ that

$$\partial_0 S^0 = \frac{\zeta}{T}\theta^2. \tag{2.155}$$

This, together with (2.154), yields

$$\boxed{\dot{\sigma} = \frac{\zeta}{nk_bT}\theta^2.} \tag{2.156}$$

Hence; for the rate of increase of entropy per particle the bulk viscosity comes into play. Next, energy conservation is on the agenda;

$${T^{\mu\nu}}_{;\nu} = 0. \tag{2.157}$$

To obtain the energy conservation equation, direct computation invoking the Christoffel symbols listed in appendix B is the obvious way. An alternative route that keeps track of the connection to thermodynamics, is to start by the thermodynamic identity, equation (2.47), reading

$$du = Tds - Pd\hat{V}. \tag{2.158}$$

Here u is the internal energy per unit volume, s is the entropy per unit volume and $\hat{V}$ is taken to be a dimensionless volume. As equation (2.157) this equation expresses energy conservation, and should yield the same result.

From (2.155) $Tds = \zeta\theta^2 dt$ may be concluded. as for the change du in the internal energy density and the work pdV done by the expansion, one must in a co-moving frame have

$$du = d(\rho a^3) \qquad and \qquad PdV = pd(a^3). \tag{2.159}$$

These two last expressions together with $TdS = \zeta\theta^2 dt$ allows for rewriting (2.158) to

$$\begin{aligned} &d(\rho a^3) = \zeta\theta^2 dt - Pd(a^3) \\ &\rightarrow \dot{\rho}a^3 + 3\rho a^2 \dot{a} = \zeta\theta^2 - 3Pa^2\dot{a}, \end{aligned} \tag{2.160}$$

from which, when using $\theta = 3\frac{\dot{a}}{a}$,

$$\boxed{\dot{\rho} + (\rho + P)\theta = \zeta\theta^2,} \tag{2.161}$$

follows. This is the **energy conservation equation** for the FRW-metric in the viscous case. The derivation and justification of equation (2.161) was perhaps a bit hand-waving. To see that this equation actually is what is obtained, the direct computation of (2.157) will now be sketched. To this end, it is useful to rewrite (2.157) via (2.14), such that

$$\begin{aligned} T^{\mu\nu}{}_{;\nu} &= \partial_\mu T^{\mu\nu} + \Gamma^\nu{}_{\nu\lambda} T^{\mu\lambda} + \Gamma^\mu{}_{\nu\lambda} T^{\lambda\nu} \overset{(2.14)}{=} \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} T^{\mu\nu}) + \Gamma^\mu{}_{\nu\lambda} T^{\lambda\nu} \\ &= g^{\mu\nu}\partial_\nu P' + \frac{1}{\sqrt{g}} \partial_\nu \left(g^{1/2}(\rho + P')U^\mu U^\nu \right) + \Gamma^\mu{}_{\nu\lambda}(P' + \rho)U^\nu U^\lambda \end{aligned} \tag{2.162}$$

where $P' = P - \zeta\theta$ for shorthand. Now, with $\Gamma^\mu{}_{00} = 0$ and $U^i = 0$ one finds three trivially satisfied equations for the spatial parts, whereas $\mu = 0$ gives the equation

$$a^3(t)\frac{d}{dt}(P - \zeta\theta) = \frac{d}{dt}\left(a^3(t)(\rho(t) + P(t) - \zeta\theta) \right), \tag{2.163}$$

which easily transforms into the form given in (2.161),

$$\dot{\rho} + (\rho + P)\theta = \zeta\theta^2. \tag{2.164}$$

So in this conservation equation one finds that the introduced bulk viscosity plays the same result as always; reducing as it is, the effective pressure.

For the conservation of particles, equation (2.124e), one finds

$$\begin{aligned} &(nU^\mu)_{;\mu} = 0 \\ &\overset{(2.14)}{\rightarrow} \frac{1}{\sqrt{g}} \partial_\mu (g^{1/2} n U^\mu) \overset{U^i=0}{=} \frac{1}{\sqrt{g}} \partial_0 (g^{1/2} n), \end{aligned} \tag{2.165}$$

which, by the determinant for the FRW metric[7] gives

$$\frac{d}{dt}\left(n(t)a^3(t) \right) = 0 \quad \rightarrow \quad n(t)a^3(t) = \text{const}. \tag{2.166}$$

That is to say; na^3 =const in a co-moving frame of reference.

[7] g is defined such that

$$g \equiv -Det(g_{\mu\nu}) \overset{\text{FRW}}{=} \frac{a^6 r^4 \sin^2\theta}{1 - kr^2}$$

Before going on, the most important formula for the viscous cosmology investigated in the present work are repeated. One has the energy-momentum tensor and the equation of energy conservation

$$T_{\mu\nu} = \rho U_\mu U_\nu + (P - \zeta\theta)\, h_{\mu\nu}, \tag{2.167a}$$

$$\dot{\rho} + (\rho + P)\theta = \zeta\theta^2, \tag{2.167b}$$

and also the **new "Friedmann" equations**[8]

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \tag{2.168a}$$

$$-3H^2 - 2\dot{H} - \frac{k}{a^2} = 8\pi G\,[P - \zeta\theta] - \Lambda, \tag{2.168b}$$

where $H = \dot{a}/a$ is the Hubble parameter.

Previously in this chapter the so-called acceleration equation, equation (2.29) was found. In the viscous case, the corresponding equation, found by combining the two new "Friedmann" equations, (2.168a) and (2.168b), reads

$$\dot{H} = -4\pi G[\rho + P - \zeta\theta] + \frac{k}{a^2}. \tag{2.169}$$

This equation determines the expansion of the universe as a function of the energy content, found on the right hand side. Note that even though the cosmological constant was included in the "Friedmann equations" from which this last equation was deduced, it does not show up here.

2.5 Observations

When doing theoretical model construction, one ought to have some information from the observational side, so that one can separate non-physical theoretical constructs from those of physical value. In the following part some important data sets of observations are given.

2.5.1 Hubble parameter

To the best of knowledge, table 2.1, taken from (Farooq and Ratra, 2013), gives the most complete compiled list of Hubble parameter measurements against redshift z. It includes 28 different (and according to the authors also independent) measurements of H in the range $z \in [0.070, 2.300]$

[8]It might not be entirely correct to call these equations Friedmann equations, but for lack of a better name, this will be the convention in this work.

Compiled data of Huble measurements			
z [−]	$H(z)$ $(km\ \ s^{-1}\ \ Mpc^{-1})$	σ_H $(km\ \ s^{-1}\ \ Mpc^{-1})$	Reference
0.070	69	19.6	5
0.100	69	12	1
0.120	68.6	26.2	5
0.170	83	8	1
0.179	75	4	3
0.199	75	5	3
0.200	72.9	29.6	5
0.270	77	14	1
0.280	88.8	36.6	5
0.350	76.3	5.6	7
0.352	83	14	3
0.400	95	17	1
0.440	82.6	7.8	6
0.480	97	62	2
0.593	104	13	3
0.600	87.9	6.1	6
0.680	92	8	3
0.730	97.3	7.0	6
0.781	105	12	3
0.875	125	17	3
0.880	90	40	2
0.900	117	23	1
1.037	154	20	3
1.300	168	17	1
1.430	177	18	1
1.530	140	14	1
1.750	202	40	1
2.300	224	8	4

Table 2.1: This table is a reproduction of Table 1 in (Farooq and Ratra, 2013), and gives the Hubble parameter versus redshift data. The last column gives the references, which respectively are 1:(Simon et al., 2005), 2:(Stern, 2010), 3:(mor), 4:(Busca, 2012), 5:(Zhang, 2014), 6: (Blake, 2012) and 7:(Chuang and Wang, 2012).

From this data, (Farooq and Ratra, 2013) conclude that the data "*requiere a currently accelerating cosmological expansion at about, or better than,* 3σ *confidence.*" In the paper, 3 models (ΛCDM, XCDM and ϕCDM), and two Hubble constant priors are considered, and the best fit (mean and standard deviation) for **the deceleration-acceleration redshift is found to be** $z_{da} = 0.74 \pm 0.05$. This is found to be in good agreement with the previous work of Busca (2012).

2.5.2 Energy partitioning in the universe

Table 2.2 lists what is thought to be the most recent data on cosmological parameters. The data is drawn from (Planck Collab.I; Aghanim, 2013), which uses a 6 paramter ΛCDM-model.

The paramters of interest are listed in the table. The table is a reproduction of parts of table 9 and 10 in (Planck Collab.I; Aghanim, 2013).

Cosmological Parameters I			
Symbol	*Best fit*	*68% limits*	*Definition*
$\Omega_b h^2$	0.022161	0.02214±0.00024	Baryon density today
$\Omega_c h^2$	0.11889	0.1187±0.0017	Cold Dark Matter density today
Ω_Λ	0.6914	0.692±0.010	Relative D. E. density today
H_0	67.77	67.80±0.77	Current expansion rate in $kms^{-1}Mpc^{-1}$
Age/Gyr	13.7965	13.798±0.037	Age of the universe today (in Gyr)

Table 2.2: This table is a reproduction of parts of table 9 and 10 in (Planck Collab.I; Aghanim, 2013). It includes cosmological parameter values for the *Planck* best-fit cosmology including external data sets ($Pl.+WP+highL+BAO$). Refer to the reference for further details. The two parameters above the horizontal line are among the 6 parameter fits, and the two above are derived from the same model (ΛCDM).

Note the astronomical unit Mpc. According to (Liddle, 2003), one has

$$1\text{pc} = 3.261 \ \ \text{l.y.} = 3.086 \cdot 10^{16}\text{m}. \tag{2.170}$$

Also, from the most recent Planck data, (Planck Collab.XIV; Aghanim, 2013), one finds the data listed in table 2.3.

Cosmological Parameters II				
Symbol	*Best fit*	*limits*	*Definition*	Data Sets
Ω_{0k}	-0.0003	$-0.0005^{+0.0065}_{-0,0066}$	present curv. param.	$Pl.+WP+highL+BAO$
Ω_{0m}	0.3183	$0.315^{+0.016}_{-0.018}$	present rel. mat. dens.	*Planck*+WP
z_{eq}	3402		Redsh. of mat.-rad. eq.	*Planck*
z_{re}	11.35		Redsh. of half reioniz.	*Planck*
ω_Λ		$-1.13^{0.13}_{-0.10}$	Dark energy EoS param.	*BAO and CMB*

Table 2.3: This data is taken from (Planck Collab.XIV; Aghanim, 2013), mostly table 2 and 10. The confidence levels are 68% and 95% for Ω_{0m} and Ω_{0k} respectively.

In the present work, the best-fit values seem to be more than accurate enough. Finally, a table of other physical constants of later use

Physical Constants				
Symbol	*Numerical Value*	Units	*Rel. std. uncert.*	*Definition*
G	$6.67384(80) \cdot 10^{-11}$	$\text{m}^3\ \text{kg}^{-1}\text{s}^{-2}$	$1.2 \cdot 10^{-4}$	Newt. grav. const.
c	299792458	ms^{-1}	Exact	Speed of lig. in vac.

Table 2.4: This data is taken from (Mohr et al., 2012).

Chapter 3

classical kinetic theory

3.1 A closer look at dissipation and viscosity

Since the aim of the present work is to find realistic models for the bulk viscosity it seems appropriate to look a bit more intently into this concept. To this end, classical kinetic theory will be investigated. One main reason for examining this field, is that galaxies move non-relativistically. Perhaps, if seeing galaxies as interacting particles in an expanding gas, one could find a reasonable estimation of the viscosity of the late universe. Also in this chapter it is sought to motivate why one should at all expect dissipative processes in the cosmic fluid from classical kinetic theory. To wit; dissipation coefficients are derived from kinetic theory and shown to be first order deviations from equilibrium. The lines to the continuum limit are drawn. Note that since this chapter is purely non-relativistically, natural units are not being used.

3.1.1 Kinetic theory

Following (Landau and Lifshitz, 1981), the starting point of the investigation will be the Boltzmann transport equation for gases.

In the following, a phase space volume element is denoted by $d\tau = dVd\Gamma$, where Γ is the part of phase space that does not correspond to spatial variables $dV = dxdydz$. I.e.; Γ represent the momentum variables. The phase space distribution function is denoted by $f(t,\mathbf{r},\Gamma)$. In the 19th century the french mathematician Liouville came up with a theorem, **Liouville's theorem**, which, when applied to the distribution function of a gas in which *the collisions are negligible*, states that

$$\frac{d}{dt}f(t,\mathbf{r},\Gamma) = \frac{\partial}{\partial t}f(t,\mathbf{r},\Gamma) + \mathbf{v}\nabla f(t,\mathbf{r},\Gamma) + \mathbf{F}\frac{\partial}{\partial \mathbf{p}}f(t,\mathbf{r},\Gamma) = 0. \tag{3.1}$$

Here, the last term on the LHS corresponds to an externally applied field. So; the time evolution of the distribution function is zero. This theorem, however, is not true when collisions matter. In such a case, a so-called collision integral $I_c(f)$ has to be added to the RHS of (3.1), accounting for the flow of $f(t,\mathbf{r},\Gamma)$ in and out of a given phase space volume element $d\tau$. For two molecules colliding at a point (x,y,z), the collision integral should contain a "loss-term", accounting for all the molecules for which the remaining coordinates were initially Γ and Γ_1 that leave these coordinates $(\Gamma,\Gamma_1 \rightarrow \Gamma'\Gamma'_1)$. Similarly, the integral should contain a "gain-term", accounting for all the reverse processes; molecules with different initial phase

space coordinates, that after the collision acquire Γ and Γ_1 ($\Gamma', \Gamma_1' \to \Gamma\Gamma_1$).
Let $\omega(\Gamma', \Gamma_1'; \Gamma, \Gamma_1)$ be the transition rate from phase space coordinates Γ, Γ_1 to Γ', Γ_1'. Then **the principle of detailed balance** can be expressed as

$$\omega(\Gamma', \Gamma_1'; \Gamma, \Gamma_1) = \omega(\Gamma, \Gamma_1; \Gamma', \Gamma_1') \tag{3.2}$$

for a system in statistical equilibrium. Applying this, it can be shown that the collision integral $I_c(f)$ becomes

$$I_c(f) = \int \omega'(f' f_1' - f f_1) d\Gamma_1 d\Gamma' d\Gamma_1'. \tag{3.3}$$

Neglecting the external field-term in equation (3.1), and inserting the above integral on the RHS, one obtains a (generally) non-linear integro-differential equation on the form

$$\boxed{\frac{\partial}{\partial t} f + \mathbf{v} \nabla f = \int \omega'(f' f_1' - f f_1) d\Gamma_1 d\Gamma' d\Gamma_1'.} \tag{3.4}$$

Here, ω' is shorthand for $\omega(\Gamma, \Gamma_1; \Gamma', \Gamma_1')$, and the suffixes on f are such that $f_1' = f(t, \mathbf{r}, \Gamma_1')$. This is the so-called **Boltzmann transport equation**.
Underneath, a few general properties that should be known about the Boltzmann equation are listed.

- The LHS represents the change in $f(t, \mathbf{r}, \Gamma)$ due to the motion of single particles ("single-particle" properties) in the absence of collisions.
- The RHS, which accounts for collisions ("multi-particle" property), contains a loss- and a gain-term.
- The integral on the RHS somehow has to account for conservation of energy and momentum in the collisions.
- The Boltzmann equation only accounts for two body collisions, resulting from the so-called *mean-field approximation.*

Size and magnitude

As a qualitative analysis, one may obtain rough but useful estimates of the inter-collisional time and length scales by use of the collision integral. Remembering that the collision integral vanishes for the equilibrium distribution f_0, a rough estimate should suggest

$$I_c(f) \sim -\frac{f - f_0}{\tau} \sim -\frac{\bar{c}}{l}(f - f_0) \tag{3.5}$$

where τ is the *mean free time*, or relaxation time, l is the *mean free path* and $\bar{c}$ is the *mean thermal velocity*[1]. In passing; the notation used here is such that

$$\mathbf{v} = \mathbf{V} + \mathbf{c} \tag{3.6}$$

where v then is the absolute velocity of a particle, $\mathbf{v}$ is the centre of mass velocity of the fluid as a whole and c is the remaining part of the velocity; the thermal velocity. So; equation (3.5) suggests the usual

$$\tau \sim \frac{l}{\bar{c}}. \tag{3.7}$$

The mean free path itself can also be estimated. Let σ be the collision cross-section (not to be confused with the entropy per particle, which also is denoted σ) and n the number density of the gas. Estimating that a molecule travelling through a volume element $\sim \sigma \cdot 1$ undergoes $n\sigma$ collisions, one finds the inverse estimate

$$l \sim \frac{1}{n\sigma}. \tag{3.8}$$

Denoting the mean intermolecular distance $\bar{r}$ and the dimension of the molecules d, it should for a gas hold true that $\bar{r} \gg d$. Taking the cross-section as $\sigma \sim d^2$, and $N \sim 1/\bar{r}^3$ this amounts in suggesting

$$l \gg \bar{r}. \tag{3.9}$$

As a final comment, note that the collisions were treated as taking place at specific points in space. For a realistic scenario, this should mean that the Boltzmann equation is applicable *only* over time-scales long compared to the collision interactions, and thus distances large compared with the "point" in space where the collision takes place.

Entropy production

Starting from the entropy of an ideal gas in a non-equilibrium macroscopic state,

$$S = \int f \log(e/f)\, d\tau, \tag{3.10}$$

one can show that

$$\boxed{\frac{dS}{dt} \geq 0,} \tag{3.11}$$

where equality occurs at equilibrium. This is in keeping with the second law of thermodynamics, and should in general be required.[2].

An ideal fluid; i.e. a fluid where dissipative processes do not occur, will undergo *adiabatic*

[1] Attempting at being consistent in notation the convention used in (Hänel, 2004) is followed here. The *mean arithmetic velocity* $\bar{c}$ is defined as

$$\bar{c} = \frac{1}{n}\int_0^\infty |c| f(c)\, d|c|$$

where f(c) is the velocity distribution function. In this section, however, the overall mean velocity $\bar{v}$ that is used in (Landau and Lifshitz, 1981) will be used mostly, since the macroscopic velocity $\mathbf{V}$ will be set to zero, which implies $\bar{v} = \bar{c}$. The troubles of multiple sources.

[2] At least for ordinary fluids. As argued in (Brevik and Grøn, 2013), it might not be obvious that Dark Energy has to obey this.

motion when moving. Assuming such an ideal fluid, and following the entropy of each fluid element as it moves around in time and space, one finds from equation (3.11) that

$$\frac{d\sigma}{dt} = \frac{\partial\sigma}{\partial t} + (\mathbf{v}\nabla)\,\sigma = 0 \tag{3.12}$$

which is the equation of **adiabatic motion**. Here, σ denotes entropy per particle. Using the continuity equation for mass (an equation to be discussed later), this can be rewritten in an equation of conservation of entropy;

$$\frac{\partial}{\partial t}\left(\sigma\rho\right) + \nabla\cdot\left(\rho\sigma\mathbf{v}\right) = 0 \tag{3.13}$$

where $\rho\sigma\mathbf{v}$ is the *entropy flux density*.

Macroscopic equations and conservation laws

The Boltzmann equation is a microscopic description, but it is possible to obtain the macroscopic fluid mechanical equations from it as the continuum limit. In this section, the relations between microscopic and macroscopic variables will be found. In order for a macroscopic description to be valid, the macroscopic variables, (temperature, density, overall velocity, etc) of the gas must vary sufficiently slowly over its volume. Since $d\Gamma$ gives the non-spatial distribution function,

$$n(t,\mathbf{r}) = \int f(t,\mathbf{r},\Gamma)\,d\Gamma, \tag{3.14}$$

must be the spatial distribution function. By such,

$$\rho = mn \tag{3.15}$$

is the mass density of the gas when m is the mass of a single molecule. Another macroscopic variable is the centre of mass velocity $\mathbf{V}$, which can be defined as the mean $\bar{\mathbf{v}}$ of the microscopic velocities $\mathbf{v}$. One has

$$\mathbf{V} = \bar{\mathbf{v}} = \frac{1}{n}\int \mathbf{v} f\,d\Gamma. \tag{3.16}$$

Requiring that the energy and momentum be conserved under each individual collision, the collisions will not alter the overall macroscopic quantities in each phase space volume element; the density, the momentum and the internal energy will be conserved in each element. Writing this in terms of the collision integral, one obtains[3]

$$\int I_c(f)\,d\Gamma = 0 \tag{3.17}$$

$$\int \epsilon I_c(f)\,d\Gamma = 0 \tag{3.18}$$

$$\int \mathbf{p} I_c(f)\,d\Gamma = 0 \tag{3.19}$$

where ϵ is the energy of a unit volume of gas and $\mathbf{p}$ its momentum. Refer to the source, (Landau and Lifshitz, 1981), for details. Now, these conditions give us a set of three macroscopic equations, when applied to the transport equation. They are applied by integrating both sides of the transport equation on component form,

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}(v_i f) = I_c(f), \tag{3.20}$$

[3]Another neat way of looking at this is to view the three integrals as moments of the distribution function. Refer to (Hänel, 2004) for an extensive, readable and good treatment.

over $md\Gamma$, $\epsilon d\Gamma$ and $p_\beta d\Gamma$ respectively. The three equations obtained are, in respective order,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \qquad \text{Conservation of mass,} \tag{3.21a}$$

$$\frac{\partial}{\partial t}(N\bar{\epsilon}) + \nabla \cdot \mathbf{q} = 0 \qquad \text{Conservation of energy,} \tag{3.21b}$$

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial \Pi_{ij}}{\partial x_j} = 0 \qquad \text{Conservation of momentum.} \tag{3.21c}$$

Here, Π_{ij} is defined such that

$$\Pi_{ij} = \int m v_i v_j f \, d\Gamma \tag{3.22}$$

and is called the **momentum flux tensor**. Also,

$$\mathbf{q} = \int \epsilon \mathbf{v} f \, d\Gamma \tag{3.23}$$

is the **energy flux** in the gas. These equations can also be written out in terms of macroscopic variables. To this end, a Galilean velocity transformation is applied so that one, in the new frame (primed), moves with the gas. The relation to the old unprimed frame of reference, where the gas had a macroscopic velocity $\mathbf{V}$ must be $\mathbf{v} = \mathbf{v}' + \mathbf{V}$. From (3.22) for the momentum tensor, it is quite straight forward to show that

$$\Pi_{ij} = \rho V_i V_j + \delta_{ij} P \tag{3.24}$$

by using the ideal gas equation of state and the equipartition theorem. Similarly, the energy flux $\mathbf{q}$ becomes

$$\mathbf{q} = \mathbf{V}(\frac{1}{2}\rho V^2 + w), \tag{3.25}$$

where $w = P + n\bar{\epsilon}'$ is the **heat function** of the gas per unit volume, P is the pressure and $n\bar{\epsilon}$ is the mean internal energy per unit volume. These expressions inserted back into (3.26) will give the more familiar set of equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \qquad \text{Continuity equation (mass cons. eq.)} \tag{3.26a}$$

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \left(\mathbf{V}(\frac{1}{2}\rho V^2 + w)\right) = 0 \qquad \text{(energy cons. eq.)} \tag{3.26b}$$

$$\frac{\partial \rho \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\left(\mathbf{V}\rho\right) = -\nabla P \qquad \text{Euler's equation (mom. cons. eq.)} \tag{3.26c}$$

With these equations a fluid-mechanical system should be completely determined, since there are five equations, (3.26) in five unknowns (V_X, V_Y and V_Z and P and ρ).[4]

Accounting for dissipation

So far what has been done is valid only for the ideal case, where there are no dissipative terms. It has been assumed so far, that the distribution function is equal to the local equilibrium distribution function f_0. This assumption is what needs be altered in the case of

[4] Remembering that w is a function of the velocity and the pressure.

dissipation. Henceforth a small perturbation of f is added, such that

$$f = f_0 + \delta f \quad , \quad \delta f = -\frac{\partial f_0}{\partial \epsilon}\chi(\Gamma). \tag{3.27}$$

The deviation is small, and indeed; the length L over which the temperature changes considerably is taken to be $L \gg l$, where l as before is the mean free path in-between collisions. Equivalently; $\delta f \ll f_0$. The equilibrium distribution function in a gas at rest ($\mathbf{V} = 0$) is the so-called Boltzmann distribution. It reads

$$f_0(\Gamma) = \text{const.} \cdot e^{\epsilon(\Gamma)/T}. \tag{3.28}$$

From this one necessarily finds

$$\delta f = f_0 \frac{\chi(\Gamma)}{T}. \tag{3.29}$$

Now; also for a distribution out of equilibrium, particle number, internal energy and momentum has to be conserved for each region in phase space (when dissipation terms are included in the balance, that is). Therefore, as $f \to f_0 + \delta f$ the integrals of f, ϵf and $\mathbf{p} f$ over Γ, should yield the same as the corresponding integrals with f_0 instead of f. Thus, in addition to the three integrals (3.18), (3.17) and (3.19), the following three conditions on δf and thus on $\chi(\Gamma)$ must hold;

$$\int f_0 \chi d\Gamma = 0 \tag{3.30}$$

$$\int \epsilon f_0 \chi d\Gamma = 0 \tag{3.31}$$

$$\int \mathbf{p} f_0 \chi d\Gamma = 0. \tag{3.32}$$

Remember that what ultimately is sought is a solution of the Boltzmann equation (3.4) in the general dissipative case.

Inserting the new distribution function $f = f_0 + \frac{\chi(\Gamma)}{T}$ (eq. (3.27) with (3.29)) into the Boltzmann equation, (3.4), the equilibrium distribution terms in the collision integral must vanish as before. The δf terms remain to give the integral

$$I_c = \frac{f_0}{T} I(\chi) \quad \text{, where} \quad I(\chi) = \int \omega' f_{01}(\chi' + \chi_1' - \chi - \chi_1) d\Gamma_1 d\Gamma' d\Gamma_1'. \tag{3.33}$$

With the three new conditions above, equations (3.30), (3.31) and (3.32), it can be shown that the remaining collision integral vanishes for the three general cases

$$\chi = \text{const.} \quad , \quad \chi = \text{const.} \cdot \epsilon \quad \text{or} \quad \chi = \text{const.} \cdot \mathbf{p} \cdot \delta \mathbf{V}.$$

This ensures mass, energy and momentum conservation, respectively.

That was the RHS of the Boltzmann equation, but the LHS must also be evaluated. To this end, it will be allowed for gradients of all macroscopic variables. This is necessary in order to account both for viscosity and thermal conduction. The gradients are assumed small, so as an approximation f_0 will be used for f. The quantities on LHS of (3.4) that need to be calculated are $\frac{\partial f}{\partial t}$ and $\mathbf{v}\nabla f$. Adding macroscopic velocity $\mathbf{V}$ to the gas, the equilibrium distribution function reads

$$f_0 = e^{\frac{\mu - \epsilon_{int}}{T}} e^{\frac{m(\mathbf{v}-\mathbf{V})^2}{2T}}. \tag{3.34}$$

Starting with first things first one finds, via the chain rule that

$$\frac{\partial f}{\partial t} = \frac{f_0}{T}\left[\left(\frac{\partial \mu}{\partial T}\right)_P - \frac{\mu - \epsilon(\Gamma)}{T}\right]\frac{\partial T}{\partial t} + \left(\frac{\partial \mu}{\partial P}\right)_T \frac{\partial P}{\partial t} + m\mathbf{v}\frac{\partial \mathbf{V}}{\partial t}. \tag{3.35}$$

Here, $\mathbf{V} = 0$ has been chosen, since it will not constrain the generality of the result; the dissipations certainly will not be affected by the motion of the fluid as a whole. Exploiting the thermodynamic relations (2.50) and (2.51) one finds

$$\frac{\partial f}{\partial t} = \frac{f_0}{T}\left\{\frac{\epsilon(\Gamma) - w}{T}\frac{\partial T}{\partial t} + \frac{1}{n}\frac{\partial P}{\partial t} + m\mathbf{v}\frac{\partial \mathbf{V}}{\partial t}\right\}. \tag{3.36}$$

In the same spirit, one obtains for $\mathbf{v}\nabla f$

$$\mathbf{v}\nabla f_0 = \frac{f_0}{T}\left\{\frac{\epsilon(\Gamma) - w}{T}\mathbf{v}\nabla T + \frac{1}{n}\mathbf{v}\nabla P + m v_i v_j V_{ij}\right\}, \tag{3.37}$$

where $V_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)$ is introduced as a shorthand notation. To continue, the route goes back to the equation of continuity(3.26a), the Euler equation(3.26c) and the equation of adiabatic motion, equation (3.13). Imposing $\mathbf{V} = 0$ on these three equations yield, respectively,

$$\frac{1}{n}\frac{\partial n}{\partial t} = -\nabla \cdot \mathbf{v} \tag{3.38}$$

(where the ideal gas equation of state (2.55) has been used), and

$$\frac{\partial \mathbf{V}}{\partial t} = -\frac{1}{nm}\nabla P \tag{3.39}$$

and lastly

$$\frac{\partial \sigma}{\partial t} = \frac{c_p}{T}\frac{\partial T}{\partial t} - \frac{1}{P}\frac{\partial P}{\partial t} = 0 \tag{3.40}$$

by the thermodynamic identities (2.53) and (2.54). These expressions allow for rewriting all the macroscopic quantities in our expressions for $\frac{\partial f}{\partial t}$ and ∇f that are derivatives of time to spatial derivatives of macroscopic quantities. In doing so, the resulting transport equation obtained from equating the sum of equations (3.36) and (3.37) with (3.33) becomes

$$\frac{\epsilon(\Gamma) - c_P T}{T}\mathbf{v}\nabla T + \left[m v_i v_j - \delta_{ij}\frac{\epsilon(\Gamma)}{c_v}\right] V_{ij} = I(\chi). \tag{3.41}$$

One last ingredient that was needed to obtain this transport equation, was the assumption that the specific heat be independent of temperature, implying $w = c_P T$. This is the only assumption done as to the temperature dependence of the thermodynamic quantities (together with the equation of state for an ideal gas). Note that the pressure gradient is not present in the equation; a pressure gradient alone cannot bring about dissipation.

Thermal conduction

It is now instructive to continue analysing the LHS of equation (3.41) term by term. Seeking an expression for the thermal conductivity, only the ∇T term is left on the LHS. One finds

$$\frac{\epsilon(\Gamma) - c_P T}{T}\mathbf{v}\nabla T = I(\chi), \tag{3.42}$$

where it must now be required that

$$\chi = \mathbf{g}\nabla T, \tag{3.43}$$

where $\mathbf{g} = \mathbf{g}(\Gamma)$ is some function of the phase space coordinates Γ, parallel to $\mathbf{v}$. The pre-factors in front of ∇T on both sides must be equal, so the equation reduces to

$$\frac{\epsilon(\Gamma) - c_P T}{T}\mathbf{v} = I(\mathbf{g}). \tag{3.44}$$

Now imposing the three conditions previously found as equations (3.30)-(3.32), only the last condition is not automatically satisfied, begging that

$$\int f_0 \mathbf{v}\cdot\mathbf{g}\, d\Gamma = 0. \tag{3.45}$$

Now, the art of this is contained in constructing the right function $\mathbf{g}$. But as soon as this function is known, and the transport equation solved, the thermal conductivity may be found by calculating the energy flux $\mathbf{q}$. Again requiring the overall motion of the gas to be zero ($\mathbf{V} = 0$), one has (equation (3.23))

$$\mathbf{q} = \int \epsilon \mathbf{v} f\, d\Gamma = \mathbf{q} \overset{(3.29)}{=} \frac{1}{T}\int \epsilon \mathbf{v}\chi(\Gamma)\, d\Gamma = \frac{1}{T}\int \epsilon \mathbf{v}\cdot(\mathbf{g}\nabla T)\, d\Gamma \tag{3.46}$$

where in the second equality the necessity of $\mathbf{q} = \int \epsilon \mathbf{v} f_0\, d\Gamma = 0$ was used. Now, since a gas in equilibrium is isotropic, the dot-products reduces such that

$$\mathbf{q} = -\kappa \nabla T \tag{3.47}$$

with

$$\boxed{\kappa \equiv -\frac{1}{3T}\int \epsilon \mathbf{v}\cdot\mathbf{g} f_0\, d\Gamma.} \tag{3.48}$$

κ is called the scalar *thermal conductivity*.

It is instructive to have a closer look at the validity of the approximation that was made to derive this result, namely that f is close to f_0. Taking the mean energy of a molecule to be $\bar{\epsilon} \sim T$, one upon inserting into (3.44) obtains the relation $\mathbf{v} \sim g/\tau \sim g\bar{v}/l$ and hence $g \sim l$. Inserting this result into (3.48) gives the well-known

$$\kappa \sim cnl\bar{v}, \tag{3.49}$$

c being the specific heat per molecule of gas. Inserting $l \sim 1/N\sigma$, $c \sim 1$ and $\bar{v} \sim \sqrt{T/m}$ one finds

$$\kappa \sim \frac{1}{\sigma}\sqrt{T/m}, \tag{3.50}$$

(remember that σ here denotes the cross-section, and not the entropy) which for temperatures high enough for the hard sphere scattering approximation, concludes that $\kappa \sim \sqrt{T}$. Note the important consequence of equation (3.50); the thermal conductivity doesn't depend on the gas density. This results as a consequence of the two-particle-collision approximation used in deriving the Boltzmann equation.

Viscosity

The procedure in this section resembles the procedure used for obtaining the thermal conductivity in the last section. This time, however, the deviation from equilibrium is assumed due to the macroscopic velocity **V** of the gas, and not the temperature gradient. From (3.41), the equation to solve is this time

$$\left[m v_i v_j - \delta_{ij} \frac{\epsilon(\Gamma)}{c_v} \right] V_{ij} = I(\chi). \tag{3.51}$$

To find expressions for the coefficients, one must then naturally choose

$$\chi = g_{ij} V_{ij}, \tag{3.52}$$

where g_{ij} is a symmetric tensor. Before going on, the momentum tensor for the ideal scenario, equation (3.24) must be extended. In the general viscous case, a term Σ' is added, such that

$$\Pi_{ij} = P\delta_{ij} + \rho V_i V_j - \Sigma'. \tag{3.53}$$

The *viscous stress tensor* Σ' contains the viscous content of the momentum tensor, and reads

$$\Sigma' = 2\eta(V_{ij} - \frac{1}{3}\nabla \cdot \mathbf{V}) + \zeta\delta_{ij}\nabla \cdot \mathbf{V}, \tag{3.54}$$

where η and ζ are viscosity coefficients to be defined. In passing, also note that the so-called *stress tensor* Σ is defined as

$$\Sigma_{ij} = -P\delta_{ij} + \Sigma'_{ij}. \tag{3.55}$$

Now, rewriting (3.51) ever so slightly, one finds

$$m v_i v_j (V_{ij} - \frac{1}{3}\delta_{ij}\nabla \cdot \mathbf{V}) + \left[\frac{1}{3} m v^2 - \epsilon(\Gamma)/c_v \right] \nabla \cdot \mathbf{V} = I(\chi), \tag{3.56}$$

which makes the connection to (3.53) with (3.55) evident, suggesting that calculations of expressions for η and ζ should be done separately.

The first viscosity: Calculating the first viscosity coefficient, η, one must therefore demand $\nabla \cdot \mathbf{V} = 0$, to distinguish this coefficient from the second viscosity coefficient ζ.[5] Equation (3.56) may then be rewritten to

$$m(v_i v_j - \frac{1}{3}\delta_{ij} v^2) V_{ij} = I(\chi) \tag{3.57}$$

by changing g_{ij} accordingly. Inserting the solution $\chi = g_{ij} V_{ij}$ into the above equation, one has

$$m(v_i v_j - \frac{1}{3}\delta_{ij} v^2) = I(g_{ij}). \tag{3.58}$$

[5] This causes no loss of generality, since what here is sought is an expression for the coefficient η, which remains in the equation.

Again the three conditions (3.30)-(3.32) have to be satisfied - which they necessarily are. Recall the definition of the stress tensor;

$$\Pi_{ij} = \int v_i v_j f \, d\Gamma. \tag{3.59}$$

Inserting $f = f_0 + \delta f = f_0 + f_0\chi(\Gamma)/T$, this must mean that one can express the viscous contribution Σ' as

$$\Sigma'_{ij} = -\frac{m}{T}\int v_i v_j f_0 \chi \, d\Gamma \equiv \eta_{ijkl} V_{kl}, \tag{3.60}$$

where η_{ijkl} is a rank 4 tensor defined such that

$$\eta_{ijkl} = -\frac{m}{T}\int v_i v_j f_0 g_{kl} \, d\Gamma. \tag{3.61}$$

Now, g_{ij} is a symmetric tensor. Therefore η_{ijkl} is symmetric in i,j and in k,l. Coming back to the requirement that the gas be isotropic, the tensor can only be expressed in terms of the delta function. This tensor shall be taken to be on the form

$$\eta_{ijkl} = \eta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}). \tag{3.62}$$

Defining η through

$$\Sigma'_{ij} \equiv 2\eta V_{ij}, \tag{3.63}$$

it is found to be the contraction of the two pairs of indices i,j and k,l. The result is

$$\boxed{\eta \equiv -\frac{m}{10T}\int v_i v_j f_0 g_{ij} \, d\Gamma,} \tag{3.64}$$

the scalar *shear viscosity coefficient*.[6]
As in the case of heat conduction, again the actual form of g_{ij} has to be worked out on physical grounds. Also for this result note that a result similar to (3.49) may be obtained by the same methods. It reads

$$\eta \sim m\bar{v}nl \quad \rightarrow \quad \eta \sim \frac{\sqrt{mT}}{\sigma}. \tag{3.65}$$

Again the result is not dependent on the density. This is a direct consequence of the Boltzmann transport equation, which only takes two body collisions into account. Remember that $\bar{v}$ here must be the mean thermal velocity (since $\mathbf{V} = 0$ one finds $\bar{v} = \bar{c}$).

Second viscosity: Finally turning to the second viscosity; the bulk viscosity, the interesting part of the transport equation reads

$$\left[\frac{1}{3}mv^2 - \epsilon(\Gamma)/c_v\right] \nabla\cdot\mathbf{V} = I(\chi). \tag{3.66}$$

This time seeking a solution of the form $\chi = g\nabla\cdot\mathbf{V}$ one finds

$$\frac{1}{3}mv^2 - \epsilon(\Gamma)/c_v = I(g). \tag{3.67}$$

[6] Also called the *dynamic viscosity*, but this name will not be used any further in this text.

Again the stress tensor (3.59) must be calculated. For the interesting variational part δf one finds

$$\Sigma'_{ij} = -\frac{m}{T}\int v_i v_j f_0 \chi d\Gamma = -\frac{m}{T}\int v_i v_j f_0 g \nabla\cdot\mathbf{V} d\Gamma = -\frac{m}{3T}\int v^2 g f_0 d\Gamma = \Xi\nabla\cdot\mathbf{V} d\Gamma \qquad (3.68)$$

where, upon comparison with equation (3.55) one finds that $\Xi = \zeta$. Therefore,

$$\boxed{\zeta \equiv -\frac{m}{3T}\int v^2 g f_0 d\Gamma} \qquad (3.69)$$

is found for the *bulk viscosity*, which is the name henceforth given to it[7]. From the definition of the viscous stress tensor Σ', it is clear that the bulk viscosity functions as to modify the pressure of the fluid. Also note that the bulk viscosity will vanish for mono-atomic gases, since the LHS of (3.67) vanishes. Refer to (Tisza, 1942) for a useful discussion of the so-called Stokes viscosity relation (which says that $\zeta = 0$) and its applicability in supersonic absorption.

In the following two sections, a solution procedure called the **Chapman-Enskog** solution will be applied to actually solving the integrals derived for the transport coefficients, but first a short word on the continuum limit.

Navier-Stokes equation: In this section it has been shown how to account for dissipative processes via kinetic theory, and also how the fluid mechanical equations arise as a macroscopic result of kinetic theoretical equations. It is evident that dissipative processes enter into the macroscopic fluid mechanical equations due to small deviations from equilibrium on microscopic level. To conclude this section by making the connection to the continuum limit absolute; note that adding the dissipative term Σ' to the momentum flux tensor changes the momentum balance (equation (3.26c)). Inserting and rearranging terms ever so slightly, one finds, following (Landau and Lifshitz, 2009),

$$\rho\left[\frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial x^k}\right] = -\frac{\partial P}{\partial x^k} + \frac{\partial}{\partial x^k}\left[\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_i^k\frac{\partial v_l}{\partial x_l}\right) + \zeta\delta_i^k\frac{\partial v_l}{\partial x_l}\right]. \qquad (3.70)$$

Here, the density ρ has been assumed constant. In fluid mechanics, this is the most general form of the equation of motion. In general, the viscosity coefficients that appear will be dependent on pressure and temperature. In the special case, however, where this is not the case, equation (3.70), in vector form, reduces to

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla P + \eta\Delta\mathbf{v} + (\zeta + \frac{1}{3}\eta)\nabla(\nabla\cdot\mathbf{v}), \qquad (3.71)$$

which one recognizes as the typical form of the **Navier-Stokes equation**. Note, that from the above Navier-Stokes equation, or also from the definition of the dissipative part of the momentum tensor, equation (3.55), it is evident that if the divergence of the velocity field is zero, i.e. $\nabla\cdot\mathbf{v} = 0$, then the bulk viscosity is gone. A fluid for which $\nabla\cdot\mathbf{v} = 0$ is called *incompressible*. **The bulk viscosity therefore pertains in a fluid that has the ability of expanding.** The shear viscosity coefficient η on the other hand, remains even for incompressible fluids (i.e. fluids where $\partial v_i/\partial x_i = 0$). Friction between adjacent layers of the fluid in motion at different

[7] It is also called the *second viscosity*, as previously mentioned. Another related quantity is *volume viscosity*, which is to be discussed later.

velocities will cause dissipation, so the factor

$$\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)$$

remains.

3.1.2 The BGK-model and Chapman-Enskog solution

In the previous sections, the macroscopic transport coefficients were found to be

$$\boxed{\kappa=-\frac{1}{3T}\int \epsilon\mathbf{v}\cdot\mathbf{g}f_0\,d\Gamma\ ,\qquad \eta=-\frac{m}{10T}\int v_iv_jf_0g_{ij}\,d\Gamma\ ,\qquad \zeta=-\frac{m}{3T}\int v^2gf_0\,d\Gamma.} \tag{3.72}$$

However, these integrals are generally hard to solve. In this section, a simplified model will be applied to obtain more useful expressions for the transport coefficients. The model bears the explanatory name **the BGK model**, after its inventors **B**hatnagar, **G**ross and **K**rook. However, the equation is solved for a mono-atomic gas only, and unfortunately, as mentioned, the bulk viscosity vanishes in this case. Literature searches for useful expressions of the bulk viscosity coefficient from the solution of Boltzmanns equation for a gas with internal degrees of freedom has proven unsuccessful. To keep things within the scope and limitations of the present work, therefore, remedy is not sought through direct solution of the Boltzmann equation itself, but rather through alternative justification of an approximative expression for the bulk viscosity. This section is extensively based on chapters 5 and 7 in (Hänel, 2004).

The simplified Boltzmann equation, from now on referred to as the BGK-Boltzmann equation, reads

$$\boxed{\frac{\partial}{\partial t}f+\mathbf{v}\nabla f=\omega(f_0-f)} \tag{3.73}$$

where here f_0 as before is the local equilibrium distribution, and $f=f(\rho,\mathbf{v},T)$ is the general non-equilibrium distribution. $\omega=\omega(\rho,T)$ is the molecular collision frequency.

Below it is to some extent demonstrated (and to some extent merely stated) that the above equation conserves the most important properties found in the original Boltzmann equation:

1) Equilibrium: According to the H-theorem, the distribution function of any system, strives towards equilibrium (The Maxwell distribution) as time goes on. In the BGK model, the collision integral on the RHS of the Boltzmann equation (3.4) is simplified in such a way that this is taken care of. In a homogeneous situation where the distribution reduces to

$$\frac{\partial}{\partial t}f=\omega(f_0-f) \tag{3.74}$$

one may solve to find

$$f-f_0=\big(f(t=0)-f\big)\,e^{-t/\tau}, \tag{3.75}$$

where $\tau=1/\omega$ is the relaxation time. In other words it is evident that the difference between f and f_0 vanishes as time goes on. This is taken to mean that the distribution function approaches the Maxwell distribution with time.

2) Conservation of mass, energy and momentum: For the BGK model to be of any use, it better conserve mass, energy and momentum, as was found in the general case. An effective way to demonstrate this, is to calculate moments of the BGK model. Let us first define the three variables

$$\phi_1 = m \quad , \qquad \phi_2 = m\mathbf{v} \quad , \qquad \phi_3 = \frac{m}{2}\mathbf{v}^2. \tag{3.76}$$

Inserting any of these three in the BGK Boltzmann (3.73) and integrating should yield the equation

$$\frac{\partial}{\partial t}\int \phi_i f d\mathbf{v} + \frac{\partial}{\partial x^j}\int \phi_i v_j f d\mathbf{v} = \omega\left(\int \phi_i f_0 d\mathbf{v} - \int \phi_i f \mathbf{v}\right), \tag{3.77}$$

where i here is meant to make reference to the three different ϕ-s listed in (3.76), whereas the index j runs over the three velocity directions. Now; the clue is that the RHS of the above equation will vanish for any of the ϕ values. Inserting one can easily validate that the three remaining equations will give the conservation equations for mass, momentum and energy, listed in (3.26).

3) Entropy: Further, the entropy production should be positive. And indeed; starting from

$$H = \int_v f\, lnf\, d\mathbf{v} = -\frac{1}{kV} \leq 0 \tag{3.78}$$

one can, by differentiating with respect to time and using that $\frac{\partial f}{\partial t} = \omega(f_0 - f)$ show that

$$\frac{\partial S}{\partial t} \geq 0 \tag{3.79}$$

which should be required to be in keeping with the thermodynamic identity.

Solving the equation

Before going on, the Boltzmann equation is rewritten in dimension-less form by defining dimensionless quantities through

$$x_i = \bar{x}_i \cdot L, \qquad t = \bar{t} \cdot \frac{L}{c_0}, \qquad v = \bar{v}c_0, \qquad f = \bar{f} \cdot \frac{n}{c_0^2}, \qquad \omega = \bar{\omega}\frac{c_0}{l}, \tag{3.80}$$

where L is the typical length of the system, c_0 is a typical thermal velocity and l is the mean free path in between collisions. Further, one defines

$$\epsilon = \frac{l}{L} \tag{3.81}$$

which indeed is nothing but the so-called **Knudsen number**. A Knudsen number $\epsilon = 0$ means that the system is in thermal equilibrium. Since what here is treated are small deviations from such an equilibrium, one must require that

$$Kn = \epsilon = \frac{l}{L} << 1 \qquad \rightarrow \qquad \epsilon = \frac{\tau}{T} << 1, \tag{3.82}$$

where in the last equation $l = c_0\tau$ has been used together with $T = L/c_0$. Therefore; a small Knudsen number is to say that many collisions will occur before a molecule has moved in any macroscopic sense, or, equivalently; **the mean free time τ between collisions is much**

shorter than a characteristic time scale T of the macroscopic system. For small ϵ, the transport coefficients should be non-vanishing and finite. In the following, the Chapman-Enskog expansion is applied: The non-equilibrium distribution is written out in a power series in ϵ such that

$$\bar{f} = \bar{f}^{(0)} + \epsilon \bar{f}^{(1)} + \epsilon^2 \bar{f}^{(2)} + ... + \epsilon^n \bar{f}^{(n)}, \qquad \epsilon << 1. \tag{3.83}$$

In the following, this series will be truncated after the first two terms, so that a first order contribution is kept. Inserting into the BGK-Boltzmann equation, using the definitions of the dimensionless quantities, one finds

$$\epsilon \frac{d}{dt}\left(\bar{f}^{(0)} + \epsilon \bar{f}^{(1)}\right) = \omega\left(f_0 - \bar{f}^{(0)} + \epsilon \bar{f}^{(1)}\right), \tag{3.84}$$

where the total derivative $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i}$ has been used to keep the notation readable. For this equation to hold true for all ϵ, one must require that the terms of the same power in ϵ can be equated alone. I.e; only keeping terms to zeroth order, one finds

$$\bar{\omega}\left(f_0 - \bar{f}^{(0)}\right) = 0 \qquad \text{and} \qquad \frac{d}{d\bar{t}}\bar{f}^{(0)} = 0, \tag{3.85}$$

which requires $\bar{f}^{(0)} = \bar{f}_0$, *and* that the solution is invariant under time translation. This is to say that the Maxwell equilibrium distribution solves the equation.
Keeping the terms up to first order in ϵ, one finds the three equations

$$\bar{\omega}\left(f_0 - \bar{f}^{(0)}\right) = 0 \quad , \quad \frac{d}{d\bar{t}}\bar{f}^{(0)} = -\bar{\omega}\bar{f}^{(1)} \quad \text{and} \quad \frac{d}{d\bar{t}}\bar{f}^{(1)} = 0, \tag{3.86}$$

which can be written in one equation as

$$\frac{d}{d\bar{t}}\bar{f}_0 = -\bar{\omega}\bar{f}^{(1)}. \tag{3.87}$$

Since $\bar{f}_0$ is known, one may now solve this equation to obtain an expression for the distribution function $\bar{f} = \bar{f}_0 + \epsilon \bar{f}^{(1)}$. When the distribution function is known, one may from there deduce the transport coefficients, as will be outlined soon.

Following the source, the distribution function to first order is rewritten in dimension-full form as

$$f = f_0 + f_0\phi, \qquad \text{where} \qquad \phi = \epsilon\frac{f}{f_0} << 1. \tag{3.88}$$

Note that with this one is back to equation(3.27); the assumption that was the starting point of the more general derivation of transport coefficients. The BGK-Boltzmann equation reads

$$\frac{\partial f_0}{\partial t} + v_i \frac{\partial f_0}{\partial x_i} = -\omega f_0 \phi, \qquad \mathrm{i} = 1,2,3. \tag{3.89}$$

From the Boltzmann distribution one obtains for an ideal gas the Maxwell velocity distribution, which, as mentioned, is the equilibrium distribution here referred to as f_0. It reads

$$f_0(\mathbf{x}, t, \mathbf{v}) = \frac{n}{(2\pi RT)^{3/2}} e^{\frac{(v_i - V_i)^2}{2RT}}. \tag{3.90}$$

Here, $R = k/m$ is the specific gas constant. and the collision frequency $\omega(\mathbf{x}, t) = \omega(\rho, T)$. The macroscopic variables are also time and space dependent;

$$n = n(\mathbf{x}, t) \quad , \qquad v_i = v_i(\mathbf{x}, t) \quad \text{and} \quad T = T(\mathbf{x}, t). \tag{3.91}$$

All in all, one may therefore solve equation (3.89) for ϕ;

$$\phi(\mathbf{x}, t, \mathbf{v}) = -\frac{1}{\omega f_0(\mathbf{x}, t, \mathbf{v})}\left(\frac{\partial f_0(\mathbf{x}, t, \mathbf{v})}{\partial t} + v_i \frac{\partial f_0(\mathbf{x}, t, \mathbf{v})}{\partial x_i}\right). \tag{3.92}$$

The next step now is to rewrite the problem through $df_0/f_0 = d(lnf_0)$, and also rewriting the spatial and temporal derivatives in terms of the macroscopic variables. Finally using the three conservation equations for mass, energy and momentum, equations (3.26), the source finds

$$\phi = -\frac{1}{\omega}\left[\left(\frac{\mathbf{c}^2}{2RT} - \frac{5}{2}\right)\frac{1}{T}c_i \cdot \frac{\partial T}{\partial x_i} + \frac{1}{RT}\left(c_i c_j - \frac{1}{3}\mathbf{c}^2 \delta_{ij}\right) \cdot \frac{\partial v_j}{\partial x_i}\right] \tag{3.93}$$

where

$$\rho = m \cdot n \quad , \qquad P = \rho RT \qquad \text{and} \qquad e = c_v T = \frac{3}{2} RT \tag{3.94}$$

has been used as well. **Note that this assumes a mono-atomic gas** ($c_v = 3R/2$). From here one finally obtains for the distribution function $f = f_0(1 + \phi)$ that

$$f = f_0\left(1 - \frac{1}{\omega}\left[\left(\frac{\mathbf{c}^2}{2RT} - \frac{5}{2}\right)\frac{1}{T}c_i \cdot \frac{\partial T}{\partial x_i} + \frac{1}{RT}c_i c_j \cdot S_{ij}\right]\right). \tag{3.95}$$

Refer to (Hänel, 2004) for more details and further references. Here, S_{ij} is the distortion tensor defined as

$$S_{ij} = S_{ji} \equiv \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\frac{\partial V_k}{\partial x_k}\delta_{ij}\right). \tag{3.96}$$

Hence is the BGK model solved to first order in the Chapman-Enskog expansion for the non-equilibrium distribution function f. One may now obtain the transport coefficients from comparison with definitions. The definition of the stress tensor Σ_{ij} (see equation (3.55)) can be taken as

$$\Sigma_{ij} = m\int c_i c_j f d\mathbf{c} = m\int c_i c_j f_0 d\mathbf{c} + m\int c_i c_j f_0 \phi d\mathbf{c} = P\delta ij - \Sigma'_{ij}, \tag{3.97}$$

where P is the pressure and Σ'_{ij} is the viscous part of the stress tensor. They are defined such that

$$P\delta_{ij} \equiv m\int c_i c_j f_0 d\mathbf{c} \tag{3.98}$$

and

$$\Sigma'_{ij} \equiv -m\int c_i c_j f_0 \phi d\mathbf{c} = -m\int c_i c_j (f - f_0) d\mathbf{c}. \tag{3.99}$$

Effectively, the splitting of terms in equation (3.97) is a division between the equilibrium part of the stress tensor (first term; the pressure P), and the non-equilibrium part (Σ'_{ij}). Actually performing the integration, one finds that in the case of the BGK model,

$$\Sigma'_{ij} = 2\frac{nkT}{\omega} \cdot S_{ij} = \frac{nkT}{\omega} \cdot \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3}\frac{\partial V_k}{\partial x_k}\delta_{ij} \right). \tag{3.100}$$

Dividing into tangential (shear) and normal stresses one finds

$$\Sigma'_{ij} = \begin{cases} \frac{nkT}{\omega} \cdot \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) & \textbf{Tangential stress} \quad i \neq j, \\ \frac{nkT}{\omega} \cdot \left(2\frac{\partial V_i}{\partial x_i} - \frac{2}{3}\frac{\partial V_k}{\partial x_k}\delta_{ij} \right) & \textbf{Normal stress} \quad i = j. \end{cases} \tag{3.101}$$

For the heat flow, one finds through the definition (3.23) that

$$q_i = -\frac{m}{2} \int c_i c^2 f \phi d\mathbf{c} = -\frac{m}{2} \int c_i c^2 f_0 d\mathbf{c} - \frac{m}{2} \int c_i c^2 f_0 \phi d\mathbf{c}. \tag{3.102}$$

Inserting for ϕ from (3.93) and performing the integration, one ends up with

$$q_i = \frac{5}{2}\frac{k}{m}\frac{nkT}{\omega}\frac{\partial T}{\partial x_i}. \tag{3.103}$$

Transport coefficients

One can now, through the so-called **Newtonian ansatz**[8] relate the macroscopic quantities η, ζ and κ – the transport coefficients in the Navier-Stokes equations (3.70) – to the microscopically obtained equations. The **Newtonian ansatz** reads

$$\Sigma'_{ij} = 2\eta S_{ij} = \eta \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{\eta_v}{\eta}\frac{\partial V_k}{\partial x_k}\delta_{ij} \right). \tag{3.104}$$

With this ansatz the viscosity coefficients are uniquely determined. Similarly, with **Fourier's heat conduction ansatz**[9]

$$q_i = -\kappa \frac{\partial T}{\partial x_i}, \tag{3.105}$$

the heat conduction coefficient κ is also determined by comparison with (3.103). All in all the end results are

$$\boxed{\begin{aligned} \eta &= \frac{nkT}{\omega} = \frac{3}{2}\eta_V, \\ \kappa &= \frac{5}{2}\frac{k}{m}\frac{nkT}{\omega}. \end{aligned}} \tag{3.106}$$

In (Hänel, 2004) it is shown that for a Maxwell distribution, the arithmetic mean thermal velocity $\bar{c}$ of the molecules will be

$$\bar{c} = \sqrt{\frac{8RT}{\pi}} \sim \sqrt{T} \tag{3.107}$$

[8] In the context of a mono-atomic gas, this is rigorously derived, and should be taken as a definition of η_V rather than as an ansatz. However, in the good old spirit of naming conventions, names remain names, and the physics in it something quite different.

[9] (Hänel, 2004) seems to be using a different sign convention than (Landau and Lifshitz, 1981), but hopefully the present work has succeeded in presenting the material both consistently and correctly. What is important in the end is that the coefficients turn out positive, which seems to be the case.

By $\omega = \bar{c}/l$, where l is the mean free path, it is then evident that all the three transport coefficients evolve proportional to the square root of the temperature;

$$\eta \sim \eta_v \sim \kappa \sim \sqrt{T}.$$

This is in agreement with the estimates done for the more general coefficients found in section 3.1.1. According to (Hänel, 2004) this reflects the fact that a solid sphere model was used.

The bulk viscosity coefficient: In the present work the bulk viscosity is of major concern. However, it doesn't show up in the list of coefficients (3.106). *The volume viscosity η_v must not be confused with the bulk viscosity.* The volume viscosity coefficient is specifically defined such as to make Σ'_{ij} trace-free[10] (this is automatically fulfilled for a mono-atomic gas). A trace-free Σ'_{ij} amounts in cancelling any pressure modifications. This can be seen from the definition of the stress tensor (3.97). Denote the viscosity modified pressure P'. Then, with (3.101) one finds from the diagonal part of the stress tensor that

$$P' \equiv \Sigma_{jj} = P - \Sigma'_{jj} \tag{3.108}$$

Now, since $\Sigma'_{ij} = 0$ whenever $i = j$[11], this gives

$$P' = P \tag{3.109}$$

It seems therefore that the viscosity-modified pressure that here was sought, vanishes. This is quite correct, and resides on the fact that an ideal mono-atomic gas assumption was made in the derivation. What really should be done, is to step a step or two back in the derivation and try to make an estimate for the bulk viscosity if this assumption is relaxed[12]. However, that would be far beyond the scope of the work at hand, and a much simpler and far more crude approximation will here be done. Noting that the factor nkT/ω seems important both for the shear viscosity and for the scalar thermal conduction, and since it is also the functional form of the volume viscosity, it will be used as a first model for the bulk viscosity:

$$\boxed{\zeta \sim \frac{nkT}{\omega}} \tag{3.110}$$

This might not be as *ad hoc* an estimate as it first looks like. In the next chapter it will be shown that the formula derived in (Zimdahl, 1996) (2.122) reduces to the above formula except for a prefactor when employed with two different fluids with about the same number densities $n_1 = n_2 \equiv n$. In the continuation, equation (3.110) therefore will be used as an approximation for ζ.

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[10]At least this is the definition used in the present work. Again there might be different conventions in the literature. For instance, the use of η_v in (Hoogeveen, 1986) is not quite understood to this end. Never mind; the point is that η_v as defined in (Hänel, 2004) must not be confused with ζ as defined in (Landau and Lifshitz, 2009) and used in the present work.

[11]This must be so, since $\mathrm{Tr}(\Sigma') = 0$ and since the universe is assumed isotropic at every point, such that $\Sigma'_{11} = \Sigma'_{22} = \Sigma'_{33}$.

[12]It is tempting to think that the prefactor 2/3 in front of the divergence part of the viscous stress tensor Σ' is due to the degrees of freedom for the mono-atomic gas assumption. If so, additional degrees of freedom ADF could be thought to yield $\zeta = ADF \cdot \eta$ and thus $\mathrm{Tr}(\Sigma' \neq 0)$. This is a mere speculation at this point.

3.1.3 Chapman-Enskog expansion of the Boltzmann equation

In the previous section, it was shown that expressions for the viscosity coefficients could be obtained from a first order Enskog-Chapman expansion of the BGK-Boltzmann equation. As was shown, this equation is a simplified version of the real Boltzmann equation. It is, however, also possible to apply the Chapman-Enskog expansion to the Boltzmann equation without the simplifying BGK model. It can then be shown that the transport coefficients only depend on the temperature T, and that for an intermolecular potential $\Phi(r) \sim r^{-\alpha}$, one will have

$$\eta = \eta_r \left(\frac{T}{T_r} \right)^{(\alpha+4)/2\alpha} \qquad \text{and} \qquad \kappa = \kappa \left(\frac{T}{T_r} \right)^{(\alpha+4)/2\alpha} \tag{3.111}$$

where η_r and κ_r are reference values defined through $\eta(T = T_r) \equiv \eta_r$ and $\kappa(T = T_r) \equiv \kappa_r$.

See (Hänel (2004),chap 7.2) and references therein for details. Note that a hard sphere model $\alpha \to \infty$ will give $\eta \sim \kappa \sim \sqrt{T}$ as before.

Chapter 4

Constraining the bulk viscosity through observations

4.1 Introduction

In the two preceding chapters, formulae for the bulk viscosity have been derived on different grounds; The Weinberg formula (2.109a) is derived by applying the Eckart formalism in cosmological context, the Zimdahl formula (2.122) originats from a rather general argument applied to cosmology, and the Hänel formula (3.110) is derived from kinetic theory.

In the present chapter, the observational side will be the main focus. As mentioned in the introduction, what here is referred to as observations are the Hubble parameter measurements $H(z)$ as a function of redshift. The data used are listed in table 2.1. Following the procedure used in (Wang and Meng, 2014), measurements of the development of the Hubble parameter as function of redshift, $H(z)$, will be compared with viscosity-modified solutions of the energy equation and Friedmann's first equation. Perhaps this can give any insight in constraining the size of the bulk viscosity. Finally, upon having obtained some constraints, an attempt at finding the cause of the viscosity on the basis of the three above mentioned formulae is performed.

Much of this chapter, therefore, is dedicated to solving the energy equation for different assumptions made for the bulk viscosity. The procedure is not based on any previous literature in particular, but standard methods for solving differential equations are applied, and the literature is referred to where appropriate.

4.2 Literature survey

Quite a bit has been done to calculate and estimate the viscosity of the early epochs of the universe's history. This is natural, since the viscosity is assumed to have been much greater during these epochs. For an extensive review to this end, refer to e.g. (Grøn, 1990) and also (Brevik and Heen, 1994). However, not so much seems to have been done to estimate the bulk viscosity of the later epochs of the universe. Table 4.1, taken from (Frampton, 2015) gives a brief overview of the history of the universe.

Cosmological Evolution			
Cosmic time	*scale factor a*	*Era*	*Redshifts*
$t = 13.8$ Gy	1	Present	0
9.8Gy< $t < 13.8$Gy	$a(t) = e^{H_0(t-t_0)}$	DE domination	-
$t = 9.8$Gy	0.75	onset of DE domination	0.25
47ky< $t < 9.8$	$a(t) \propto t^{2/3}$	matter domination	-
$t = 47$ky	$1.2 \cdot 10^{-4}$	onset of matter domination	3400
$t < 47$ky	$a(t) \propto t^{1/2}$	radiation domination	-
$t = 10^{-10}s$	$1.7 \cdot 10^{-15}$	electroweak phase transition	-
10^{-44}s< $t < 10^{-10}$s	$a(t) \propto t^{1/2}$	Possible inflation or bounce	-
$t < 10^{-44}$	$1.7 \cdot 10^{-32}$	Planck time	$5.9 \cdot 10^{31}$

Table 4.1: Overview over cosmological time as function of redshift. The first three columns are based on (Frampton, 2015), and the last column contains a few useful approximate redshifts.

One interesting article dealing with the later epochs of cosmic evolution is found, however. In the following section this article (Wang and Meng, 2014) is discussed. Thereafter, in sections to come, more general results are sought formulated, using Wang and Meng's procedure.

4.3 Article by Wang and Meng

In their article (Wang and Meng, 2014), Wang and Meng "*extend the concept of temperature-dependent viscosity from classical statistical physics to observational cosmology.*" In particular "*the cosmological effects with possibility of the existence for two kinds of viscosity forms, which are described by the Chapman's relation and Sutherland's formula respectively*" are examined. Some of the detailed calculations done in this seem to be based on assumptions that are not necessarily the best ones. In the following, Wang and Meng's calculations will be redone and extended with what seems necessary.

The starting point in (Wang and Meng, 2014) is an energy-momentum tensor of the form

$$T_{\mu\nu} = \rho U_\mu U_\nu + (P - 3H\zeta)h_{\mu\nu}, \tag{4.1}$$

where H is the Hubble parameter $H = \dot{a}/a$. This is in agreement with equation (2.167a). From this, and with the FRW-metric for a **spatially flat** universe, the Einstein equations are solved to give

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda), \tag{4.2a}$$

$$H^2 + \dot{H} = -\frac{4\pi G}{3}\left[\rho_m + \rho_\Lambda + 3P'\right], \tag{4.2b}$$

which is the starting point in (Wang and Meng, 2014), and in agreement with (2.168). Note that they are assuming a fluid description of the cosmological constant, such that

$$\Lambda = 8\pi G \rho_\Lambda. \tag{4.3}$$

The overall equation of state is taken to be

$$P' = \sum_i \omega_i \rho_i - \zeta\theta = -\rho_\Lambda - \zeta\theta, \tag{4.4}$$

since $\omega_\Lambda = -1$ and $\omega_m = 0$.

The viscosity is in the paper taken to have the form

$$\zeta(T) = \zeta_0 T^\alpha, \tag{4.5}$$

where $\alpha = \frac{1}{2}$ or $\frac{3}{2}$ refers to Chapman and Sutherland viscosity, respectively. I.e.; the above formula is a combination of

1 the **Chapman-Enskog equation** for dilute multi-component gas mixtures (to first approximation), which in simplified form reads

$$\zeta = \frac{x_i^2}{Ax_i^2 + B} T^{1/2}, \tag{4.6}$$

where A and B according to the paper are generalized temperature-independent factors including collision diameter, collision integral and molecular mass, and x_i is the fraction of each gas component.

2 the **Sutherland's** formula, which reads

$$\zeta = \zeta' \left(\frac{T}{T'}\right)^{3/2} \frac{T' + S}{T + S}, \tag{4.7}$$

where S is Sutherland's constant and the prime represents reference values. Refer to the paper for details and further references therein.

In the following, only equation (4.5) will be used.

In their paper, the full derivation of the equation that was integrated numerically, was (quite naturally) not included. In the following, however, the derivation of the equation - as far as understood by undersigned - is given.

4.3.1 Derivation

By direct insertion of equation (4.2a) into equation (4.2b) one finds

$$\dot{H} = -4\pi G(\rho + P'). \tag{4.8}$$

Inserting the EoS (4.4) then gives

$$\dot{H} = -4\pi G\rho_m + 12\pi GH\zeta \tag{4.9}$$

in natural units ($c = 1$). ρ_Λ is now gone from the equation, and the exact same result would have been obtained by omitting ρ_Λ from all the equations. Therefore, omitting ρ_Λ in (4.2) one finds[1]

[1]This step is found to be rather strange. Rewriting in terms of H at this point should have included a term of ρ_λ, (since $\rho = \rho_\lambda + \rho_m$), but this does not seem to correspond with the final result in the paper, so it seems as if

$$\dot{H} = -\frac{3}{2}H^2 + 12\pi G H\zeta. \tag{4.10}$$

Now, to continue, Wang and Meng have rewritten in terms of the dimensionless $E = H/H_0$ (equation (2.44)), using the relation

$$E^2(z) \equiv \Omega_{0k}(1+z)^2 + \sum_i \Omega_{i0}(1+z)^{3(1+\omega_i)}. \tag{4.11}$$

Here Ω_{0i} is the present relative density of a cosmic fluid with EoS $p_i = \omega_i \rho_i$, and Ω_{0k} is the curvature parameter. With $k = 0$ and $i \rightarrow m$ it follows that

$$E^2(z) = \Omega_{0m}(1+z)^3. \tag{4.12}$$

Refer to (Grøn and Hervik (2007), p. 124) and further equation-references to see that equation (4.11) (and thus also (4.12)) are valid for an ideal cosmological fluid only; i.e. for $\zeta = 0$ in the thermodynamic identity, equation (2.161). This will be discussed more intently when general solutions are sought. For the moment proceeding in ignorance, however, one finds by inserting (4.12) into (4.10) that

$$\dot{H} = -\frac{3}{2}\Omega_{0m}H_0^2(1+z)^3 + 12\pi G H\zeta, \tag{4.13}$$

which is equation (9) in (Wang and Meng, 2014).

To go on from here, equation (2.45) is utilized (refer back to see the relation between expansion and redshift).

$$\dot{H} = -(1+z)H_0^2 E\frac{dE}{dz}. \tag{4.14}$$

Also, equation (4.13) contains a viscosity term, for which the evolution over cosmic time is needed. This is taking on the very core of this dissertation. Wang and Meng has assumed a viscosity of the form (4.5), and need to relate this viscosity to the expansion a. To do so, they assume that

$$\rho_{rad} \propto T^4 \qquad \text{and} \qquad \rho_{rad} \propto \frac{1}{a^4} \tag{4.15}$$

which together gives the crucial relation

$$T \propto \frac{1}{a}. \tag{4.16}$$

Denoting the proportionality constant T_0^α the viscosity $\zeta(T)$ can be expressed in terms of the redshift parameter z as

$$\zeta(z) = \zeta_0 T_0^\alpha (1+z)^\alpha. \tag{4.17}$$

Inserting equation (4.14) and equation (4.17) into equation (4.13) and rearranging, the final result becomes

---

ρ_Λ has been dropped. What *actually* has been done in the paper if this is correct, is to solve with only one fluid component; pressure-less matter. However, the possibility that undersigned has misunderstood something is kept open.

$$\boxed{\frac{\partial E}{\partial z} = \frac{A}{E}(1+z)^2 - \frac{B}{H_0}(1+z)^{\pm\frac{1}{2}},} \tag{4.18}$$

where the constants are such that $A = \frac{3}{2}\Omega_{m0}$ and $B = 12\pi G\zeta_0 T_0^{\alpha}$. Again; $\alpha = \frac{1}{2}$ or $\frac{3}{2}$.

This is the equation that was solved numerically for $E(z)$ in Wang and Mengs article.

4.3.2 Method

In the present work, 4th order Runge-Kutta method was used to solve (4.18) numerically. the Matlab was used for the implementation, and the code is appended as appendix E.

4.3.3 Results

Figure 4.1 presents the numerical solutions of equation (4.18) plotted against numerical data sets found in table 1 in Wang and Mengs paper (Wang and Meng, 2014). Different colors in the plots suggest different sources for the data. All the sources are referenced in Wang and Meng's paper, and they are the same as those listed in table 4.1 except for the last data point at $z = 2.36$[2]

[2]This measurement is not included in the compiled list found in (Farooq and Ratra, 2013) and will not be carried over to the rest of the thesis, since it is important that all measurements be independent.

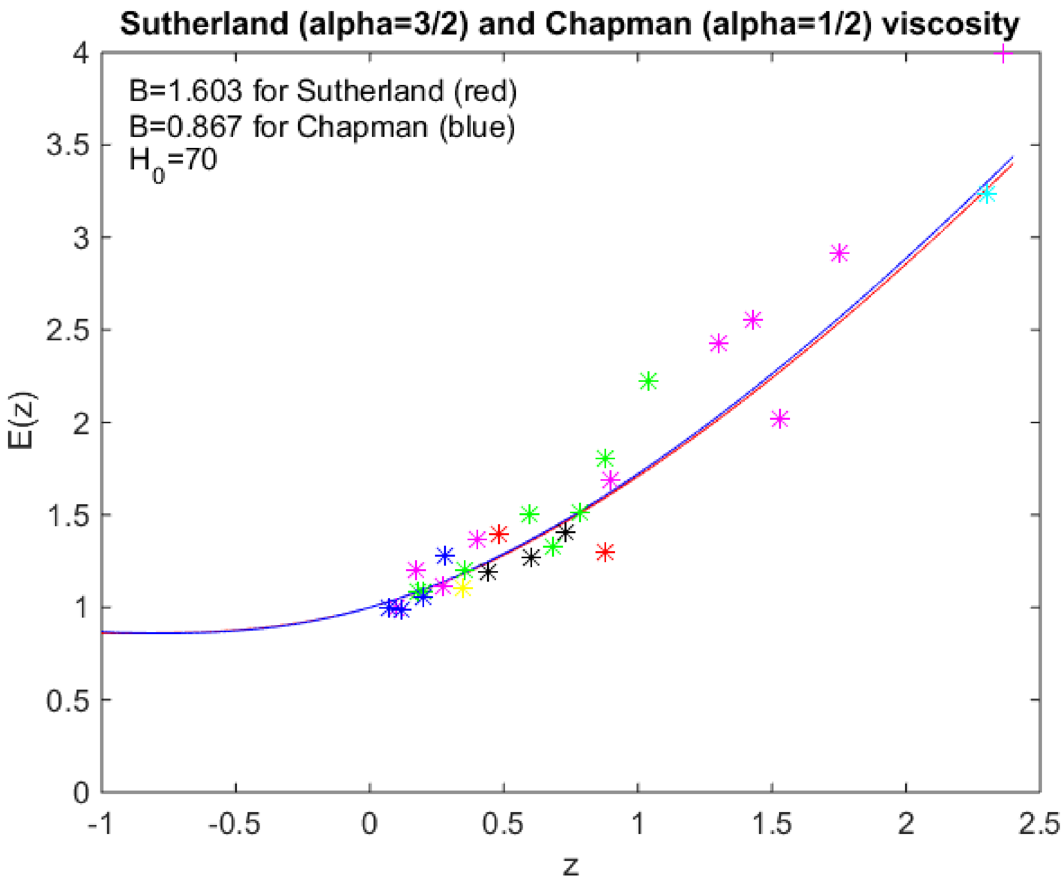

Figure 4.1: Reproduced plot from (Wang and Meng, 2014), showing the numerical solution of equation (4.18) for $E(z)$ with Chapman and Sutherland ($\alpha = 1/2$ and $\alpha = 3/2$, respectively) viscosities, plotted against data sets obtained from observations. Different colors suggest different data sets.

4.4 General solutions for redshift-dependent viscosity

In the following, general expressions for $E(z)$ are sought when viscosity is taken into account. First the energy equation, (2.161) (restated below), is solved for $\rho(a,\zeta)$. Thereafter this solution is used in the Friedmann equations. Where appropriate, comparison with the solutions found by (Wang and Meng, 2014) will be performed. Before starting out, note the definition

$$B \equiv 12\pi G\zeta_0, \tag{4.19}$$

where ζ_0 is the viscosity at present, and natural units are being used as before (divide by $1/c^2$ to convert). This definition will be used throughout this chapter. For further reference it is also useful to know the conversion factor between the viscosity-parameter B and the viscosity ζ_0 itself. From the values listed in table 2.4 one finds

$$\zeta_0 = b\cdot 1.16\cdot 10^6 \text{ Pa s}, \tag{4.20}$$

where b here is taken to be the dimensionless numerical value of the parameter B when given in units $\text{kms}^{-1}\text{Mpc}^{-1}$.

Also, from section 2.1.4, recall the definitions

$$\Omega_{0i} \equiv \frac{\rho_{0i}}{\rho_{c0}} \text{ where } \rho_{c0} \equiv H_0^2\frac{3}{8\pi G} \quad \text{and} \quad \Omega_{0k} \equiv -\frac{k}{H_0^2 a^2} \quad \text{and} \quad \Omega_{0\Lambda} \equiv \frac{\Lambda}{3H_0^2}. \tag{4.21}$$

Note that the subscript 0 means that the quantities are calculated at present time (i.e. $a = a_0 = 1$).

4.4.1 Solving the energy equation

The energy equation, without any interaction terms, reads

$$\dot\rho + (\rho + P)\theta = \zeta\theta^2 \qquad \textbf{assumption 0.} \tag{4.22}$$

Rewriting it in terms of the scalar expansion a instead[3], one has

$$a\partial_a\rho(a) + 3(\rho(a) + P) = 3\zeta\theta. \tag{4.23}$$

It is in the following assumed[4] a cosmic fluid with n components

$$\rho = \sum_i^n \rho_i \qquad \textbf{assumption 0'} \tag{4.24}$$

[3] Parametrizing with any monotonically changing scalar field should be OK. One could wonder how to define time in a precise manner in the first place. As Weinberg puts it; "*Choose any one of these* [monotonically decreasing scalar fields], *say a scalar S, and let the time of any event be any definite decreasing function $t(S)$ of the chosen scalar, when and where the event occurs. The coordinates* $\mathbf{x}, t$ *so defined will be called the cosmic standard coordinate system.*" (Weinberg (1972), p 409).

[4] The two assumptions 0 and 0′ appear as quite natural choices. Especially it is hard to think of a replacement for assumption 0′. Thus these two assumptions are denoted as assumptions 0 and 0′. However, as shown in appendix G, assumption 0′ can be relaxed (at least on mathematical grounds). Also assumption 0 can be changed by e.g. adding explicit interaction terms. As it is, however, explicit inclusion of interaction terms is not a pursued part in the work at hand.

for which the equation of state reads

$$P = \sum_i^n \omega_i \rho_i \qquad \textbf{assumption 1.} \tag{4.25}$$

I.e.; each component i contributes linearly to the overall pressure P.[5] This is an assumption that in sections to come will be relaxed. With this assumption, however, the energy equation is easily verified to have the homogeneous solution (i.e. $\zeta = 0$)

$$\rho_H(a) = \sum_i \rho_{0i} a^{-3(\omega_i+1)}, \tag{4.26}$$

where ρ_{0i} are the present densities ($a = a_0 = 1$). Following standard procedure, the next step is to let the general solution be a sum of a homogeneous and a particular one. This gives

$$\rho(a) = \sum_i \rho_H(a)_i + \rho_p(a)_i = \sum_i \rho_H(a)_i \left[1 + u_i(a)\right] = \sum_i \rho_{0i} a^{-3(\omega_i+1)} \left[1 + u_i(a)\right], \tag{4.27}$$

where $u_i(a)$ are functions to be determined by substituting equation (4.27) for ρ in the energy equation (4.23). Doing so, one finds the differential equation

$$\sum_i \rho_{Hi} \partial_a u_i = 3\frac{\zeta(a)}{a}\theta. \tag{4.28}$$

To continue, one may insert for $\theta = 3H$ from the first Friedmann equation (2.168), which includes the cosmological constant into the solution;

$$\sum_i \rho_{Hi} \partial_a u_i(a) = 9\frac{\zeta(a)}{a}\sqrt{\frac{8\pi G}{3}\sum_i \rho_i \left[1 + u_j(a)\right] - \frac{k}{a^2} + \frac{\Lambda}{3}}. \tag{4.29}$$

Now; this equation is not particularly illuminating. One could of course attempt at solving for one component $u_i(a)$, but since the equation is non-linear in ρ, the superposition principle cannot be used to find the solution for a multi-component fluid $\rho(a)$. Since it is so far not known whether the viscosity components are independent or not, and since the cosmic fluid is treated on a phenomenological level, it seems natural to simplify such as to require one function $u(a)$ for all the components. In this way, the non-linearity of (4.29) in ρ is avoided. Equation (4.27) now becomes

$$\rho(a) = \sum_i \rho_{0i} a^{-3(\omega_i+1)} \left[1 + u(a)\right] \qquad \textbf{assumption 2} \tag{4.30}$$

and equation (4.29) can be rewritten to solve for $u(a)$;

$$\boxed{\partial_a u(a) = 9\frac{\zeta(a)}{a\sum_i \rho_{Hi}}\sqrt{\frac{8\pi G}{3}\sum_i \rho_{Hi}\left[1 + u(a)\right] - \frac{k}{a^2} + \frac{\Lambda}{3}}.} \tag{4.31}$$

The equation is boxed, since it contains the most general formulation of the problem at hand, after having posed the two preceding assumptions. Since the ρ_is are functions of a, it should in principle be solvable if k and Λ are known. But since the equation is non-linear in $u(a)$, general analytic solutions will not be sought. In the following, a special case solution is found.

[5] the upper limit n in the sums will in the continuation be suppressed to simplify notation.

The case k=0

Since observations seem to suggest that the curvature parameter k is quite close to 0, (refer back to table 2.3), it is in the present work seen as sufficiently general to set

$$k = 0 \qquad \textbf{assumption 3}. \tag{4.32}$$

Now, imagine for a moment that the only component in the cosmological fluid ρ is a cosmological constant ($\rho \to \rho_\Lambda$) obeying

$$P = -\rho_\Lambda \qquad \textbf{assumption 4}. \tag{4.33}$$

Then, since $\rho_\Lambda = \text{const}$, the energy equation reduces to

$$\zeta_\Lambda \theta^2 = \zeta_\Lambda \left(\frac{-k}{a^2} + \frac{\Lambda}{3} \right) = 0, \tag{4.34}$$

where the first Friedmann equation was used in the last equality. This leaves us with two options; either the parenthesis equals zero, such that $\Lambda = 3k/a^2$, or $\zeta = 0$. Imposing $k = 0$, as is done in this section, one is left only with the last option; $\zeta_\Lambda = 0$. As a digression, this seems to suggest that

In flat space a cosmological fluid entirely consisting of a cosmological constant (i.e.; $\omega = -1$), cannot be viscous.

This however, as far as is understood, does not imply that Λ cannot affect the viscosity through interacting with other fluid components in a multicomponent cosmological fluid. To actually solve (4.31), the cosmological constant is incorporated in $\rho(a)$, by defining

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} \tag{4.35}$$

and incorporating this component in the sum. With $k = 0$, equation (4.31) reduces to

$$\partial_a u(a) = 9 \frac{\zeta(a)}{a \sum_i \rho_{Hi}} \sqrt{\frac{8\pi G}{3} \sum_i \rho_{Hi} \left[1 + u(a)\right]}. \tag{4.36}$$

This equation is easy to solve, and the solution is

$$u(a) = \left[\frac{9}{2} \sqrt{\frac{8\pi G}{3}} \int \frac{\zeta(a)}{a\sqrt{\rho_H}} da + C_0 \right]^2 - 1, \tag{4.37}$$

where $\rho_H \equiv \sum_i \rho_{Hi}$. From equation (4.27), this gives

$$\boxed{\rho(a, \zeta(a)) = \sum_i \rho_{0i} a^{-3(\omega_i + 1)} \left[\frac{9}{2} \sqrt{\frac{8\pi G}{3}} \int \frac{\zeta(a)}{a\sqrt{\rho_H}} da + 1 \right]^2,} \tag{4.38}$$

where $C_0 = 1$ was determined from $\rho(a_0, \zeta = 0) = \sum_i \rho_{0i}$. Again recall that this equation is **valid only for** $k = 0$. Also, if $\omega_\Lambda = -1$, it requires $\zeta_\Lambda = 0$ in order to be a solution of the energy equation (4.23).

4.4.2 Obtaining general expressions for E(z)

In order to say anything about how $E(z)$ evolves, and by such be in position to compare with measurements, (Wang and Meng, 2014) uses the second of the Friedmann equations, which contains a pressure term, and therefore will be affected by the inclusion of viscosity ($P \to P - \zeta\theta$). Slightly rearranged equation (2.168b) reads

$$\dot{H} = -\frac{3}{2}H^2 - 4\pi G(P - \zeta\theta) - \frac{k}{2a^2} + \frac{\Lambda}{2}. \tag{4.39}$$

As displayed in the last section, Wang and Meng integrated this equation numerically. In the more general case, however, it seems unnecessary to invoke the second Friedman equation at all, since when calculated more carefully, the viscosity modification appears in the expression for ρ, equation (4.38), and thus it is already included[6]. In the following general scenario, therefore, the first Friedman equation will be used instead. This makes the calculations easier, since no numerics need to be involved. Restating equation (2.168a) it reads

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}. \tag{4.40}$$

Since the dimensionless $E(z)$ is the aim, equation (4.40) is rewritten by use of the dimensionless quantities found in section 2.1.4 and restated in terms of present day values in (4.21). Taken together with assumption 2; $\rho(a) = \sum_i \rho_{0i} a^{-3(\omega_i+1)}\left[1 + u(a)\right],$ the first Friedman equation now reads

$$E^2(a) = \frac{H^2(a)}{H_0^2} = \Omega_{0k}a^{-2} + \Omega_{0\Lambda} + \sum_i \Omega_{0i} a^{-3(\omega_i+1)}\left[1 + u(a)\right], \tag{4.41}$$

where $u(a)$ is given by equation (4.37). It is useful to rewrite this equation in terms of an observable parameter. since $1/(1+z) = a$, one also has

$$\boxed{E^2(z) = \frac{H^2(z)}{H_0^2} = \Omega_{0k}(1+z)^2 + \Omega_{0\Lambda} + \sum_i \Omega_{0i}(1+z)^{3(\omega_i+1)}\left[1 + u(z)\right].} \tag{4.42}$$

In particular; note that equation (4.11), which was used in (Wang and Meng, 2014) is recovered in the limit $k = \Lambda = \zeta = 0$. Letting $\zeta \to 0$ is the same as letting the entropy production in the energy equation, equation (4.23), vanish. Expression (4.38) reduces to the homogenous solution given by equation (4.26). This is precisely as it should be, and it emphasizes that the particular solution is needed in order to account for the viscosity in a proper manner. Finally note that the integral in the expression for $u(z)$ (4.37) now needs to be evaluated over z. The integral becomes

$$u(z) = \left[1 - \frac{9}{2}\sqrt{\frac{8\pi G}{3}}\int \frac{\zeta(z)}{(1+z)\sqrt{\rho_H}}dz\right]^2 - 1, \tag{4.43}$$

where $\rho_H \equiv \sum_i \rho_{Hi}$.

[6] No information seems to be lost by omitting the second Friedmann equation, since the first Friedmann equation is used together with the energy equation, and these are not independent of the second Friedmann equation.

4.5 Solutions with constant bulk viscosity

The simplest non-trivial scenario of bulk-viscosity is that of a constant coefficient ζ =const. In this section the above developed formalism will be implemented with a constant coefficient, seeking constraints on its size upon comparison with observations, as was done under other assumptions in Wang and Meng (2014). Also, as before; the curvature parameter k is set to zero: $k = 0$.

4.5.1 Matter domination in flat space

In table 2.3 one finds Hubble parameter measurements back to redshift ~ 2.3. According to table 4.1 this stretches deep into the matter dominated epoch. Later, at redshift $z = 0.25$ dark energy became the main constituent. All in all, therefore, it seems natural as a first approach to assume a one-component dust universe ($\omega = 0$) when using the above given formalism. Dust because the matter has been more or less pressure-less since its time of domination[7]. The cosmological constant, however, which at present is believed to constitute about 70% of the overall energy content of the universe, is kept. Treating the cosmological constant as a component in the fluid, one all in all ends up with a **two-component fluid** consisting of matter[8] and dark energy. Letting $k \to 0$ and $\rho(z) \to \rho_m(z) + \rho_\Lambda$ equation (4.42) reduces to

$$E^2(z) = \Omega_{\Lambda 0} + \Omega_{m0}(1+z)^{3(\omega_m+1)}\left[1 + u(z)\right], \tag{4.44}$$

where subscript m denotes matter. In the following it is assumed that the matter takes the form of dust; i.e. $\omega = 0$. $u(z)$ is given by equation (4.43), which when inserted into the above equation gives

$$E^2(z) = \Omega_{\Lambda 0} + \Omega_{m0}(1+z)^{3(\omega_m+1)}\left[1 - \frac{9}{2}\sqrt{\frac{8\pi G}{3}}\int \frac{\zeta}{(1+z)\sqrt{\rho_H}}dz\right]^2, \tag{4.45}$$

where now $\rho_H = \rho_\Lambda + \rho_{m0}(1+z)^3$ (equation (4.26)).

Note for future reference that this equation by the standard definitions can be rewritten to

$$E(a) = \sqrt{\Omega}\left[1 + \frac{B}{H_0}\int \frac{1}{a\sqrt{\Omega}}da\right], \tag{4.46}$$

where here $\Omega \equiv \sum_i \Omega_i a^{-3(\omega_i+1)}$.

Returning to our present business; rewriting equation (4.45) in terms of relative densities and calculating the integral by use of Wolfram Alpha gives the solution

$$\begin{aligned} E(z) = &\sqrt{\Omega_{\Lambda 0} + \Omega_{m0}(1+z)^3} \\ &\cdot\left[1 + \frac{2B}{3H_0\sqrt{\Omega_{\Lambda 0}}}\text{arctanh}\left(\sqrt{\frac{\Omega_{\Lambda 0} + \Omega_{m0}(1+z)^3}{\Omega_{\Lambda 0}}}\right) + I_0\right], \end{aligned} \tag{4.47}$$

[7] According to (Grøn and Hervik, 2007), p 279; "*The transition from a radiation to a dust dominated model, is believe to have happened around $t = 44000 years$. Since this time, the dynamics of the universe has been driven by matter and vacuum energy.*"

[8] Matter is here meant to refer to the combination of dark matter and ordinary matter.

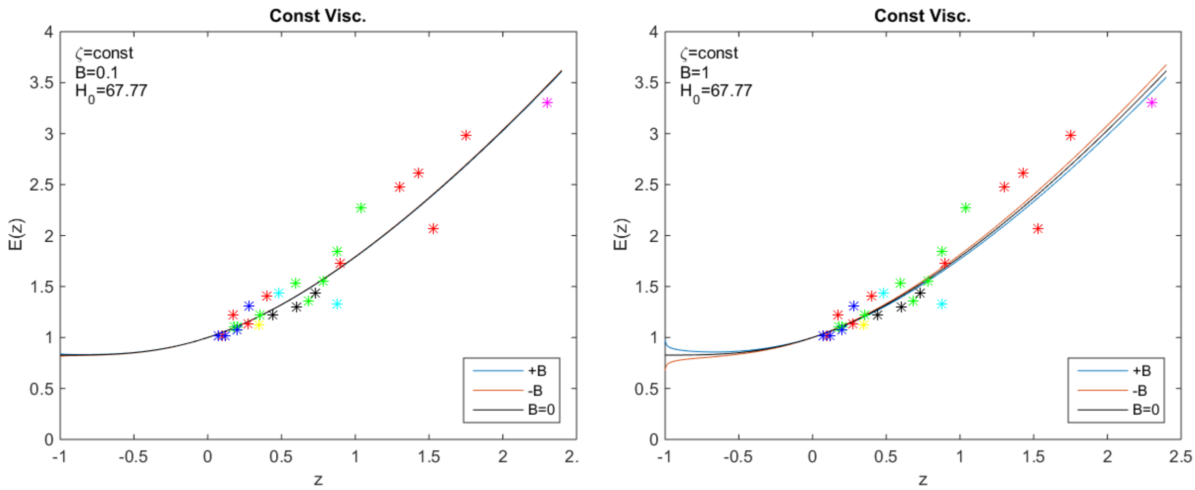

Figure 4.2: Solution of Friedmanns 1. equation for $E(z)$ with constant bulk viscosity ζ. The B-values are $B = \pm 0.1$ kms^{-1}Mpc^{-1} (left) and $B = \pm 1$ kms^{-1}Mpc^{-1} (right). The blue curve corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity.

where the parameter $B = 12\pi G\zeta_0$ as always. The integration constant I_0 is determined from the condition that $E(z=0) = 1$;

$$I_0 = \frac{1}{\sqrt{\Omega_{\Lambda 0} + \Omega_{m0}}} - 1 - \frac{2B}{\sqrt{3\Omega_{\Lambda 0}}H_0} \operatorname{arctanh}\left(\sqrt{\frac{\Omega_{m0}}{\Omega_{\Lambda 0}} + 1}\right). \tag{4.48}$$

The above result for a constant bulk viscosity was implemented and plotted against data by use of Matlab. The code used is appended in Appendix F. Figure 4.2 show the results with parameter choices $B = 0.1$ kms^{-1}Mpc^{-1} (left) and $B = 1$ kms^{-1}Mpc^{-1} (right). The data used for the Hubble parameter as a function of redshift is that which is listed in table 2.1. Also, from table 2.2 and 2.3 the best-fit values

$$H_0 = 67.77 \quad , \quad \Omega_{0m} = 0.3183 \quad \text{and} \quad \Omega_\Lambda = 0.6914$$

are adapted.

The two plots in figure 4.2 give solutions of the equations with both positive and negative viscosities. Note that a constant negative viscosity might seem to cause a negative singularity of $E(z)$ as $z \to -1$. In this chapter, however, only positive values for z are considered; values that correspond to the past and for which measurements therefore exist. From the two figures it looks as if the parameter choice $B = 0.1$kms^{-1}Mpc^{-1} coincides so well with the scenario of no viscosity that one by eye inspection hardly can tell them apart.

It seems, therefore, as if the parameter choice $B = 1$ kms^{-1}Mpc^{-1} corresponds to an acceptable estimate for the upper bound on ζ as far as the Hubble parameter measurements are concerned. To leading order of magnitude one then finds from equation (4.20), with $b = 1$, that a viscosity within the boundaries

$$\boxed{|\zeta_0| \leq 10^6 \text{ Pa s}} \tag{4.49}$$

will not change the predictions of the Hubble parameter evolution much. The numerical pre-factor was cancelled in order to give a more correct picture of the accuracy.[9]The attentive reader might not agree that the constraints on ζ_0 be wrapped up in absolute signs such as to include negative viscosity. But forgetting about thermodynamics for a moment, just trying to fix the value such that it best represents the observational data, one has to admit negative solutions as well as positive. Also, from eye sight it seems adding negative viscosity would correspond better with observations than adding the same positive amount. As it is, however, careful statistical analysis should be performed in order to conclude anything on this point.

4.6 General solutions with density dependent viscosity

In the last section, a constant bulk viscosity was assumed. In this section, following e.g. (Brevik and Gorbunova, 2005) and (Brevik I. and Timoshkin, 2014), a viscosity depending on the density ρ will be assumed. In particular, ζ is given the general form

$$\zeta(\rho) = \zeta_0 \left(\frac{\rho}{\rho_0}\right)^{\lambda}. \tag{4.50}$$

Assuming a viscosity on this form, actually throws the formalism all the way back to level zero, since the variable on which ζ now depends is different, which in turn will lead to different solutions of the energy equation. Starting from Friedmann's first equation and the energy equation again (as stated below), general solutions for $E(z)$ are found.

The energy equation (4.23) with the ansatz (4.50), reads

$$a\partial_a\rho + 3(\rho + P) = 3\zeta_0 \left(\frac{\rho}{\rho_0}\right)^{\lambda} \theta. \tag{4.51}$$

Note that ρ and P are functions of a. In particular, assumptions 1 and 2 from section 4.4 are adopted;

$$P = \sum_i^n \omega_i \rho_i. \qquad \textbf{assumption 1} \tag{4.52}$$

and

$$\rho(a) = \sum_i \rho_{0i} a^{-3(\omega_i+1)} \left[1 + u(a)\right] \qquad \textbf{assumption 2.} \tag{4.53}$$

Using these assumptions, and also rewriting $\theta = 3H$, where H as always is the Hubble parameter, one obtains, by using Friedmann's first equation (4.40), a differential equation for $u(a)$, just like in section 4.4;

$$\sum_i \rho_{Hi}\partial_a u = \frac{9\zeta_0}{a\rho_0^{\lambda}}\sqrt{\frac{8\pi G}{3}} \left\{\sum_i \rho_{Hi}(1+u)\right\}^{\lambda+1/2}, \tag{4.54}$$

where u as before is taken to be a function of a. Again $k = 0$ has been assumed. This equation is separable in $u(a)$;

[9]The uncertainty in the Hubble parameter measurements prevents this inequality from being absolute. To wit; the viscosity might be bigger. The point here, however, is to give a plausible estimate. At least plausible in the sense that the predictions made by adding no viscosity are not violated noteworthy if the viscosity is contained within the given boundaries.

$$\frac{1}{(1+u)^{\lambda+1/2}}\frac{du}{da} = \frac{9\zeta_0}{a\rho_0^\lambda}\sqrt{\frac{8\pi G}{3}}\left\{\sum_i \rho_{Hi}\right\}^{\lambda-1/2}, \tag{4.55}$$

and the solution is by straight forward solving found to be

$$u(a) = \begin{cases} \left\{9\sqrt{\frac{8\pi G}{3}}\frac{\zeta_0}{\rho_0^\lambda}\int\frac{\rho_H^{\lambda-1/2}}{a}da - I_0\right\}^{\frac{1}{1/2-\lambda}}(1/2-\lambda)^{\frac{1}{1/2-\lambda}} - 1 & \text{for} \lambda \neq \frac{1}{2} \\ I_0 a^{\left(9\sqrt{\frac{8\pi G}{3}}\frac{\zeta_0}{\rho_0^{1/2}}\right)} - 1 & \text{for} \lambda = \frac{1}{2}. \end{cases} \tag{4.56}$$

For $\lambda = 1/2$ the equation is rather messy. Tidying a bit, and inserting into *assumption 2*, (equation (4.53)), gives

$$\rho(a,\zeta_0) = \begin{cases} \rho_H\left\{9\left(\frac{1}{2}-\lambda\right)\sqrt{\frac{8\pi G}{3}}\frac{\zeta_0}{\rho_0^\lambda}\int\frac{\rho_H^{\lambda-1/2}}{a}da + 1\right\}^{\frac{1}{1/2-\lambda}} & \text{for} \lambda \neq \frac{1}{2} \\ a^{\left(9\sqrt{\frac{8\pi G}{3}}\frac{\zeta_0}{\rho_0^{1/2}}\right)}\left(\sum_i \rho_{0i}a^{-3(1+\omega_i)}\right) & \text{for} \lambda = \frac{1}{2} \end{cases} \tag{4.57}$$

As before, $\rho_H \equiv \sum_i \rho_{Hi}$. The initial condition; $\rho(a = a_0 = 1) = \sum_i \rho_{Hi}$ was used to determine the integration constant I_0. The next step is to insert the above expressions for ρ into Friedmanns first equation. This gives, when properly rewritten in terms of relative densities, that

$$\boxed{E(a) = \begin{cases} \sqrt{\Omega}\left\{(1-2\lambda)\frac{B}{H_0}\int\frac{1}{a\sqrt{\Omega}^{1-2\lambda}}da + 1\right\}^{\frac{1}{1-2\lambda}} & \text{for} \lambda \neq \frac{1}{2}, \\ a^{\left(\frac{9}{2}\sqrt{\frac{8\pi G}{3}}\frac{\zeta_0}{\rho_0^{1/2}}\right)}\cdot\sqrt{\Omega} & \text{for} \lambda = \frac{1}{2}, \end{cases}} \tag{4.58}$$

where $B = 12\pi G\zeta_0$ and, for brevity,

$$\Omega \equiv \sum_i \Omega_{0i}a^{-3(1+\omega_i)}. \tag{4.59}$$

Also, it has been used that $\rho_0 = \rho_{0c}$ when $k = 0$.

The general solution of equation (4.58) will not be sought here. Rather, the three most interesting special cases are considered.

4.6.1 Special case 1; constant viscosity

In this case, the constant viscosity-case treated previously is recovered with $\zeta = \zeta_0$. Equation (4.58) reduces to

$$E(a) = \sqrt{\Omega}\left\{1 + \frac{B}{H_0}\int\frac{1}{a\sqrt{\Omega}}da\right\}, \tag{4.60}$$

which is the same as was obtained in the constant viscosity case, eq. (4.46).

4.6.2 Special case 2; viscosity proportional to H

This case is treated in detail in Brevik and Gorbunova (2005), a paper in which the case $\zeta \propto H$, is seen as a natural assumption. It seems instructive to have a closer look at this case. Going back to the energy equation one has with $\lambda = 1/2$ and 1st Friedman equation ($k = 0$)

$$a\partial_a\rho + 3(\rho + P) = 3\zeta_0\left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}} \cdot 3\sqrt{\frac{8\pi G}{3}}\sqrt{\rho} = \frac{9\zeta_0}{\rho_0^{1/2}}\sqrt{\frac{8\pi G}{3}}\rho \equiv -3\omega_\zeta\rho, \tag{4.61}$$

where

$$-3\omega_\zeta \equiv \frac{9\zeta_0}{\rho_0^{1/2}}\sqrt{\frac{8\pi G}{3}}.$$

By the definition $B \equiv 12\pi G$ one finds

$$\omega_\zeta = -\frac{2}{3}\frac{B}{H_0}. \tag{4.62}$$

Again invoking $P = \sum_i \omega_i\rho_i$ gives

$$\sum_i a\partial_a\rho_i + 3(1 + \omega_\zeta + \omega_i)\rho_i = 0. \tag{4.63}$$

Now, this is the same equation that was solved to find the homogeneous solution, except for the constant ω_ζ. The homogeneous solution is by insertion, or by comparison with (4.26) easily verified to be

$$\rho(a) = \sum_i \rho_{0i}a^{-3(\omega_i+\omega_\zeta+1)} \qquad \textbf{(Ansatz)}, \tag{4.64}$$

where ρ_{0i} as before are the present densities ($a = a_0 = 1$). This energy density will by Friedmann's first equation lead to

$$E^2(a) = \sum_i \Omega_{0i}a^{-3(1+\omega_\zeta+\omega_i)} \qquad . \tag{4.65}$$

If one defines

$$\Omega_\zeta \equiv \sum_i \Omega_{0i}a^{-3(1+\omega_\zeta+\omega_i)}$$

one has the same form as before;

$$E(a) = \sqrt{\Omega_\zeta}. \tag{4.66}$$

This solution is in agreement with the solution just found in (4.58) for $\lambda = 1/2$, just like it ought be. However, the derivation just performed shows more explicitly that the simple form of the solution is due to the fact that the RHS of the energy equation $3\zeta\theta$ gets the same dependence on ρ as does the bracket on the LHS; $3(\omega_i + 1 + \omega_\zeta)\rho$. This allows for bringing the viscosity-term over to the LHS, interpreting it on the same level as ω_i, except that it is constant for all is. This solution has some interesting interpretations, partly investigated in Appendix G.

Obtaining upper bounds

As before ρ is simplified such that $\rho \to \rho_m + \rho_\Lambda$. With (4.66) one then finds

$$E(z) = \sqrt{\Omega_{0m}(1+z)^{3(1+\omega_\zeta)} + \Omega_\Lambda(1+z)^{3\omega_\zeta}}. \tag{4.67}$$

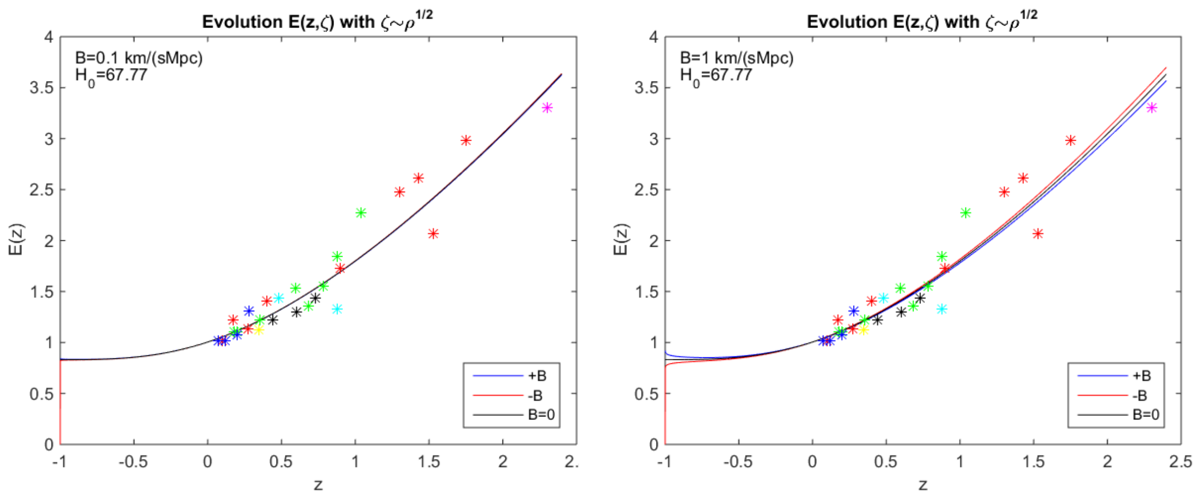

Figure 4.3: Solution of Friedmanns 1st equation for $E(z,\zeta)$ with $\zeta \propto \rho^{1/2}$. The B-values are $B = \pm 0.1$ km/sMpc (left) and $B = \pm 1$ km/sMpc (right). The blue curve corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity.

Figure 4.3 shows the results with $B = \pm 0.1\,\mathrm{kms^{-1}Mpc^{-1}}$ (left) and $B = \pm 1\,\mathrm{kms^{-1}Mpc^{-1}}$ (right). From eye inspection of the graphs, it seems like no exaggeration to say that $B = 0.1\,\mathrm{kms^{-1}Mpc^{-1}}$ is more or less undistinguishable from the case of no viscosity. From equation (4.20) one then finds (with $b = 1$), to leading order,

$$\boxed{|\zeta_0| \leq 10^6 \text{ Pa s},} \tag{4.68}$$

which is the same result as was found with constant viscosity. As mentioned; this should by no means be interpreted as strict boundaries, but rather as a suggestion based on how large viscosity can be included before the predictions made by the viscosity-less model are violated significantly[10].

Alternative ansatz

A second ansatz also seemed worth mentioning. The homogeneous energy equation is linear in ρ_i (which indeed is the very reason why also $\rho = \sum_i \rho_i$ is a solution), and so it is tempting to try an ansatz in which ρ is redefined as to incorporate a new fluid component, ρ_ζ. The solution would be

$$\rho_H(a) = \sum_j \rho_{0j} a^{-3(\omega_j+1)} \qquad \textbf{(Alternative ansatz)}, \tag{4.69}$$

where the sum now includes ω_ζ as well. That is to say; the viscosity is considered a fluid on its own. The question then consists of whether ω_ζ can be chosen (by picking a suitable ζ) such that this ansatz solves the energy equation. Inserting the ansatz, equation (4.69) into equation (4.63), one finds

[10]'significantly' is obviously not a well defined term here, but it seems to be hard to be more precise without performing any statistical analysis and comparing with uncertainties in the measurements. Unfortunately there has been found place for no such calculations in the present work.

$$\sum_j a\partial_a\left(\rho_{0j}a^{-3(\omega_j+1)}\right)+3(1+\omega_j)\rho_{0j}a^{-3(\omega_j+1)}+3\omega_\zeta\rho_{0j}a^{-3(\omega_j+1)}=0. \tag{4.70}$$

The first two brackets cancel each other out, and one is left with

$$3\omega_\zeta\left(\sum_i \rho_{0j}a^{-3(\omega_j+1)}\right)=0. \tag{4.71}$$

Not pursuing the path of negative energy densities[11], this equation is fulfilled if and only if

$$\omega_\zeta=0$$

.

The interpretation that the viscosity takes a fluid description on its own seems to therefore not be available to us.

4.6.3 Special case 3; viscosity proportional to H^2

In this case, equation (4.58) reduces to

$$E(a)=\sqrt{\Omega}\left\{1-\frac{B}{H_0}\int\frac{\sqrt{\Omega}}{a}da\right\}^{-1}. \tag{4.72}$$

Changing variables to ζ and using Wolphram Alpha to compute the integral, the solution becomes

$$E(z)=\sqrt{\Omega}\left\{\sqrt{\Omega_0}+\frac{2B}{3H_0}\left[\sqrt{\Omega}-\sqrt{\Omega_{0\Lambda}}\,\mathrm{arctanh}\sqrt{\frac{\Omega}{\Omega_{0\Lambda}}}\right]\right\}^{-1}, \tag{4.73}$$

where the initial condition $E(z=B=0)=1$ was used.

Figure 4.4 shows the results. The B-values are $B=\pm0.1$ km/sMpc (left) and $B=\pm1$ km/sMpc (right). The blue curve corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity.

[11] Even if doing so, the requirement that here had to be made, would be that the total density is zero, and the energy equation would reduce to adding up partial pressures to zero; $\sum_j P_j=0$, which seems to correspond to no interesting scenario.

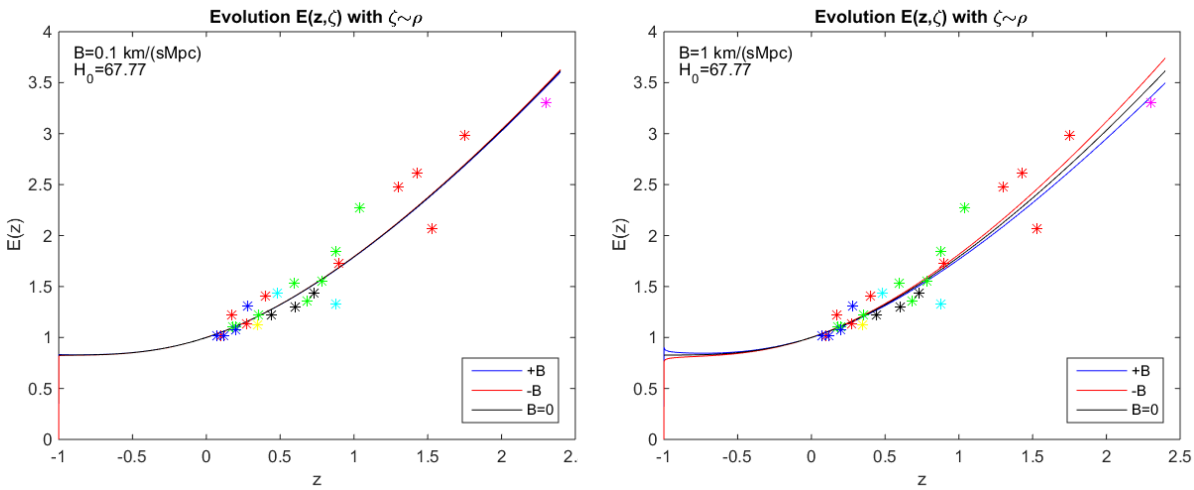

Figure 4.4: Solutions of Friedmans 1st equation for $E(z,\zeta)$ with $\zeta \propto \rho$. The B-values are $B = \pm 0.1$ km/sMpc (left) and $B = \pm 1$ km/sMpc (right). The blue curve corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity.

The conclusion for the two-component fluid is so far, when looking at the graphs in figures 4.2, 4.3 and 4.4 that a viscosity corresponding to $B = \pm 1$ km/sMpc seems to be allowed by all the three scenarios $\zeta = \zeta_0$, $\zeta \sim \rho^{1/2}$ and $\zeta \sim \rho$, although the dependence on the parameter B at high redshifts seems to be more and more sensitive as the power of ρ increases. Also, if any of the curves are to be preferred, that would be the red curves; corresponding to negative viscosity. For instance, a positive $B = 1$ km/sMpc seems to just barely be allowed for $\zeta \sim \rho$.

4.7 Solutions with temperature dependent viscosity

An another perhaps intuitively compelling choice, is to choose the thermodynamical variable T as the variable for ζ. As has been seen, this is in agreement with (Weinberg, 1971), (Wang and Meng, 2014) and also others, like (Hoogeveen, 1986), (Misner, 1967), (Brevik and Heen, 1994). As shown in the last chapter, (Hänel, 2004) has derived temperature dependent viscosity coefficients on kinetic theoretical grounds (3.111) as well. In order to actually solve equation (4.42) and obtain an expression for the evolution of the Hubble parameter, a relation between the temperature T and the scale factor a (and thus the redshift z), is needed. To this end the Weinberg formalism ((Weinberg, 1971)) layed out in section 2.2 and 2.3 comes in handy. Doing as is partly done in (Brevik I., 1996), and more carefully treated in a personal note from Brevik to undersigned, our starting point is the thermodynamic identity, equation (2.48), in dimensionless form;

$$kTd\sigma = d\left(\frac{\rho}{n}\right) + pd\left(\frac{1}{n}\right) = \frac{1}{n}\left[d\rho - \frac{\rho+p}{n}dn\right]. \tag{4.74}$$

Now assuming that the density ρ is a function of the number density n and the temperature T, as independent variables, one finds for its derivative $\dot{\rho}(n,T)$

$$\dot{\rho} = \left(\frac{\partial\rho}{\partial n}\right)_T \dot{n} + \left(\frac{\partial\rho}{\partial T}\right)_n \dot{T}. \tag{4.75}$$

This allows for rewriting equation (4.74) to

$$k\dot{\sigma} = \frac{1}{nT}\left\{\left[\left(\frac{\partial\rho}{\partial n}\right)_T \dot{n} + \left(\frac{\partial\rho}{\partial T}\right)_n \dot{T}\right]\dot{n} + \left(\frac{\partial\rho}{\partial T}\right)_n \dot{T}\right\}. \tag{4.76}$$

Also, equation (4.75), slightly rearranged, and by invoking the energy equation, can be rewritten to

$$\dot{T} = \frac{1}{(\partial\rho/\partial T)_n}\left[\dot{\rho} - \left(\frac{\partial\rho}{\partial n}\right)_T \dot{n}\right] \overset{(4.23)}{=} \frac{1}{(\partial\rho/\partial T)_n}\left[-\rho - P + \zeta\theta + \left(\frac{\partial\rho}{\partial n}\right)_T \theta\right], \tag{4.77}$$

where $\theta = -\dot{n}/n$ has been utilized[12].

Since equation (4.74) is an absolute differential,one has the relation (2.103)

$$\frac{\partial}{\partial T}\left[\frac{1}{nT}\left(\frac{\partial\rho}{\partial n}\right)_T - \frac{\rho+P}{n}\right]_n = \frac{\partial}{\partial n}\left[\frac{1}{nT}\left(\frac{\partial\rho}{\partial T}\right)_n\right]_T. \tag{4.78}$$

An actual calculation of the LHS and RHS of the above equation yields

$$T\left(\frac{\partial P}{\partial T}\right)_n = \rho + P - n\left(\frac{\partial\rho}{\partial n}\right)_T. \tag{4.79}$$

Inserting this into equation (4.77) gives, with a hint of aesthetic make over, that

$$\frac{\dot{T}}{T} = -\left(\frac{\partial P}{\partial\rho}\right)_n \theta + \frac{\zeta\theta^2}{T(\partial\rho/\partial T)_n}, \tag{4.80}$$

which is equation 10 in (Brevik I., 1996). Also, note that this is the same relation as is given in equation (2.115), except for the last term incorporating the entropy production (through bulk viscosity).
Here comes a crucial step: assuming as before $P = \omega\rho$ the first term on the RHS reduces to

$$-\left(\frac{\partial P}{\partial\rho}\right)_n \theta = 3\omega\frac{\dot{a}}{a}.$$

Now exploiting the beauty of logarithms, equation (4.80) becomes

$$\boxed{\frac{d}{dt}\ln\left(Ta^{3\omega}\right) = \frac{\zeta\theta^2}{T(\partial\rho/\partial T)_n}} \qquad \text{when } (\ P = \omega\rho\ ). \tag{4.81}$$

At this point one might seem to have hit the wall, since again one has to know ζ. But assuming that the viscosity contribution in general is small, however, which seems to be true in the later stages of the cosmic evolution, it might be a good approximation to set $\zeta = 0$,

[12]From (2.128) one has $\theta = \theta^\mu{}_{;\mu} = U^\mu{}_{;\mu}$ In co-moving coordinates $U^\mu{}_{,\mu} = 0$, and one finds

$$\theta = U^\mu{}_{;\mu} = \Gamma^\mu{}_{0\mu}U^0 = 3\frac{\dot{a}}{a} = 3H$$

For the FRW-metric one finds from $N^\mu{}_{;\mu} = 0$ that $na^3 = \text{const}$, and thus that $\dot{n}/n = 3\dot{a}/a$. All in all this means that

$$\theta = -\frac{\dot{n}}{n}$$

so that the RHS of equation (4.81) vanishes, and it follows that

$$\boxed{T = T_0 a^{-3\omega}} \qquad \text{when } (\,\zeta = 0\,), \tag{4.82}$$

where T_0 is a proportionality constants.

Last Comments on Wang and Meng's Article

At this point one might compare with (Wang and Meng, 2014). Requiring $\omega = 1/3$, which is to assume radiation domination, gives the relation used in their article, equation (4.16);

$$T = T_0 \frac{1}{a}. \tag{4.83}$$

It therefore seems clear that the conditions under which the results in Wang and Meng's paper are valid are:

Conditions for (Wang and Meng, 2014): $\zeta = 0$ and $\omega = 1/3$; Radiation domination.

These might, however, correspond to good approximations, if the viscosity and the temperature change are both sufficiently small.

4.7.1 Obtaining $\zeta(z)$

The equation of state used for the matter content of the universe has so far been $P = 0$; dust, that is ($\omega = 0$). From equation (4.82), it is now clear that this corresponds to constant temperature T;

$$T = T_0 a^{-3\cdot 0} = T_0. \tag{4.84}$$

This means a constant bulk viscosity, for which results are already obtained.

4.8 Effective one-fluid description found in literature

In the previous sections, the equation of state parameter ω has been assumed constant, in agreement with the assumptions of perfect gas with a homogeneous equation of state. In the present section, however, this assumption is relaxed, and a generally density dependent equation of state will be assumed instead. The main reason for this, is that a one-component fluid will be used. Remembering that the dominating fluid component (radiaton, matter, dark energy) is assumed to have changed throughout the history of the universe, it seems natural to require a varying effective ω when looking at the fluid as a whole. Exactly *how* the equation of state parameter would vary, seems hard to know, since the fluid components in general might interact with each other in complicated ways. An assumption is therefore needed for the functional form of $\omega(\rho)$. To this end the literature is sought for help. The model used in this section is discussed in (Brevik, 2013), (Nojiri et al., 2005), (Brevik and Gorbunova, 2005) and (Stefancić, 2005). The model seems to suggest the following general case:

$$\omega(\rho) = -1 - \alpha \left(\frac{\rho}{\rho_0}\right)^{\beta-1} \qquad \text{and} \qquad \zeta(\rho) = \zeta_0 \left(\frac{\theta}{\theta_0}\right)^{2\beta-1}. \tag{4.85}$$

This functional form of $\omega(\rho)$ is meant to be a model of the behaviour of the cosmic fluid in the late stages of the universe, when dark energy is the dominating energy constituent (thus the term -1). As seen from table 4.1 this has been the case since $z = 0.25$. From table 2.1 one finds measurements of the Hubble parameter between $z = 0$ and $z = 0.25$. This ansatz for $\omega(\rho)$ will therefore be sought extended to the period for which observations exist. The question is to what extent one can make the model fit the data set when the viscosity parameter B is chosen to this end.

With these assumptions, the energy equation (4.23) reads

$$a\partial_a \rho - 3\gamma\rho^\beta = 3\tau\theta^{2\beta}, \tag{4.86}$$

where the new constants

$$\tau \equiv \frac{\zeta_0}{\theta_0^{2\beta-1}} \quad \text{and} \quad \gamma \equiv \frac{\alpha}{\rho_0^\beta} \tag{4.87}$$

have been introduced to simplify notation. This equation is quite straight forward to solve. Indeed the form of the viscosity was chosen such as to give a mathematically simple expression (that at the same time seems plausible, explains (Brevik, 2013)).

When $\beta \neq 1$ separation of variables gives the solution as

$$\rho(a) = \left\{\left[3\tau(3\kappa)^\beta + 3\gamma\right]\left(1-\beta\right)\ln(a) + C\right\}^{1/(1-\beta)}, \qquad \beta \neq 1, \tag{4.88}$$

where $\kappa = 8\pi G$ has been used, and C is an integration constant that must be determined by initial conditions. By the first Friedmann equation, which here is written as $\theta^2 = 3\kappa\rho$, the above solution gives the dimensionless Hubble parameter $E(a)$ as

$$E(a) = \sqrt{\frac{\kappa}{3H_0^2}}\left\{\left[3\tau(3\kappa)^\beta + 3\gamma\right]\left(1-\beta\right)\ln(a) + C\right\}^{1/(2-2\beta)}, \qquad \beta \neq 1. \tag{4.89}$$

The integration constant is as before determined from $E(a = a_0 = 1) = 1$ and becomes

$$C = \frac{3H_0^2}{\kappa}^{1-\beta} = \rho_c^{1-\beta}. \tag{4.90}$$

When $\beta = 1$ the solution is even simpler. One finds in the same way

$$E(a) = \sqrt{a^{3\tau(3\kappa)^\beta + 3\gamma}}. \tag{4.91}$$

The two most useful cases seem to be $\beta = 3/2$ and $\beta = 1$. From equation (4.85) one sees that $\beta = 1$ corresponds to the previously discussed case of a constant equation of state parameter ω and a viscosity proportional to $\sqrt{\rho}$. Similarly, by inserting into the energy equation and rearranging, it can also be shown that choosing $\beta = 3/2$ corresponds to $\omega(\rho) \sim \sqrt{\rho}$ and $\zeta \sim \rho$.

4.8.1 Selecting α and plotting

In the following, both $\beta = 1/2$ and $\beta = 3/2$ has been implemented in Matlab. As always rewriting in terms of redshift z, relative densities and the viscosity parameter B, the two following equations are plotted:

$$E(z) = \begin{cases} (1+z)^{-\frac{B}{H_0}-\frac{3}{2}\alpha} & \text{for} \quad \beta = 1, \\ \sqrt{\Omega_0}\left\{\left[\frac{2B}{H_0}+3\alpha\right]\ln(\sqrt{1+z})+\sqrt{\Omega_0}\right\}^{-1} & \text{for} \quad \beta = 3/2. \end{cases} \tag{4.92}$$

The definitions in (4.87) have also been inserted.

Figure (4.5) show the solutions of Friedmans 1st equation for $E(z,\zeta)$ for a dark energy fluid ($\alpha = 0$) with $\zeta \sim \rho$ (left figure) and $\zeta \sim \rho^{1/2}$ (right figure). B-values $B = -35$ km/sMpc (left) and $B = -65$ km/ sMpc (right) seem to correspond well with the data. As before the blue curve corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity. The solution to the left seem to fit the data far better than the solution to the right, which really cannot explain the whole range of measurements.

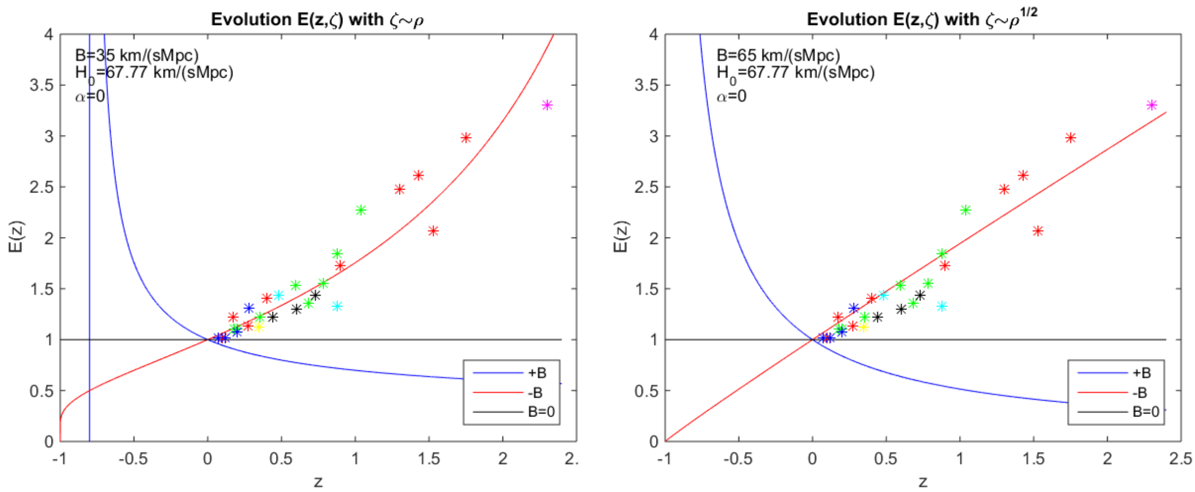

Figure 4.5: Solutions of Friedmanns 1st equation for $E(z,\zeta)$ with $\zeta \sim \rho$ (left figure) and $\zeta \sim \rho^{1/2}$ (right figure) for a one fluid description. The fluid is a dark energy fluid ($\alpha = 0$). B-values $B = -35$km/sMpc (left) and $B = -65$km/sMpc (right) seem to fit the data well. The blue curve corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity. The solution to the left seem to fit the data far better than the solution to the right, which really cannot explain the whole set of data.

Figure (4.6) show the solutions of Friedmanns 1st equation for $E(z,\zeta)$ for a fluid with $\alpha = -1$ with $\zeta \sim \rho$ (left) and $\zeta \sim \rho^{1/2}$ (right). The best fit B-values are $B = 67$ km/sMpc (left) and $B = 36$ km/sMpc (right). As before the blue curve corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity. The solution to the left seem to fit the data far better than the solution to the right, which really cannot fit the whole set of data very well.

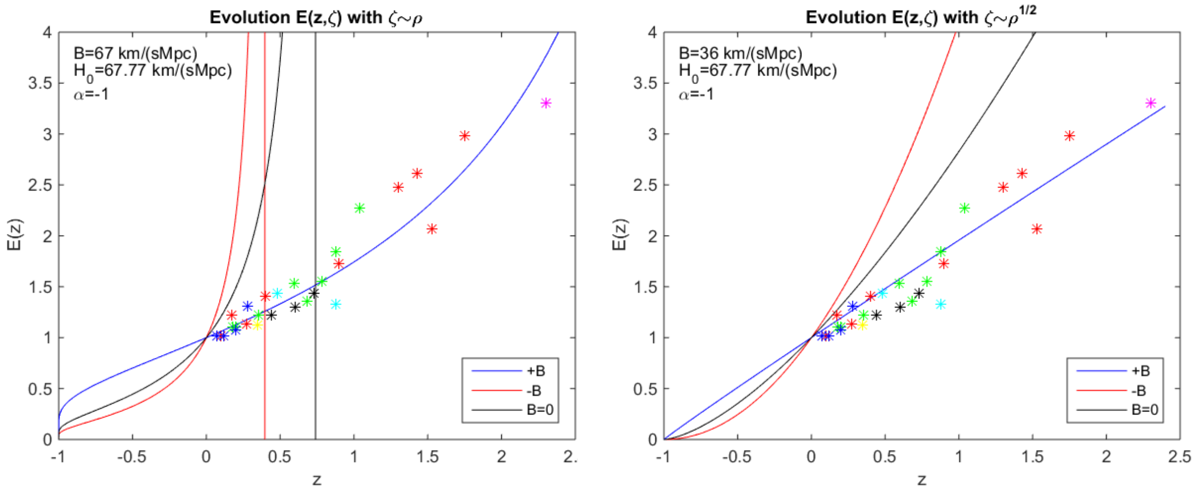

Figure 4.6: Solutions of Friedmans 1st equation for $E(z,\zeta)$ with $\zeta \sim \rho$ (left) and $\zeta \sim \rho^{1/2}$ (right) for a one fluid description. The fluid is such that $\alpha = -1$. The best fit B-values are $B = 67$km/sMpc (left) and $B = 36$km/sMpc (right). The blue curve corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity. The solution to the left seem to fit the data far better than the solution to the right, whcih really cannot fit the whole set of data very well.

There are a few key features to note from 4.5 and 4.6 in comparison with the previously obtained multi-component fluid results (figures 4.2, 4.3):

- Firstly; $\zeta \sim \rho$ seems to correspond much better with observations than $\zeta \sim \rho^{1/2}$.
- With the above parameter choices there is no way a non-dissipative theory can account for the Hubble parameter observations. The viscosity required can be both negative and positive. For $\alpha = 0$ the viscosity parameter is negative, whereas it is positive for the case $\alpha = -1$.
- Even though not displayed in the figures, the same "fine-tuning" or delicate dependence on B as in the multi-component solutions is found. Change B by ± 10, and the fit is clearly worse off.
- The red curve in figure 4.5 resembles the blue curve in figure 4.6 to a great extent. For the plots to the left in the mentioned figures this means; a dark energy solution $\alpha = 0$ with a negative bulk viscosity parameter $B = -35$ km/sMpc seems to have a very similar fit with an evolving equation of state starting from dust ($\alpha = -1$) and evolving towards $\omega = -1$ with $B = 67$ km/sMpc.
- As a parenthesis touching into end scenarios of the universe (which is not discussed any further in this work): If interpreting redshift $z = -1$ as the infinite future[13], the case $\zeta \sim \rho$ seems to correspond to $E(z) \to -\infty$ as $z \to -1$, whereas $\zeta \sim \sqrt{\rho}$ is not all that conclusive.

[13] a negative redshift should in general be a blue shift, and one could wonder whether or not the futuristic interpretation is valid. However, taking it as a pure mathematical rewriting of a this seems to work fine. Anyway the past (positive z) is the concern in the present work.

From table 4.1 one finds that dark energy becomes the dominating constituent at redshift $\zeta = 0.25$. But even so, the matter content has been fairly large as well ever since the matter-radiation equilibrium at $z \sim 3400$. It seems, therefore, as if a two-component solution should be favoured. This in turn, is then expected to explain why the parameter value of B is so much larger in a one-component solution. *Increasing B to account for observations is done instead of adding another fluid to the mixture.* As mentioned; whereas in the multi-component solutions $B = 0$km/sMpc (black line) was a observationally valid solution, it is not a permitted solution in any of the above one-fluid solutions considered.

A better choice for the pre-present universe, however, seems to be to choose an evolution of $\omega(\rho)$ in correspondence with what already is known about the homogeneous equations of state of the dominating components of the cosmological fluid. Trying with the functional form $\omega(\rho)$ from equation (4.85), which, as mentioned, originally was meant to be a good model for the late universe, where $\omega(\rho) \to -1$, and not the pre-present universe, one has

$$\omega(\rho) = \omega(z) = -1 - \alpha \left(\frac{\Omega}{\Omega_0}\right)^{\beta-1}, \tag{4.93}$$

where $\Omega = \rho/\rho_c$ has been used. It should come as no surprise that $\beta = 1$ was in bad correspondence with observations, since it assumes a constant equation of state $\omega = -1$, which indeed is not believed to be a good description of the pre-present universe. In this section, therefore, $\beta = 3/2$ will be assumed. Assuming that the viscosity is small and that the viscosity-less evolution of the universe supplies us with a good approximation, one may use equation (4.38) with $u = 0$ and $\rho = \rho_m + \rho_\Lambda$ to write the evolving energy density $\Omega(z)$ as

$$\Omega = \Omega_m(1+z)^3 + \Omega_\Lambda.$$

Using $\Omega_0 = 1$ and solving the condition $\omega(z = 0.25) = -0.5$ for α one finds

$$\omega(\rho) = -1 + 0.44 \left(\frac{\Omega}{\Omega_0}\right)^{1/2}. \tag{4.94}$$

The reason for imposing the condition $\omega(z = 0.25) = -0.5$ was motivated by the hypothesis that dark energy and matter both can be described by homogeneous equations of state. Adding linearly to the total equilibrium pressure one finds

$$P = \omega\rho = \omega_m\rho_m + \omega_\Lambda\rho_\Lambda \to \omega = \frac{\omega_\Lambda\Omega_\Lambda + \omega_m\Omega_m}{\Omega}.$$

Since these two constituents are the dominating ones, one might, at the equilibrium redshift $z = 0.25$ approximate $\Omega_m(z = 0.25) \sim \Omega_\Lambda(z = 0.25) \sim 0.5$, which then, with $\Omega(z = 0.25) \sim 1$ gives the overall $\omega(z = 0.25) = -1 \cdot 0.5 + 0 \cdot 0.5 = -0.5$.
Of course this is nothing but an approximation, and for the present day universe it gives $\omega = -0.56$, which is not quite correct, but at the same time not too far off the value $\omega(z = 0) \sim (-1.04 \cdot 0.692 + 0 \cdot 0.318)/1 \sim -0.72$ that otherwise by the same means would be estimated for the present day universe (refer to tables 2.2 and 2.3 for numbers). Requiring $\omega(z = 0) = -0.72$ instead, gives

$$\omega(\rho) = -1 + 0.28 \left(\frac{\Omega}{\Omega_0}\right)^{1/2}. \tag{4.95}$$

Note that also in this model $\omega(\rho) \to -1$ as $\rho \to 0$. That ρ becomes smaller and smaller should at least be a good approximation up to redshift $z = 0$, though not necessarily for the future universe, where different kinds of strange singularities might or might not appear – all depending on the model used (as for instance seen from the figures). Figure 4.7 shows the evolution of $E(z)$ for the usual span of redshift.

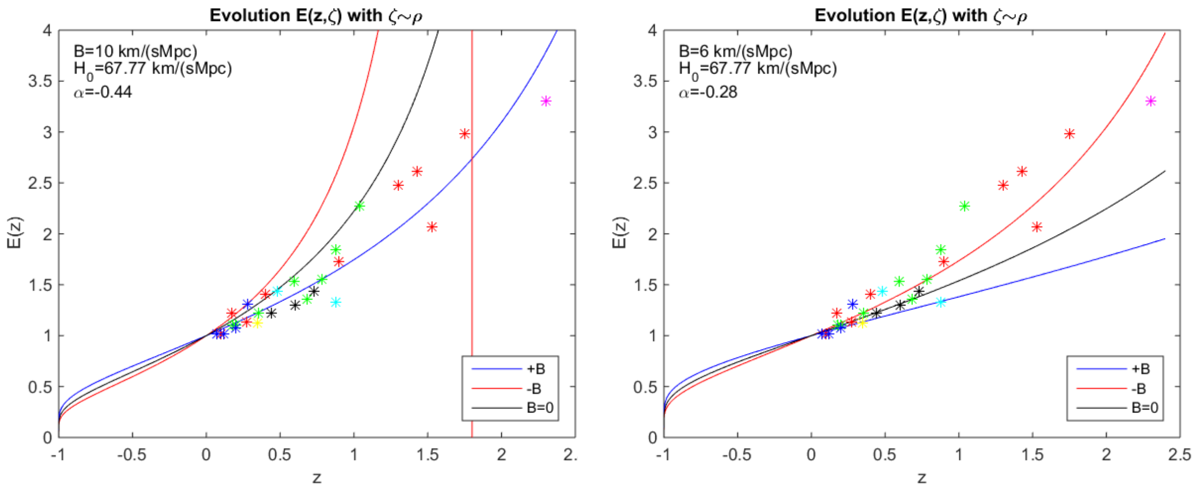

Figure 4.7: Solutions of Friedmann's first equation for $E(z,\zeta)$ with $\zeta \sim \rho$ (left figure) and $\zeta \sim \rho$ (right figure) for a one fluid description. The fluids are such that $\alpha = -0.44$ for the one to the left and $\alpha = -0.28$ for the one to the right. B-values $B = 10$ km/sMpc Pa s (left) and $B = -6$km/sMpc Pa s (right) seem to fit the data well. The blue curve corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity.

Note that changing the evolution $\omega(\rho)$ changed the amount of viscosity needed to explain the Hubble parameter measurements.

A natural next question is what value the parameter α has to take in order for the $B = 0$ to be a good solution. By eye-sight this value is found to be such that

$$\omega(\rho) = -1 + 0.34\left(\frac{\Omega}{\Omega_0}\right)^{1/2}. \tag{4.96}$$

This solution is plotted in figure 4.8 accompanied with viscous solutions $B = 0.1$ km/sMpc and $B = 1$ km/sMpc as before.

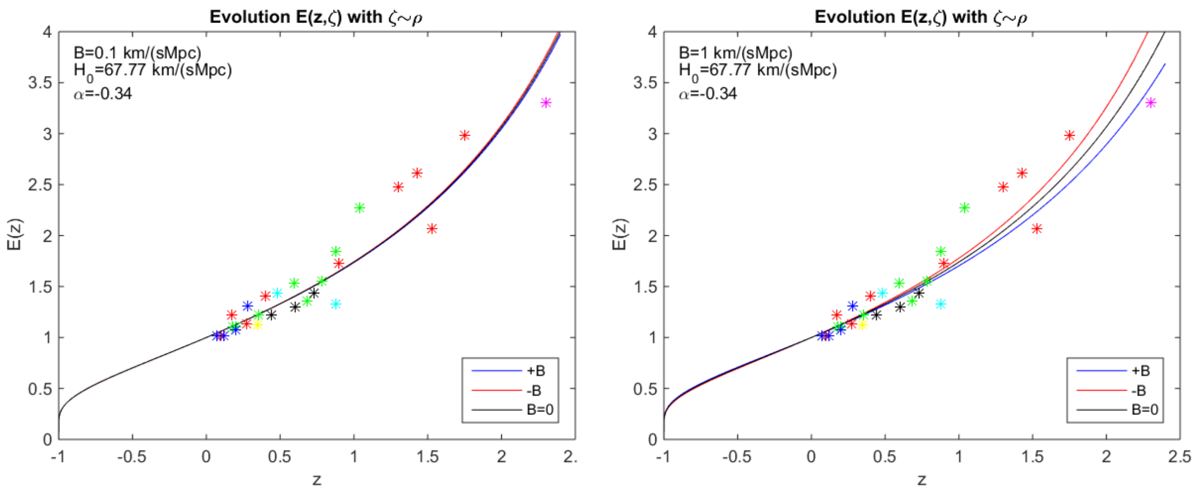

Figure 4.8: Solutions of Friedmanns 1st equation for $E(z,\zeta)$ with $\zeta \sim \rho$ for a one fluid description. The fluid is such that $\alpha = -0.34$. In the left figure $B = 0.1$ km/sMpc, whereas in the right figure $B = 1$ km/sMpc. The blue curve corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity, and in this case also the best fit among the three.

So; an important point seems to be that **it is possible to transform away any viscosity by choosing the constants in the equation of state ansatz to this end.**

Also, note that from figure 4.8 one sees the same trend as before; adding a viscosity $B = 0.1$ km/sMpc is hardly distinguishable from the $B = 0$ km/sMpc case. For $B = 1$ km/sMpc the bare eye can see the difference, but it is not big - though it seems to be a slightly bigger difference than in the multi-component solutions.

4.8.2 Making sense of the one-component description

The conclusion for the one-fluid considered seems to be that the amount of viscosity that must be added into the equations in order to account for the Hubble parameter measurements, is highly dependent on the model used. For the model here considered, in which

$$\omega(\rho) = -1 - \alpha \left(\frac{\Omega}{\Omega_0}\right)^{1/2}. \tag{4.97}$$

It seemed that one actually did not need that extra parameter which a non-zero viscosity gave, since from the above figures, it was found that a parameter choice B corresponding to both negative, positive and zero viscosity could explain the Hubble parameter measurements equally well.

It seems therefore, as if **an inhomogeneous equation of state (** $\omega \rightarrow \omega(\rho)$**) can replace the viscosity term in the one-fluid here considered.** This is quite correct, and is easily demonstrated by going back to equation (4.86);

$$a\partial_a \rho - 3\gamma\rho^\beta = 3\tau\theta^{2\beta}. \tag{4.98}$$

Since $\rho^\beta \propto \theta^{2\beta}$ one immediately recognizes that the same dependence on ρ is found in ω as in ζ. So; using

$$\omega(\rho) = -1 - \alpha \left(\frac{\rho}{\rho_0}\right)^{\beta-1} \quad \text{and} \quad \zeta(\rho) = \zeta_0 \left(\frac{\theta}{\theta_0}\right)^{2\beta-1} \tag{4.99}$$

is the same as inserting

$$\omega = -1 \quad \text{and} \quad \zeta(\rho) = \zeta_0 \left(\frac{\theta}{\theta_0}\right)^{2\beta-1}, \tag{4.100}$$

where ζ_0 now takes a different value:

$$\zeta_0 = \gamma + (3\kappa)^\beta \tau. \tag{4.101}$$

Now; this seems to be the natural way to define the viscosity in this context. The above section shall therefore be taken to have demonstrated a rather important point:

If the cosmological fluid can be described by an inhomogeneous equation of state and zero viscosity, it can also be brought to the form of a homogeneous equation of state by interpreting the inhomogeneous part as a viscosity contribution instead. What here will be done, therefore, is to interpret any modification to the pressure that is not linear in ρ as phenomenological bulk viscosity. This seems to be in agreement with e.g. (Cardone et al., 2006), in which a perfect fluid is defined to have a homogeneous equation of state.

In the scenario here considered, the constant part of the equation of state parameter was set to -1. This is a natural choice when investigating the late universe, where dark energy seems to become more and more dominating. Another question that then comes to mind is whether or not there exists a value ω =const that fits the data set as well without any viscosity as with viscosity (for a one-fluid that is). Or, can it on the other hand generally be stated that a homogeneous equation of state *requires* a viscosity term in order to account for observations? This will be investigated in the coming section.

4.8.3 Further tries with the one-fluid model

As a last take on the one fluid model, the following is considered:

$$\omega = \text{const} = \delta \quad \text{and} \quad \zeta(\rho) = \zeta_0 \left(\frac{\theta}{\theta_0}\right)^{2\beta-1}. \tag{4.102}$$

The energy equation now becomes

$$a\partial_a \rho + 3(1+\delta)\rho = 3\zeta_o \left(\frac{\theta}{\theta_0}\right)^{2\beta-1} \theta. \tag{4.103}$$

Now; the cases $\beta = 1$ and $\beta = 3/2$ correspond to the cases $\lambda = 1/2$ and $\lambda = 1$ respectively in section 4.6 (figures 4.3 and 4.4 respectively), except when previously treated it was with a two-component fluid.

Using the previously found solutions, equation (4.67) gives for the case $\beta = 1$ that

$$E(z) = \sqrt{\Omega_{0x}(1+z)^{3(1+\omega_\zeta+\delta)}}, \tag{4.104}$$

where x here denotes the overall one-fluid with an unknown equation of state parameter δ. Normalizing such that $E(z=0) = 1$ requires $\Omega_{0x} \equiv 1$. For $\beta = 3/2$ one similarly finds from equation (4.72) and (4.59) that

$$E(z) = \sqrt{\Omega_{0x}(1+z)^{3(1+\delta)}} \left\{ 1 + \frac{B}{H_0} \int \frac{\sqrt{\Omega_{0x}(1+z)^{3(1+\delta)}}}{1+z} dz \right\}^{-1}. \tag{4.105}$$

Solving the integral one finds

$$E(z) = \sqrt{\Omega_{0x}(1+z)^{3(1+\delta)}} \left(\sqrt{\Omega_{0x}} + \frac{2B}{3H_0(\delta+1)} \sqrt{\Omega_{0x}(1+z)^{3(1+\delta)}} \right)^{-1}. \tag{4.106}$$

Hence one finds that the evolution for $B = 0$ km/sMpsc is the same for $\beta = 1$ and $\beta = 3/2$, which must be true (since β only modifies the evolution of the viscosity). Figure 4.9 show the evolution for what is found to be the best suiting value for δ without viscosity. One can clearly see that the fit is bad. It is possible to fit it such that it leaps in the centre of the observational data, or to one of its outskirts, but it doesn't look like a convincing fit. This result is found to be rather strange, and one could wonder if some of the calculations have gone wrong at some point. If trusting the calculations, however, it seems as if one ought conclude that there is no parameter value $\delta \neq -1$ that one can employ to account for observations with the ansatz that here is used for ζ. This is quite surprising because it seemed to work very well for $\delta = -1$ and the same ansatz for the viscosity (treated previously). The qualitative difference between $\delta = -1$ and $\delta \neq -1$ is that a term proportional to ρ remains in the energy equation in the latter case, whereas it vanishes in the former (since $P = -\rho$). This then, is thought to be responsible for the qualitative difference in functional form.

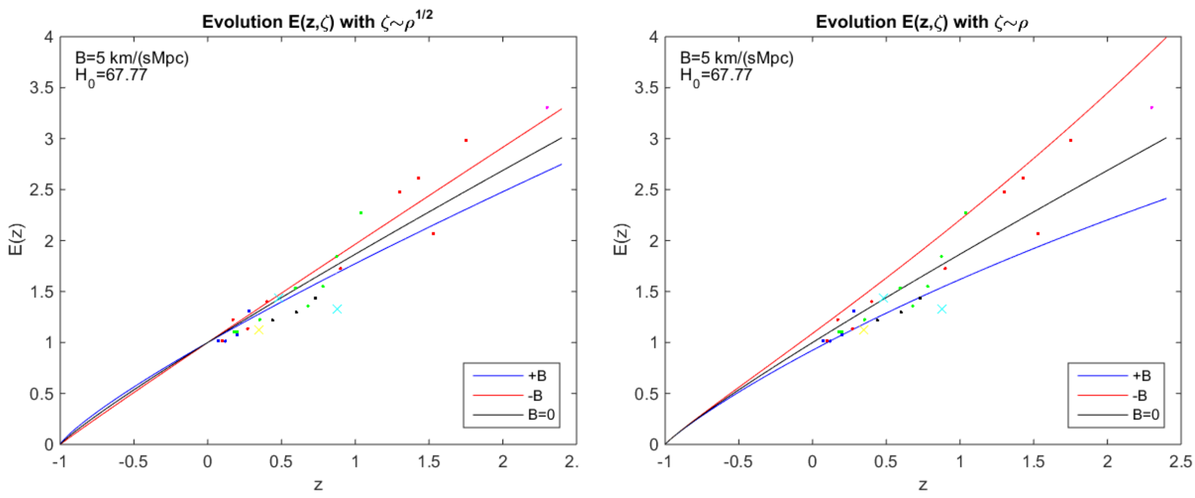

Figure 4.9: Solutions of Friedmanns 1st equation for $E(z,\zeta)$ with $\zeta \sim \sqrt{\rho}$ (left) and $\zeta \sim \rho$ (right) for a one fluid description. The fluid has constant equation of state parameter ω. The viscosity free solutions are of main interest, but solutions with $B = \pm 5\text{km/sMpc}$ are added as well. The blue curves corresponds to positive viscosity whereas the red curve reveals the development with correspondingly negative viscosity. The black curve is the evolution with zero viscosity.

The conclusion for the one-fluid case is therefore that when requiring a homogeneous equation of state there are two ways of accounting for observations when the above ansatz for ζ (4.85) is used:

- One must choose $\omega = -1$ and add the appropriate amount of viscosity with the evolution corresponding to $\beta = 3/2$. From figure 4.5 this amount is found to be $B = -35$ km /sMpc, and thus on thermodynamically shaky ground (since it is negative and thus suggests negative entropy production or negative temperature).

- One must have more than one component in the fluid. The viscosity is then superfluous in accounting for Hubble parameter observations.

If allowing for a inhomogeneous equation of state, the RHS of the energy equation can always be brought to the LHS and be interpreted as part of the equation of state, where now $\omega \rightarrow \omega(\rho)$. However, this seems to just be another way of describing what otherwise could be described as bulk viscosity on a phenomenological level.

4.9 Beyond phenomenology

The present work emphasizes the phenomenological study of the bulk viscosity in the universe[14]. This should be the place to start when so little is known about the major energy constituents of the universe in the first place. What is dark energy, and how does it - if at all - interact with the other components? What with dark matter? The questions line up. Yet; is there anything sensible that could be said beyond a phenomenological level from the little

[14] Viscosity must in a sense be said to be phenomenological in its essence, so what here is meant is that the actual physical mechanisms *causing* what phenomenologically is described as bulk viscosity has not been a major point of investigation in the present work.

that is known? In the following, the formulae for bulk viscosity that was given by Weinberg and extended by Zimdahl will be applied to estimate an order of magnitude of the viscosity. Finally, the formula derived from kinetic theory and the BGK Boltzmann equation will be applied.

4.9.1 How big visc. can Zimdahl account for?

In this section the Weinberg and Zimdahl cases will be considered. Zimdahl (1996) seems to have given the more general case of viscosity, even if it also is the less rigorously derived case. From equation (2.122), one finds that Zimdahl arrives at

$$\zeta = -\tau T \frac{\partial \rho}{\partial T}\left(\frac{\partial P_1}{\partial \rho_1} - \frac{\partial P}{\partial \rho}\right)\left(\frac{\partial P_2}{\partial \rho_2} - \frac{\partial P}{\partial \rho}\right), \tag{4.107}$$

where τ is the mean free time between interaction of the two components and $\rho = \rho_1 + \rho_2$ is a two-component fluid effectively treated as one fluid. Also,

$$\frac{\partial P_i}{\partial \rho_i} \equiv \frac{\partial P_i / \partial T}{\partial \rho_i / \partial T}.$$

In the following, it is, following the paper, assumed that

$$P_i = \omega_i \rho_i \qquad \text{with energy density} \qquad \rho_i = \frac{n_i k T}{\omega_i} + n_i m_i c^2. \tag{4.108}$$

The treatment therefore, is classical kinetic theory, with two interacting perfect fluids ρ_1 and ρ_2. Using $P(n,T) \equiv P_1(n_1,T) + P_2(n_2,T)$ and calculating derivatives, one finds

$$\zeta = -\tau T \frac{\partial \rho}{\partial T}\left(\omega_1 - \frac{\omega_1 \partial_T \rho_1 + \omega_2 \partial_T \rho_2}{\partial_T \rho}\right)\left(\omega_2 - \frac{\omega_1 \partial_T \rho_1 + \omega_2 \partial_T \rho_2}{\partial_T \rho}\right), \tag{4.109}$$

where, for the sake of transparency, $\partial \rho_i / \partial_T$ is abbreviated $\partial_T \rho_i$ according to standard notation. Using equation (4.108) to calculate the fraction inside the first bracket, one finds that it reduces to

$$\omega_1 \omega_2 \frac{n_1 + n_2}{n_1 \omega_2 + n_2 \omega_1}. \tag{4.110}$$

Doing the same for the second bracket, and writing out $\partial \rho_T$ one finds upon some straight forward but tedious algebra that the bulk viscosity arrives at

$$\zeta = \tau T k (\omega_1 - \omega_2)^2 \frac{n_1 n_2}{n_1 \omega_2 + n_2 \omega_1}, \tag{4.111}$$

where the negative sign has disappeared. Refer to appendix H for a alternative route to justifying the functional form here found. One can notice that for $n_1 \sim n_2 \sim n$ one finds

$$\zeta \sim \frac{(\omega_1 - \omega_2)^2}{2(\omega_1 + \omega_2)} \tau k T n, \tag{4.112}$$

which seems to coincide well with what is suggested by the formula derived from the BGK Boltzmann equation;

$$\zeta \sim \tau k T n, \tag{4.113}$$

where $\omega = 1/\tau$ has been used to obtain the above equation from (3.106). On the other side however; if one of the components are far less dense than the other ($n_1 << n_2$) the density of

the less dense component will dominate in the expression for ζ. It therefore seems plausible that for this sort of viscosity to contribute significantly, n_1 and n_2 must be of comparable sizes.
Another observation to make is that the only way the fluids are distinguished from each other in the above formalism is by their different equations of state. From (4.117) one finds it necessary for the two fluid components to have significantly different equations of state. Otherwise; if $\omega_1 \rightarrow \omega_2$, then $\zeta \rightarrow 0$.

Regarding candidates; which fluid components could this formula be applied to? The case where one of the components is radiation the Zimdahl formalism reduces to the Weinberg formalism, which was derived on more rigorous grounds. The case of radiation slightly out of equilibrium with a matter-component will therefore be treated with the Weinberg formula in the next section.

Another scenario could be ordinary matter interacting with dark matter. The problem with this candidate, however, is that both matter and dark matter are often hypothesized to have the same equation of state parameter $\omega = 0$ (dust; pressure-less matter). More thorough studies could be done to this end in another less phenomenological study. In the present work, however, this candidate is not regarded for further investigation. As it is, another perhaps just as dubious candidate can not yet be written off; cold matter interacting with dark energy. The problem would be to determine the number density of the latter. Perhaps, and even likely so, it doesn't even make sense to talk about number density of dark energy. Since the calculations in this chapter to a great extent has assumed a mixture of dark energy and cold matter, however, a last try is done to this end by rewriting equation (4.111) in terms of known quantities. This is done in the following section.

Rewriting in terms of known quantities

It seems useful to convert this into the notation used so far. And if the two components are both matter-components, one should be able to rewrite the number density n_i such that

$$n_i = \frac{\rho_i}{kT/\omega_i + m_i c^2} \approx \frac{1}{m_i c^2}\frac{3H_0^2 c^2}{8\pi G}\Omega_{i0}(1+z)^{3(1+\omega_i)}, \tag{4.114}$$

where $\rho_i = \rho_c \Omega_i$ has been used and it was assumed that the thermal part of the energy is far less than the rest energy; $kT/\omega_i << m_i c^2$. Also a factor c^2 has been added in going from natural to SI units. That means the factors of c^2 cancels out all in all, and the above equation allows for rewriting

$$\frac{n_1 n_2}{n_1\omega_2 + n_2\omega_1} = \frac{3H_0^2}{8\pi G}\frac{\Omega_{01}\Omega_{02}(1+z)^{3(1+\omega_1)+3(1+\omega_2)}}{\Omega_{01}\omega_2 m_2(1+z)^{3(1+\omega_1)} + \Omega_{02}\omega_1 m_1(1+z)^{3(1+\omega_2)}}. \tag{4.115}$$

Inserting this into equation (4.111) and calculating one ends up with the general expression

$$\zeta = \tau T k(\omega_1 - \omega_2)^2 \frac{3H_0^2}{8\pi G}\frac{\Omega_{01}\Omega_{02}(1+z)^{3(1+\omega_1)+3(1+\omega_2)}}{\Omega_{01}\omega_2 m_2(1+z)^{3(1+\omega_1)} + \Omega_{02}\omega_1 m_1(1+z)^{3(1+\omega_2)}}. \tag{4.116}$$

This expression might not be very illuminating in itself, but is more interesting when implemented with the below case of dark energy and pressure-less matter.

Dark energy and dust: It is time for implementing the derived formula with a specific universe. Making the so far standard choice of filling space with matter ($\omega_m = 0$) and a dark energy fluid with $\omega_\Lambda = -1$, equation (4.117) reduces to

$$\zeta = -\tau \frac{Tk}{m_\Lambda} \frac{3H_0^2}{8\pi G} \Omega_{0\Lambda} \approx -\left[\tau \frac{Tk}{m_\Lambda}\right] \cdot 10^{-26} \frac{\text{kg}}{\text{m}^3}, \tag{4.117}$$

which, surprisingly enough, is constant[15].

What can be concluded from the above formula? Perhaps not so much. The mass m_Λ of a dark energy particle sounds like a highly speculative quantity. The minus sign might not mean that the viscosity is negative, since the temperature of dark energy could be taken to be negative instead. In a thermodynamic approach this might perhaps just be two choices equally valid (see (Brevik and Grøn, 2013) for an interesting comment to this end). Note that unless the temperature T of the system is very large (which could be imagined when comparing to say galaxies) and the "mass" of the dark energy "particles" is sufficiently small, this contribution to the viscosity will be negligible.
Finally; recall that τ is taken to be the characteristic time between interaction between both fluids, such that "*during a time interval τ, the perfect fluid components may be considered as effectively insulated from each other.*" ((Zimdahl, 1996)). Dark energy is thought to fill space homogeneously, like the rest of the energy content in the universe. However, this homogeneity might not be found down to an arbitrary small scale (in this case τ would vanish), but is thought to hold on cosmic scale ($\sim$ 100Mpc). In such a case finite $\tau > 0$ might be allowed. However, if dark energy is something quite different than particles, then it might not even be clear what is meant by "mean free time", its "mass" or its "temperature". It seems best to leave the discussion at this point.

The conclusion so far for the Zimdahl formula seems to be that it doesn't give any good candidates for viscosity on the superficial investigation here performed, except perhaps photons in interaction with matter, which will be treated with the Weinberg formalism instead. However, (Zimdahl, 1996) is of great value, since it shows that two expanding ideal fluids will not be ideal when seen as one fluid. Of course this raises some questions about where the large viscosity found from the one-fluid description originates from. It is still possible, perhaps, that it can be accounted for by some of the candidates investigated above, but it seems as if more information is needed to find out.

4.9.2 How big viscosity can Weinberg account for?

From Weinberg one has (2.109a) that

$$\zeta = 4bT^4\tau \left[\frac{1}{3} - \left(\frac{\partial P}{\partial \rho}\right)_n\right]^2, \tag{4.118}$$

where (equation (2.100))

$$b = \begin{cases} a, & \text{for} \quad \text{photons or gravitons,} \\ \frac{7}{8}a, & \text{for} \quad \nu_e \text{ and } \bar{\nu}_e \text{ or } \nu_\mu \text{ and } \bar{\nu}_\mu. \end{cases} \tag{4.119}$$

[15] Perhaps not so strange after all. The evolution with z will cancel out for dark energy, which reflects the fact that its density stays constant, and the remaining evolution with z cancels out since $\omega_m = 0$.

and $a = \frac{8}{15}\pi^5 k^4 h^{-3}$ is the Stefan Boltzmann constant.

Gravitons

According to (Weinberg, 1972) gravitons have a mean free time given by

$$\tau = \frac{1}{16\pi G\eta}, \tag{4.120}$$

where the shear viscosity η is given by equation(2.109b) and reads

$$\eta = \frac{4}{15} bT^4\tau. \tag{4.121}$$

Inserting this one finds

$$\tau = \sqrt{\frac{15}{64\pi GaT^4}} = \frac{1.4\cdot 10^8}{|T|^2}\frac{\mathrm{s}^{5/2}}{\mathrm{m}^{3/2}}\cdot c^{3/2}, \tag{4.122}$$

where the absolute value again is used to symbolize a dimensionless quantity. Note that multiplication by $\cdot c^{3/2}$ had to be done to obtain correct units. If the temperature is supposed to be of the order of unity, this suggests a mean free time $\tau \sim 10^{13}$ years, which exceeds the predicted age of the universe by a factor 10^3.
If, however, the calculation is carried through despite our reluctance, one ends up with a bulk viscosity

$$\zeta_G = 6|T|^2 \cdot 10^4 \mathrm{Pa\ s}. \tag{4.123}$$

Unless the temperature that should be assigned to the gravitons is much less than 1, this viscosity is not negligible. However, there are several major concerns with this result. As mentioned the mean free time exceeds the age of the universe by such a great amount that one should expect the effect of the bulk viscosity to be zero. There simply is not enough time for this dissipative process to take place in any effective way. This is also seen through the second concern of the above result: The expression for the mean free time, equation (4.122), includes the shear viscosity, which should vanish in a homogeneous and isotropic universe. If it vanishes, the mean free time goes to infinity. As explained, this should imply the somewhat contradictory result of an infinitely large bulk viscosity that does not cause any dissipation due to lack of time for the process to occur.

Hence; nothing is concluded on this point either.

Photons

After the recombination, the photons don't interact much with matter. Let us for a moment consider the microwave background radiation at about $3K$ slightly out of equilibrium with pressure-less matter. Since the recombination is thought to have happened very early, the mean free time is estimated to be the Hubble time; $\tau = 1/H_0$. In this case (4.118) gives

$$\zeta_P \sim 10^4 \mathrm{Pa\ s} \tag{4.124}$$

for the bulk viscosity ζ_P of photons.

This result is far more edifying than the value obtained from gravitons. However, the dubious side of this result, is that the formula used is derived from radiation slightly out of equilibrium with another component. If thinking of galaxies, this seems strange, since an estimation of the galaxy "temperature" would give

$$10^{30}\text{kg/galaxy} \cdot 10^{11}\text{galaxies} \cdot 10^{2\cdot 6}\text{m}^2\text{s}^{-2} \sim kT \quad \rightarrow \quad T \sim 10^{30}\text{K},$$

which of course is far from $T_{photons} \sim 3$K. However, the point is perhaps that the radiation must at some point interact with matter. Due to the enormous temperature difference that does not happen very often. That is precisely why the mean free time was set to the Hubble time in the first place. So, if believing that radiation will interact with matter at *some* point, then perhaps the estimate is in the ballpark after all. Also remembering that the mean free time τ should be far less than the typical time T of the system, a better estimate would perhaps be to reduce the mean free time to 10% of the Hubble time. One finds

$$\boxed{\zeta_P \sim 10^3\text{Pa s.}} \tag{4.125}$$

This result seems to be the best candidate so far, and could, as mentioned, also have been obtained from the Zimdahl formalism.

Neutrinos

One could wonder whether or not neutrinos could cause some viscosity of the late universe. Their role in the early epochs have been investigated by (Misner, 1967) and (Hoogeveen, 1986) along with many others. One ought remember, however, that neutrinos interact weakly, and not electromagnetically. As it is, weak interactions require high energies to occur. Sufficient energies indeed might have been present in the early, immensely hot and dense universe. However, these conditions are not met in the late universe, and therefore neutrinos do not seem to be a good candidate.

4.9.3 The BGK formula

Before ending the section that goes beyond the otherwise phenomenological spirit of this study, a last aim is taken with the formula derived from the BGK Boltzmann equation. From equation (3.106) one finds

$$\zeta \sim \frac{nkT}{\omega}. \tag{4.126}$$

Galaxies

The applicability of the above equation is limited in the same way as the Zimdahl equation; it heavily resides on a particle description. But there seems to be one good candidate left that is worth studying and that seems best studied through equation (4.126): Galaxies. In this case it should be possible to suggest estimates for the above mentioned quantities.

In the following it is assumed that galaxies with internal degrees of freedom collide. In this respect, the galaxies should all be seen as different from each other. One can here not expect more than obtaining the correct order of magnitude, so whether one uses equation (4.111) (Zimdahl) or (4.113) should not matter. However, the latter seems easier to handle in this case.

In the continuation the following data will be used:

- The **mass** M_G **of a galaxy** is taken to be
  $M_G = 10^{11} \cdot 10^{30}$kg (Source: (Liddle, 2003)).
- The **mean velocity** $\bar{c}$ **of a galaxy** is taken to be
  $\bar{c} = 10^6$m/s (Source: (Spitzer L. Jr., 1950)).
- The **diameter** D **of a galaxy** is taken to be
  $D = 2R = 100\,000$ l.y.= $9.5 \cdot 10^{20}$m (Source: According to (Liddle (2003), p. 4) the Milkey Way galaxy has $D \approx 80000$ l.y.
- The **number density** n_G **of galaxies** is taken to be
  $n_G = 10^{-69}/\text{m}^3$ (Source: (Lotz M. J., 2011)) .

The collision frequency $\omega = 1/\tau$, where τ is the mean free time. Using

$$l = \frac{1}{\sqrt{2}\pi R^2 n_G}, \tag{4.127}$$

(Hänel, 2004) where R is the radius of the galaxies, $\tau = l/\bar{c}$ gives

$$\tau = \frac{1}{n_G\sqrt{2}\pi R^2 \cdot \bar{c}}. \tag{4.128}$$

With

$$\bar{c} = \frac{8kT}{\pi M_G} \qquad \rightarrow \qquad kT = \frac{\pi}{8} M_G \bar{c}^2. \tag{4.129}$$

Inserting into equation (4.126) one finds

$$\zeta = \frac{\sqrt{2}}{24}\frac{M_G}{R^2}\bar{c} \tag{4.130}$$

Notice that the bulk viscosity turns out to be independent of the number density n. This is because the mean free time τ is proportional to $1/n$, and thus it cancels out in (4.126). Inserting the above listed numbers, one ends up with

$$\zeta \sim 10^4 \text{Pa s}. \tag{4.131}$$

Here, then, is a non-vanishing magnitude again. In fact, such a magnitude is ridiculously large compared to typical magnitudes obtained from molecular gasses. In the next section, however, it will be shown that this is not a large number on cosmological scale.

Again the result should be treated with some care, however, because the mean free time is way larger than the time scale of the macroscopic system; $\tau >> 1/H_0$

$$\tau = \frac{1}{n_G\sqrt{2}\pi R^2 \cdot \bar{c}} \sim 10^{21} s \sim 1000 H_0^{-1}. \tag{4.132}$$

One immediately sees that reducing the mean free time by a factor 10000, so that it is on the order $\sim 10^9$y will give

$$,\zeta \sim 1 \text{Pa s} \tag{4.133}$$

which, taking into account the fact that galaxies first started to form a few hundred million years after the initial singularity, seems to be the best one may hope for in this case.

Estimating the mean free time between galaxy collisions from collisions happening in galaxy clusters instead, one finds from (Spitzer L. Jr., 1950) that the rate is

$$\omega = \frac{20}{3\cdot 10^9 y} = 2.1\cdot 10^{-16}\frac{1}{s} \quad \rightarrow \quad \tau = \frac{1}{\omega} = 4.7\cdot 10^{15}\mathrm{s} \sim 1.5\cdot 10^8\mathrm{y} \tag{4.134}$$

which will give

$$\zeta \sim 0.1\mathrm{Pa\ s}. \tag{4.135}$$

It seems therefore, that galaxy collisions are unable to account for all of the viscosity that otherwise theoretically could be allowed for ($\sim 10^5$Pa s).

Note, however, that galaxy collisions are inelastic. Two galaxies collide to yield one new, bigger galaxy. This should suggest something more complicated than a vanishing flux number density divergence; $N^{\mu}_{;\mu} = 0$. Perhaps if following this path, the viscosity due to galaxy collisions would not be that negligible after all?

Other candidates

There might very well exist other plausible candidates worth investigating. Not knowing what these would be, however, this section is ended here.

4.9.4 Bulk viscosity as a first order modification of the pressure

A final word of justification of the previously found magnitudes of the viscosity seems to be on its place. The question asked in this section is not what causes the viscosity, but rather how big it could possibly be before the formalism for viscosity in cosmological context fails. After all it has been shown how the transport coefficients are obtained as the continuum limit of first order deviations in classical kinetic theory. The right restriction therefore seems to be

$$|P| = |\omega\rho| >> |\zeta\theta|, \tag{4.136}$$

where P here is taken to be the pressure of the overall cosmological fluid, and ρ is the density. To estimate any bounds on this basis, the critical density will be used. From (Liddle (2003), page 47) one has

$$\rho_c \sim 10^{-26}\frac{\mathrm{kg}}{\mathrm{m}^3}. \tag{4.137}$$

Estimating that the equation of state parameter ω is of the order of unity or less, the above restriction reduces to

$$|10^{-26}\frac{\mathrm{kg}}{\mathrm{m}^3}| >> |\zeta\theta|. \tag{4.138}$$

Now using $\theta = 3H$ and inserting the present expansion rate $H_0 \sim 20\cdot 10^{-19}s^{-1}$ (from table 2.2) for H one finds

$$|\zeta| << \frac{10^{-26}\,\frac{\text{kg}}{\text{m}^3}}{3\cdot 20\cdot 10^{-19} s^{-1}} \sim 2\cdot 10^{-9}\,\frac{kgs}{m^3} \overset{1=c^2}{\sim} 10^8\text{Pa s}. \tag{4.139}$$

This result reveals that the viscosity coefficient actually can be extremely high compared to the intuition given by kinetic theory applied on atomic and molecular scales. Requiring that the pressure due to the viscosity is not more than 1% of the equilibrium pressure P one finds

$$\boxed{|\zeta| \sim 10^6 \text{ Pa s}} \tag{4.140}$$

or even greater magnitudes before the apparatus brakes down. This is just as expected, since it already has been pointed out throughout this chapter that no differences were seen compared to the non-viscous cases when $\zeta \sim 10^5$ Pa s was added to the equations.

Chapter 5

Summary and recommendations for further work

5.1 Summary and conclusions

Existing literature was surveyed for interesting formulae in chapters 2 and 3. Not too much was found to have been done on the subject of viscous cosmology in the late universe. However, a few central sources was found and discussed intently. In chapter 2 the foundation for cosmology in general, and viscous cosmology in particular, was laid out. The Weinberg formalism was discussed extensively, and supplied with Zimdahls general argument for a non-vanishing phenomenological viscosity in multi-component perfect fluid expansion (Section 2.3).
In chapter 3 transport coefficients were derived from purely classical kinetic theory with the Boltzmann equation as starting point. What was denoted the BGK formula for bulk viscosity (3.110) was derived from applying Chapman-Enskog expansion to a simplified Boltzmann equation (denoted the BGK-Boltzmann equation). Refer to section 3.1.2 for details.

Chapters 2 and 3 make up a substantial and important part of the work, and should be a good look up reference also for future works. Together, these two chapters are meant to have met objectives 1 and 2.

Objective 3 was sought met in chapter 4, which must be seen as the very core of the work at hand. The chapter was not based on any source in particular and quite a bit of independent work was done. References were made whenever existing literature was found. The main results are presented through the following paragraphs.

General solutions of the energy equation (2.167b) were found for flat space ($k = 0$) for different ansatzes for the viscosity ζ. The viscosity was modelled both as a function of redshift ($\zeta(z)$) and as a function of the energy density ($\zeta(\rho)$). This was done in sections 4.4 and 4.6, respectively, and the solutions were used in Friedmanns first equation (2.168a) to obtain general expressions for the dimensionless Hubble parameter $E(z)$. By such, **the solutions found in (Wang and Meng, 2014) were generalized.** Following the same procedure as in the mentioned paper, the expressions for $E(z)$ were compared with a compiled list of Hubble parameter measurements (table 2.1). These general solutions were implemented with a two-component fluid and a one-fluid, as summarized below. Also, in section 4.3, the results found in (Wang and Meng, 2014) were discussed in some detail and used as a motivation for finding more general results. It was found that

- The general solutions found for $E(z)$ ((4.42) for $\zeta(z)$ and (4.58) for $\zeta(\rho)$) reduced correctly to the non-viscous solution in the limit $\zeta \to 0$. It was found that this was the limit that had been taken in (Wang and Meng, 2014).

- It was also pointed out that (Wang and Meng, 2014) had assumed a temperature-dependence that corresponded to radiation domination ($T \sim 1/a$).

Results for two-component fluid: In sections 4.5 and 4.6 a two-component cosmological fluid was considered. For a cosmological fluid chosen to consist of matter and dark energy it has been shown, by theoretical estimates in comparison with observations, that the present value of the bulk viscosity can be as large as $|\zeta_0| \sim 10^5$ Pa s without altering the Hubble parameter predictions of the non-viscous model enough for it to be seen from the plots. For ζ_0 one order of magnitude higher the difference is starting to become apparent, though still not large. Since the theory seems to predict observations really well when not adding any viscosity, it was on these grounds found reasonable to assume that

$$\boxed{|\zeta_0| \leq 10^6 \text{Pa s}} \tag{5.1}$$

for a two-component fluid model consisting of matter and a cosmological constant. **This was found to be true for many functional forms of the viscosity; ζ =const, $\zeta \sim \rho^{1/2}$ and $\zeta \sim \rho$.** Refer to figures 4.2, 4.3 and 4.4, respectively. Note that positive as well as negative solutions were found to be allowed for on these grounds. However, this only says that it is possible to include this much viscosity (positive as well as negative) before any notable change is seen, and not that this viscosity necessarily is present. To say anything to this end, one would have to show that such viscosity explains observations better, or also to calculate and prove through a more fundamental theory (e.g. kinetic theory) that the content of the universe must have such an amount of viscosity. Nevertheless it is one of the central results of the thesis, and it is in good agreement with the results found in (Wang and Meng, 2014). **The validity of the results in this paper is therefore meant to extensively have been improved on through the present work.**

Results for a phenomenological one-fluid: In section 4.8 a cosmological one-fluid was investigated. In this case the boundaries on the viscosity parameter proved to be highly dependent on the parameters in the model. The model used was such that (4.85)

$$\omega(\rho) = -1 - \alpha\left(\frac{\rho}{\rho_0}\right)^{\beta-1} \quad \text{and} \quad \zeta(\rho) = \zeta_0\left(\frac{\theta}{\theta_0}\right)^{2\beta-1} \tag{5.2}$$

and was taken from the literature, where it was found applied to the future universe. What was tried out in the present work, however, was to implement it with the past universe, for which there exists observations. **The following interesting properties were pointed out:**

- The possibility of adding a large positive as well as negative viscosity was found (figure 4.6 and 4.5, respectively).

- From the same plots it is clearly seen that in general $\zeta \sim \rho$ corresponds to a much better fit than $\zeta \sim \sqrt{\rho}$.

At the same time it was found

- possible to explain observations equally well by transforming away the viscosity all together (figure 4.8) by picking suitable boundary conditions for $\omega(\rho)$ (i.e. picking a suitable parameter α).

That the viscosity could be transformed away altogether was found to be a trivial consequence of the fact that the evolution assumed for $\zeta(\rho)$ and $\omega(\rho)$ over ρ seemed to implement a degeneracy between the two. This caused two terms in the energy equation to acquire the same functional form. Even more general, however; the inhomogeneous part of the equation of state can always be brought over to the RHS of the energy equation and be taken as a phenomenological bulk viscosity. After all bulk viscosity is a phenomenological way to describe modification of pressure due to deviation from equilibrium. A monotonically varying equation of state parameter should be nothing but such a variation, inevitably altering the pressure. In the discussion it was therefore pointed out that a less chaotic way to deal with the whole business of parameters is to require a constant equation of state parameter ω. **This was found to be an important clarification done by the present work** and brings the discussion to the next major result:

- It was found that requiring a homogeneous equation of state with the above assumption for $\zeta(\rho)$ did not allow for any other equation of state parameter than $\omega = -1$ in order to account for observations (again refer to figure 4.5). As before one then had to choose $\zeta \sim \rho$ along with a negative value for the viscosity parameter $B = -35$ km /sMpc, which corresponds to $\zeta_0 = 4.1 \cdot 10^7$ Pa s. This is not necessarily easily reconcilable with thermodynamics, which in general requires a non-negative entropy production for a system as a whole.

If the functional form of $\omega(\rho)$ had been chosen different from that of $\zeta(\rho)$, there would have been no degeneracy between the parameters α and ζ_0. That is to say; if choosing a more complex form of ζ, one could perhaps explain the Hubble parameter measurements with another constant equation of state parameter than $\omega = -1$.

Specifying a last important result obtained for the one-fluid case: In agreement with (Zimdahl, 1996), the models here investigated seem to suggest that **a one-component cosmological fluid with a homogeneous equation of state cannot be non-viscous in order to account for Hubble parameter observations.** Even if a good motivation for investigation, it seems, however, that it is not clear whether or not the Zimdahl formula is able to explain all of the observed viscosity. It was therefore speculated in whether the viscosity could be accounted for on other grounds, like adding an interaction term in the energy equation, or perhaps other phenomena like particle production.

First order modification to pressure: For the viscosity-modified pressure to be far less (1%) than the pressure of the universe (and thus within the boundaries of a first order perturbation theory) the bulk viscosity can be as large as

$$|\zeta| \sim 10^6 \text{ Pa s}.$$

This is in perfect agreement with what was found by testing how much viscosity could be added to the equations before a difference was seen from the plots by the bare eye - both for the two-component non-viscous fluid and for the non-viscous one-fluid.

Cause of the viscosity: A few candidates were investigated in section 4.9. They seemed to all lie within the positive half of the above boundary. The most conclusive candidates that were found were

- **Photons in interaction with matter:** From the Weinberg formula (or the Zimdahl formula) was found that
$$\zeta_P \sim 10^3 \text{ Pa s}$$
- **Galaxy collisions:** Applying the BGK formula one found for galaxies in collisional interaction that
$$\zeta_G \sim 1 \text{ Pa s}$$

Other candidates were investigated as well, but without any conclusive estimates. The second candidate, ζ_G, is quite small, but could not be any greater in order for the mean free time to be small enough. It was set to be 1/10th of the Hubble age, so that the theory should be valid on kinetic theoretical grounds[1].

Since no candidate with negative viscosity was found[2] for the two-component fluid, one might on these grounds set the following loose bounds:

$$\boxed{10^3 \text{Pa s} \leq \zeta_0 \leq 10^6 \text{Pa s}} \tag{5.3}$$

for the viscosity of the present day universe.

5.2 Recommendations for further work

Interaction terms in the energy equations for the fluid components has not been under investigation in the present work. This is done in e.g. (Brevik I. and Timoshkin, 2014). Also, a two-component fluid was assumed, and the radiation component was neglected. The general formulae is already there, ready to be implement with a third component; radiation. Radiation was thought to make up such a small fraction of the energy density that it would not contribute noteworthy to the overall evolution of the Hubble parameter. Perhaps this is correct, but one must note that the viscosity due to radiation out of equilibrium with pressureless matter was the largest candidate found as cause for viscosity. Thus, in retrospective it seems as if it would have been natural to include it after all. A natural continuation of the present work could therefore be to

- add interaction terms in the energy equations for the different fluid components.
- add another component to the energy density; $\rho \to \rho_m + \rho_\Lambda + \rho_{rad}$ for the multi-component solution.

The present study has sought to lay out some of the foundation for investigating the causes of viscosity. However, it is believed that on this point the surface has barely been scratched, and a lot more could be done to this end. A long term future continuation could therefore be to

[1] How relativistic effects potentially would come into play here has not been investigated. If galaxies at some point have been moving with relativistic velocity this could cause deviating results from what here is found.

[2] This is another point of concern. If knowing more about dark energy, perhaps a candidate could be found. Perhaps the overall viscosity would then be vanishing?

- continue the search for viscosity-generating candidates. This might very likely include extensive derivations of background formulae. For instance it could be an idea to go through the derivation of the BGK-formula (3.110) and establish its validity more rigorously. Perhaps the correct prefactor could be found as well?

What is more, the present study has used a rather simple approach when it comes to comparison of viscosity-including models with the non-viscous models. Although found to be sufficient in the present work, it seems appropriate to at some point

- perform some statistical analysis,

especially if more accurate models are developed and need to be distinguished from each other.

Taking everything a step further one could also apply the constraints suggested in the present work to the formalism developed in papers like e.g. (Brevik and Gorbunova, 2005) and (Brevik, 2013) for the late universe. In this way one could see whether the found boundaries may aid in determining the way the universe will end its days: Perhaps some future singularity can be favoured?

Appendix A

Acronyms

A list of abbreviations used in the work:

EoS: Equation of state

RHS: Right hand side

LHS: Left hand side

Λ**CDM:** standard cosmological model.

p / pp / chap: page / pages / chapter

Appendix B

Christoffel Symbols for the FRW-metric

The explicit formula for the Christoffel symbols is, according to (Grøn and Hervik, 2007), s. 100;

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\delta\alpha} \left(g_{\gamma\delta,\beta} + g_{\beta\delta,\gamma} - g_{\gamma\beta,\delta} \right) \tag{B.1}$$

The FRW-metric may be written in the form

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = -dt^2 + a(t)^2 \gamma_{ij}(x) dx^i dx^j, \tag{B.2}$$

where γ is a diagonal $3x3$ matrix defined such that

$$\gamma_{11} = \frac{1}{1-kr^2} \quad , \quad \gamma_{22} = r^2 \quad , \quad \gamma_{33} = r^2 \sin^2(\theta). \tag{B.3}$$

For the metric tensor itself one may then write

$$g_{00} = -1 \quad , \quad g_{ij} = a^2 \gamma_{ij} \quad , \quad \text{and} \quad g^{00} = 1 \quad , \quad g^{ij} = \frac{1}{a^2} \gamma^{ij}. \tag{B.4}$$

Now it is time to apply (B.1) in calculating the actual Christoffel symbols. In doing so, remember that $g_{\mu\nu}$ is diagonal, and remember that the time derivatives of the spatial components vanish in a co-moving frame of reference. Also, note that the Christoffel symbols are symmetric with respect to interchange of the two lower indices ($\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta}$). A complete list, therefore, should be as follows:

- The upper entry in Γ is time:

$$\Gamma^{0}{}_{\beta\gamma} = \frac{1}{2} g^{00} \left(g_{\beta 0,\gamma} + g_{\gamma 0,\beta} - g_{\beta\gamma,0} \right) = -\frac{1}{2} g_{\beta\gamma,0} \tag{B.5}$$

  which, since $\gamma_{00,0} = 0$, reduces to

$$\Gamma^{0}{}_{ij} = -\frac{1}{2} g_{ij,0} = -a\dot{a}\gamma_{ij} \tag{B.6}$$

- The upper entry in Γ is spatial:

$$\Gamma^{i}{}_{\alpha\beta} = \frac{1}{2} g^{ij} \left(g_{\alpha j,\beta} + g_{\beta j,\alpha} - g_{\alpha\beta,j} \right). \tag{B.7}$$

  This branches into the two following scenarios:

– one of the lower entries in Γ is time:

$$\Gamma^{i}{}_{0k} = \frac{1}{2} g^{ij} \left(g_{0j,k} + g_{kj,0} - g_{0k,j} \right) = \frac{1}{2} g^{ij} g_{kj,0} = \frac{\dot{a}}{a} \delta^{i}{}_{k} \tag{B.8}$$

– all entries in Γ are spatial:

$$\Gamma^{i}{}_{lk} = \frac{1}{2} g^{ij} \left(g_{lj,k} + g_{kj,l} - g_{lk,j} \right) = \frac{1}{2} \gamma^{ij} \left(\gamma_{lj,k} + \gamma_{kj,l} - \gamma_{lk,j} \right) \tag{B.9}$$

From equations (B.6), (B.8) and (B.9) the Christoffel symbols are computed and found as hence listed:

$$\boxed{\begin{array}{l}
\Gamma^{0}{}_{11} = -\dfrac{a\dot{a}}{1-kr^2} \quad , \quad \Gamma^{0}{}_{22} = -a\dot{a}r^2 \quad , \quad \Gamma^{0}{}_{33} = -a\dot{a}r^2 \sin^2\theta \\
\Gamma^{1}{}_{10} = \Gamma^{1}{}_{01} = \Gamma^{2}{}_{20} = \Gamma^{2}{}_{02} = \Gamma^{3}{}_{30} = \Gamma^{3}{}_{03} \dfrac{\dot{a}}{a} \\
\Gamma^{1}{}_{11} = \dfrac{kr}{1-kr^2} \quad , \quad \Gamma^{1}{}_{22} = -r(1-kr^2) \quad , \quad \Gamma^{1}{}_{33} = -r(1-kr^2)\sin^2\theta \\
\Gamma^{2}{}_{21} = \Gamma^{2}{}_{12} = \dfrac{1}{r} \quad , \quad \Gamma^{2}{}_{33} = -\sin\theta\cos\theta \\
\Gamma^{3}{}_{31} = \Gamma^{3}{}_{13} = \dfrac{1}{r} \quad , \quad \Gamma^{3}{}_{32} = \Gamma^{3}{}_{23} = \cot\theta
\end{array}} \tag{B.10}$$

Also, to ease the look-up work, let me restate the three general equations from which the above results followed; equations (B.6),(B.8) and (B.9):

$$\boxed{\begin{array}{l}
\Gamma^{0}{}_{ij} = -a\dot{a}\gamma_{ij} \\
\Gamma^{i}{}_{0k} = \dfrac{\dot{a}}{a}\delta^{i}{}_{k} \\
\Gamma^{i}{}_{lk} = \dfrac{1}{2}\gamma^{ij}\left(\gamma_{lj,k} + \gamma_{kj,l} - \gamma_{lk,j}\right)
\end{array}} \tag{B.11}$$

Appendix C

Classical analogues for some of the tensors in the text

C.1 Classical decomposition

This section is based on chapter 14 in (Grøn and Hervik, 2007). Consider the total time derivative of a velocity field $\mathbf{v}(x^i, t)$ in a cartesian coordinate system. The acceleration of a particle along a trajectory $x^i(t)$ with a velocity $v^j(t)$ over that trajectory, is

$$\frac{D}{Dt}\mathbf{v} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}, \tag{C.1}$$

where the first part expresses the change in velocity due to time (*local derivative*), and the second term expresses the change in velocity due to change of location (*convective derivative*). The convective derivative can also be expressed in matrix form;

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = M\mathbf{v} \tag{C.2}$$

where M is a matrix that must be defined such that

$$M = \begin{bmatrix} \partial_x v^x & \partial_y v^x & \partial_z v^x \\ \partial_x v^y & \partial_y v^y & \partial_z v^y \\ \partial_x v^z & \partial_y v^z & \partial_z v^z \end{bmatrix} \tag{C.3}$$

Further on, this matrix can be serparated into a symmetric and an anti-symmetric part. Defining

$$\begin{aligned} (\theta^i{}_j) &= \frac{1}{2}\left(M + M^T\right) \qquad \textbf{Symmetric part} \\ (\omega^i{}_j) &= \frac{1}{2}\left(M - M^T\right) \qquad \textbf{Anti-symmetric part} \end{aligned} \tag{C.4}$$

one can rewrite

$$M = v^i{}_j = (\theta^i{}_j) + (\omega^i{}_j) \tag{C.5}$$

ω is called the **vorticity tensor** or **rotation tensor**. θ_{ij} is called the **expansion tensor**, and can further be split into two parts

$$\theta^i{}_j = \frac{1}{3}\theta\delta^i{}_j + \sigma^i{}_j, \tag{C.6}$$

where

$$\begin{aligned}\theta &= Tr(M) = v^i{}_i \\ (\sigma^i{}_j) &= \frac{1}{2}\left(M + M^T\right) - \frac{1}{3}\delta^i{}_j Tr(M)\end{aligned}, \tag{C.7}$$

and $Tr(M)$ means the trace of M. All in all, it is possible to rewrite the convective derivative to

$$\boxed{v_{i,j} = \frac{1}{3}\theta\delta_{ij} + \sigma_{ij} + \omega_{ij}.} \tag{C.8}$$

Also, the total derivative (C.1)

$$\frac{Dv^i}{Dt} = v^j v^i{}_{,j} + \frac{\partial v^i}{\partial t} \tag{C.9}$$

then becomes

$$\frac{Dv^i}{Dt} = v^j\left(\frac{1}{3}\theta\delta^i_j + \sigma^i_j + \omega^i_j\right) + \frac{\partial v^i}{\partial t} \tag{C.10}$$

C.2 Relativistic decomposition

The decomposition in the previous section can be straight forward generalized to four-dimensional space time. Consider now a four-velocity $U(x^\mu)$ in space-time with metric $g_{\mu\nu}$. The four-acceleration A is given by

$$A = \frac{dU}{d\tau} \to A_\mu = U_{\mu;\nu}U^\nu \equiv \dot{U}_\nu \tag{C.11}$$

The projection operator

$$h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu \tag{C.12}$$

projects tensors onto the plane of simultaneity orthogonal to the four-velocity U^μ. With this tool, one finds that the relativistic decomposition analogous to what was found in the classical case, becomes

$$\begin{aligned}\theta &= U^\mu{}_{;\mu} \\ \sigma_{\alpha\beta} &= \frac{1}{2}\left(U_{\mu;\nu} + U_{\nu;\mu}\right)h^\mu{}_\alpha h^\nu{}_\beta - \frac{1}{3}U^\mu{}_{;\mu}h_{\alpha\beta} \\ \omega_{\alpha\beta} &= \frac{1}{2}\left(U_{\mu;\nu} - U_{\nu;\mu}\right)h^\mu{}_\alpha h^\nu{}_\beta\end{aligned} \tag{C.13}$$

All in all, the covariant derivative of the four-velocity can then be written as

$$\boxed{U_{\alpha;\beta} = \frac{1}{3}\theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} - \dot{U}_\alpha U^\beta} \tag{C.14}$$

Appendix D

Viscosity generation by perfect fluids

The following is a self-deviced attempt at proving that an expanding universe consisting of many ideal fluids cannot be seen as ideal when employing a phenomenological one-fluid description. This is consistent with (Zimdahl, 1996), for which this proof perhaps could be considered a simplified and less rigorous version.

The energy equation, resulting from $T^{\mu\nu}{}_{;\nu}$, for a universe with bulk viscosity ζ reads

$$a\partial_a\rho + 3(\rho + P) = 3\zeta\theta \tag{D.1}$$

Consider now a fluid ρ that is assumed to have equation of state

$$P = \omega\rho. \tag{D.2}$$

In this case, the ansatz

$$\rho(a) = \rho_0 a^{-3(1+\omega)} \tag{D.3}$$

solves equation (D.1) with a vanishing RHS ($\zeta = 0$).

Now; assume further on that it is known that the fluid ρ in itself consists of several components ρ_i of perfect fluids, and that ω in equation (D.2) is unknown, whereas ω_i of all the components are known. To keep the argument simple, only two components will be assumed. Let

$$\rho = \rho_1 + \rho_2, \tag{D.4}$$

where, for the two components, equation (D.1) is fulfilled as follows;

$$\begin{aligned} a\partial_a\rho_1 + 3(1+\omega_1)\rho_1 = 0 \\ a\partial_a\rho_2 + 3(1+\omega_2)\rho_2 = 0 \end{aligned} \tag{D.5}$$

In other words; the two components are perfect fluids with pressures $P_i = \omega_i \rho_i$. Interaction and reaction terms are not considered. Inserting (D.4) into (D.1) one finds

$$\begin{aligned}
&\sum_i a\partial_a \rho_i + 3(1+\omega_i)\rho_i = 0 \\
&\to a\partial_a \rho + 3(\rho + \omega_1\rho_1 + \omega_2\rho_2) = 0 \\
&\to a\partial_a \rho + 3(\rho + \omega_1\rho_1 + \omega_2\rho_2 + \omega\rho) = 3\omega\rho \\
&\to a\partial_a \rho + 3(1+\omega)\rho = 3\omega\rho - 3\omega_1\rho_1 - 3\omega_2\rho_2 \\
&\to a\partial_a \rho + 3(1+\omega)\rho = 3\sum_i(\omega - \omega_i)\rho_i
\end{aligned} \tag{D.6}$$

Now; by equation (D.5), the LHS of the last equality above is identically zero. The only non-trivial way this equation can be satisfied for a multi-component fluid is by requiring

$$\omega = \frac{\sum_i \rho_i \omega_i}{\rho}, \tag{D.7}$$

which indeed is nothing but Dalton's law for partial pressures;

$$\omega\rho = \sum_i \rho_i\omega_i \to P = \sum_i P_i \tag{D.8}$$

So; it all hinges on this law. If it is broken, there will be an additional contribution to the pressure balance, which can be interpreted as viscosity.

Homogeneous equation of state: Observe that equation (D.7) cannot hold for ω =const if ω_1 and ω_2 are both constant, but different from each other. From equation (D.3) it is evident that $\omega_1 \neq \omega_2$ implies that ρ_1 evolve different with time compared with ρ_2.

Since for an ideal fluid one has ω=const. [1], this proves that even though ρ_1 and ρ_2 are ideal, $\rho = \rho_1 + \rho_2$ cannot be seen as one effective ideal fluid. The argument should be equally valid for more than two components.

The argument also works the other way around: If requiring that the two components are not ideal fluids, i.e. $\omega_1 \to \omega(\rho_1)$ and $\omega_2 \to \omega_2(\rho_2)$, the fluid as a whole could still end up as a phenomenological ideal fluid. One could construct a scenario in which

$$\omega_1\rho_1 = \omega_2\rho_2 \cdot Const.$$

With

$$\rho_1 = f(x) \cdot \rho_2$$

The only way to assure a constant ω seems (by insertion into (D.7)) to be by choosing

$$\omega_2 = f(x) + 1 \qquad \text{and} \qquad \omega_1 = (f(x)+1)f(x)$$

Otherwise, for a phenomoenological one-fluid consisting of two components, $\omega \neq$const. Thus; although a mathematical possibility, it occurs a priory unlikely that the cosmological fluid actually consists of non-ideal components that happen to become ideal when modelled as a one-fluid.

[1] At least this clearly seems to be the requirement made for an ideal fluid in (Cardone et al., 2006).

D.1 Functional form and interpretation

Now, equating this term with $3\zeta\theta$ one finds

$$\zeta = \frac{\sum_i (\omega - \omega_i)\rho_i}{\sqrt{3\kappa\rho}} \tag{D.9}$$

Which, for the two-component fluid becomes

$$\zeta = \frac{1}{\sqrt{3\kappa}}\left[\omega\sqrt{\rho} - \omega_1\frac{\rho_1}{\sqrt{\rho}} - \omega_2\frac{\rho_2}{\sqrt{\rho}}\right] \tag{D.10}$$

This path is not followed any longer here, but could contain potential for further studies. What does not seems so correct with it, is that it contains an evolution $\sim \sqrt{\rho}$. However, if ρ_1 and ρ_2 have sufficiently different functional forms, perhaps the overall functional form of ζ can become interesting.

Appendix E

Forth order Runge-Kutta method implemented in Matlab

E.1 Main Script (plotting and calling the numerical routine)

The main Matlab script used in order to reproduce the numerical solutions found in (Wang and Meng, 2014).

```
% %%Plotting formula (10) in Wang-Meng article
% using 4th Order Runge-Kutta method.
%Definitions:
%E_pos corresp to alpha=3/2 and E_neg corresp to alpha=1/2


clear all;
%Least upper and lower bounds and stepsize dz:
ub=2.4; lb=-1; dz=0.0001;
%Initializing constants:
%for alpha=3/2:
A_pos=0.427;    %Dimensionless
B_pos=1.603;    %km/(s*Mpc)

%for alpha=1/2
A_neg=0.427;    %Dimensionless
B_neg=0.867;    %km/(s*Mpc)
H_0=70;     %km/(s*Mpc)
%For lambdaCDM model:
A=(3/2)*0.287;  %Dimensionless
H_std=69.32;    %H-values used for the standard lambdaCDM model.

%Using the function Hubble_generator.m to solve the equation and generate
%output:
[E_pos,E_neg,z,E]=Hubble_generator(ub,lb,dz,A_pos,B_pos,A_neg,B_neg,A,H_0);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Arrays of experimental data with corresponding z-values. Since the values
%are from different sources, they are in separate arrays.
%Units: km/(s*Mpc)
H_37=[69 83 77 95 117 168 177 140 202]./H_std;
z_37=[0.100 0.170 0.270 0.400 0.900 1.300 1.430 1.530 1.750];
```

```
H_38=[97 90]./H_std;
z_38=[0.480 0.880];

H_39=[75 75 83 104 92 105 125 154]./H_std;
z_39=[0.179 0.199 0.352 0.593 0.680 0.781 0.875 1.037];

H_40=[76.3]./H_std;
z_40=[0.35];

H_41=[69.0 68.6 72.9 88.8]./H_std;
z_41=[0.07 0.12 0.20 0.28];

H_42=[82.6 87.9 97.3]./H_std;
z_42=[0.44 0.60 0.73];

H_43=[224.0]./H_std;
z_43=[2.30];

H_44=[226]./H_std;
z_44=[2.36];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%PLOTTING

figure;
%Plotting Sutherland (red) and Chapman (blue):
plot(z,E_pos,'red',z,E_neg,'blue');
grid off;
hold on;
plot(z_37, H_37,'magenta *');
plot(z_38, H_38, 'red *');
plot(z_39, H_39, 'green *');
plot(z_40, H_40, 'yellow *');
plot(z_41, H_41, 'blue *');
plot(z_42, H_42, 'black *');
plot(z_43, H_43, 'cyan *');
plot(z_44, 4, 'magenta +');
%Making axis and title + div pimping
x=-1:0.05:2.5;
line(x,1);
y=-2:0.001:5;
line(0,y);
title('Sutherland (alpha=3/2) and Chapman (alpha=1/2) viscosity');
xlabel('z');
ylabel('E(z)');
text(-0.9,3.8,'B=1.603 for Sutherland (red)');
text(-0.9,3.6,'B=0.867 for Chapman (blue)');
TeXString = texlabel('H_{0}=70');
text(-0.9,3.4,TeXString);
axis([-1 2.5 0 4]);
hold off;



%Plotting the relative difference of the standard lambdaCDM model and the
%new viscous models:
```

```
%Sutherland:
H=E*H_std;
dH=E_pos*H_0-H;
r=dH./H;
figure;
plot(z,r);
%line(z,0);
%line(0,z);
title('Sutherland rel. diff.');
xlabel('z');
ylabel('dH/H');
text(1,0.1,'B=10');
axis([-1 2 -0.08 0.15]);


%Chapman:
H=E*H_std;
dH=E_neg*H_0-H;
r=dH./H;
figure;
plot(z,r);
%line(z,0);
%line(0,z);
title('Chapman rel. diff.');
xlabel('z');
ylabel('dH/H');
text(1.2,0.6,'B=20');
axis([-1 2 -0.1 0.8]);
```

E.2 The function

The function Hubble_generator.m used to implement the RK4 method:

```
%A function that calculates the E-values of LambdaCDM-model and the two new
%viscous models by use of RK4, and returns 4 arrays (E_pos,E_neg,z and E) of
%values:

function [E_pos,E_neg,z,E]=Hubble_generator(ub,lb,dz,A_pos,
B_pos,A_neg,B_neg,A,H_0)

%Since the initial condition we have is at E(z=0), we ensure that that 0 is
%somewhere in the z-array (id est; z(negLength+1)=0):
negLength=ceil(abs(lb)/dz);
posLength=ceil(abs(ub)/dz);
z_0=-dz*negLength;
z=z_0:dz:(dz*posLength);

%Vectors of E-values:
E_pos=zeros(1,length(z));           % alpha=1/2
E_pos(negLength+1)=1;               %Initial condition

E_neg=zeros(1,length(z));           % alpha=-1/2
E_neg(negLength+1)=1;               %Initial condition

E=zeros(1,length(z));
E(negLength+1)=1;                   %Initial condition

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Implementing the RK4 method itself

%For positive z-values first:
for i=(negLength+1):length(z)-1

    %for E_pos (Sutherland, alpha=3/2):
    k1=dz*(A_pos*(1+z(i))^2/E_pos(i) - (1+z(i))^(1/2)*B_pos/H_0);
    k2=dz*(A_pos*(1+z(i)+dz/2)^2/(E_pos(i)+k1/2)
    - ((1+z(i)+dz/2)^(1/2))*B_pos/H_0);
    k3=dz*(A_pos*(1+z(i)+dz/2)^2/(E_pos(i)+k2/2)
    - ((1+z(i)+dz/2)^(1/2))*B_pos/H_0);
    k4=dz*(A_pos*(1+z(i)+dz)^2/(E_pos(i)+k3)
    - ((1+z(i)+dz)^(1/2))*B_pos/H_0);
    E_pos(i+1)=E_pos(i)+k1/6+k2/3+k3/3+k4/6;

    %for E_neg (Chapman, alpha=1/2):
    k1=dz*(A_neg*(1+z(i))^2/E_neg(i) - ((1+z(i))^(-1/2))*B_neg/H_0);
    k2=dz*(A_neg*(1+z(i)+dz/2)^2/(E_neg(i)+k1/2)
    - ((1+z(i)+dz/2)^(-1/2))*B_neg/H_0);
    k3=dz*(A_neg*(1+z(i)+dz/2)^2/(E_neg(i)+k2/2)
    - ((1+z(i)+dz/2)^(-1/2))*B_neg/H_0);
    k4=dz*(A_neg*(1+z(i)+dz)^2/(E_neg(i)+k3)
    - ((1+z(i)+dz)^(-1/2))*B_neg/H_0);
    E_neg(i+1)=E_neg(i)+k1/6+k2/3+k3/3+k4/6;

    %for the standard lambdaCDM-model:
    k1=dz*A*(1+z(i))^2/E(i);
    k2=dz*A*(1+z(i)+dz/2)^2/(E(i)+k1/2);
```

```
    k3=dz*A*(1+z(i)+dz/2)^2/(E(i)+k2/2);
    k4=dz*A*(1+z(i)+dz)^2/(E(i)+k3);
    E(i+1)=E(i)+k1/6+k2/3+k3/3+k4/6;


end
%For negative values of z:
for i=(negLength+1):-1:2
    %for E_pos (Sutherland, alpha=3/2):
    k1=dz*(A_pos*(1+z(i))^2/E_pos(i) - ((1+z(i))^(1/2))*B_pos/H_0);
    k2=dz*(A_pos*(1+z(i)-dz/2)^2/(E_pos(i)-k1/2)
    - ((1+z(i)-dz/2)^(1/2))*B_pos/H_0);
    k3=dz*(A_pos*(1+z(i)-dz/2)^2/(E_pos(i)-k2/2)
    - ((1+z(i)-dz/2)^(1/2))*B_pos/H_0);
    k4=dz*(A_pos*(1+z(i)-dz)^2/(E_pos(i)-k3)
    - ((1+z(i)-dz)^(1/2))*B_pos/H_0);
    E_pos(i-1)=E_pos(i)-k1/6-k2/3-k3/3-k4/6;

    %for E_neg (Chapman, alpha=1/2):
    k1=dz*(A_neg*(1+z(i))^2/E_neg(i) - ((1+z(i))^(-1/2))*B_neg/H_0);
    k2=dz*(A_neg*(1+z(i)-dz/2)^2/(E_neg(i)-k1/2)
    - ((1+z(i)-dz/2)^(-1/2))*B_neg/H_0);
    k3=dz*(A_neg*(1+z(i)-dz/2)^2/(E_neg(i)-k2/2)
    - ((1+z(i)-dz/2)^(-1/2))*B_neg/H_0);
    k4=dz*(A_neg*(1+z(i)-dz)^2/(E_neg(i)-k3)
    - ((1+z(i)-dz)^(-1/2))*B_neg/H_0);
    E_neg(i-1)=E_neg(i)-k1/6-k2/3-k3/3-k4/6;

    %for the standard lambdaCDM-model:
    k1=dz*A*(1+z(i))^2/E(i);
    k2=dz*A*(1+z(i)-dz/2)^2/(E(i)-k1/2);
    k3=dz*A*(1+z(i)-dz/2)^2/(E(i)-k2/2);
    k4=dz*A*(1+z(i)-dz)^2/(E(i)-k3);
    E(i-1)=E(i)-k1/6-k2/3-k3/3-k4/6;
end
```

Appendix F

Script used for plotting solutions of Friedmann's first Equation

F.1 Main script

Typical script used for plotting the Hubble parameter as function of redshift.

```
%%CORRECTING WANG MENG EQ 10 WITH THE FOLLOWING:
%%1) En.eq. (1/3)\theta d_a\rho+\theta(\rho+P)=\zeta\theta^2
%%2)zeta=const!
%%3)Solving FIRST FRIEDMANN EQ.




clear all;
%Least upper and lower bounds and stepsize dz:
ub=2.4; lb=-1; dz=0.001;
%Initializing constants:
Omega_m=0.3183;     %Dimensionless
Omega_Lambda=0.6914;

H_0=67.77;      %km/(s*Mpc)
%For lambdaCDM model:
H_std=67.77;    %H-values used for the standard lambdaCDM model.

B=1;    %%km/(s*Mpc)[B=8*pi()*G*zeta ]
%Using the function Hubble_generator.m to solve the equation and generate
%output:
[E_ViscPos,E_ViscNeg,z,E]=
HubGen_Fr1_ConstVisc(ub,lb,dz,Omega_m,Omega_Lambda,B,H_0);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Arrays of experimental data with corresponding z-values. Since the values
%are from different sources, they are in separate arrays.
%Units: km/(s*Mpc)
H_37=[69 83 77 95 117 168 177 140 202]./H_0;
z_37=[0.100 0.170 0.270 0.400 0.900 1.300 1.430 1.530 1.750];

H_38=[97 90]./H_0;
z_38=[0.480 0.880];
```

```
H_39=[75 75 83 104 92 105 125 154]./H_0;
z_39=[0.179 0.199 0.352 0.593 0.680 0.781 0.875 1.037];

H_40=[76.3]./H_0;
z_40=[0.35];

H_41=[69.0 68.6 72.9 88.8]./H_0;
z_41=[0.07 0.12 0.20 0.28];

H_42=[82.6 87.9 97.3]./H_0;
z_42=[0.44 0.60 0.73];

H_43=[224.0]./H_0;
z_43=[2.30];

%H_44=[226]./H_0;
%z_44=[2.36];

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%PLOTTING


%Plotting
figure;
plot(z,E_ViscPos,z,E_ViscNeg,z,E, 'black');
legend('+B','-B','B=0','Location','southeast');
grid off;
hold on;
plot(z_37, H_37,'cyan . ');
plot(z_38, H_38, 'red .');
plot(z_39, H_39, 'green .');
plot(z_40, H_40, 'yellow .');
plot(z_41, H_41, 'blue .');
plot(z_42, H_42, 'black .');
plot(z_43, H_43, 'magenta .');
%plot(z_44, 4, 'cyan .');
%Making axis and title + div pimping
x=-1:0.05:2.5;
line(x,1);
y=-2:0.001:5;
line(0,y);
title('Const Visc.');
xlabel('z');
ylabel('E(z)');
TeXString = texlabel('zeta=const');
text(-0.9,3.8,TeXString);
text(-0.9,3.6,'B=1');

TeString = texlabel('H_{0}=67.77');
text(-0.9,3.4,TeString);
axis([-1 2.5 0 4]);


%Plotting the relative difference of the standard lambdaCDM model and the
%new viscous models:

% H=E.*H_std;
```

```
% dH=E_ViscPos.*H_0-H;
% r=dH./H;
% figure;
% plot(z,r);
% line(z,0);
% line(0,z);
% title('Chapman rel. diff.');
% xlabel('z');
% ylabel('dH/H');
% text(1.2,0.6,'B=20');
% axis([-1 2 -0.1 0.8]);
```

F.2 The function

The function called by main script and used in evaluating

```
%%This function evaluates E(z) with CONST VISCOSITY ZETA
%
%EXACT SOLUTIONS
%

function[E_ViscPos,E_ViscNeg,z,E]=
HubGen_Fr1_ConstVisc(ub,lb,dz,Omega_m,Omega_Lambda,B,H_0)

%Since the initial condition we have is at E(z=0), we ensure that  0 is
%somewhere in the z-array (id est; z(negLength+1)=0):
negLength=ceil(abs(lb)/dz);
posLength=ceil(abs(ub)/dz);
z_0=-dz*negLength;
z=z_0:dz:(dz*posLength);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Calculating the viscosity-corrected E(z):
C_Visc=1/sqrt(Omega_Lambda+Omega_m)-1-(B/
(sqrt(Omega_Lambda)*H_0))*atanh(sqrt(Omega_m/Omega_Lambda+1));
rho_O=(Omega_Lambda+Omega_m.*(1+z).^3).^(1/2);
E_ViscPos=rho_O.*(1+(B./((Omega_Lambda).^(1/2).*H_0).*atanh(rho_O./
(Omega_Lambda).^(1/2)))+C_Visc);


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Calculating E(z) without viscosity:
C=1/sqrt(Omega_Lambda+Omega_m)-1;
rho_O=(Omega_Lambda+Omega_m.*(1+z).^3).^(1/2);
E=rho_O.*(1+C);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Calculating the viscosity-corrected E(z):
C_Visc=1/sqrt(Omega_Lambda+Omega_m)-1-(-B/
(sqrt(Omega_Lambda)*H_0))*atanh(sqrt(Omega_m/Omega_Lambda+1));
rho_O=(Omega_Lambda+Omega_m.*(1+z).^3).^(1/2);
E_ViscNeg=rho_O.*(1+(-B./((Omega_Lambda).^(1/2).*H_0).*atanh(rho_O./
(Omega_Lambda).^(1/2)))+C_Visc);
```

Appendix G

Alternative ansatz for the energy equation

While playing around with the equations, another solution to the energy equation was found. However, it is not known whether it has any physical interpretation.

The equation in the viscosity-free case reads

$$a\partial_a \rho + 3\left(\rho + P\right) = 0 \tag{G.1}$$

with solution

$$\rho(a) = \sum_i \rho_{0i} a^{-3(1+\omega_i)}. \tag{G.2}$$

Then, by the superposition principle,

$$\rho(a) = \sum_i \rho_{0i} a^{-3(1+\omega_i)} \tag{G.3}$$

is also a solution. However, it is also possible to construct another solution:

$$\begin{aligned}\rho(a) =& \Pi_i^N \rho_{0i} a^{-3(1+\omega_i)} = \rho_{01}\rho_{02} \cdot \ldots \cdot \rho_{0N} a^{-3(1+\omega_1)} a^{-3(1+\omega_2)} \cdot \ldots \cdot a^{-3(1+\omega_N)} \\ &\equiv \rho_0^{TOT} a^{-3(N+\omega_{TOT})}\end{aligned} \tag{G.4}$$

where $\rho_{TOT} = \Pi_i^N \rho_{0i}$ and $\omega_{TOT} = \sum_i^N \omega_i$. Now; inserting this into the energy equation (G.1) one may determine P such that the constructed solution solves the equation. One finds

$$a\partial_a \left(\rho_0^{TOT} a^{-3(1+\omega_{TOT})}\right) + 3\left(P + \rho_0^{TOT} a^{-3(1+\omega_{TOT})}\right) = 0 \tag{G.5}$$

which solves to give

$$P = (N + \omega_{TOT} - 1)\,\rho \tag{G.6}$$

G.1 Interpretation

The reason for including this solution in the present work was to show that there are other possibilities than

$$\rho = \sum_i \rho_i \tag{G.7}$$

However, in the present work the discussion ends here, since it has not been investigated any further whether the new ansatz

$$\rho(a) = \Pi_i^N \rho_{0i} a^{-3(1+\omega_i)} \tag{G.8}$$

has any physical interpretation. (G.7), labelled as assumption $0'$ in the text (4.24), was assumed as the natural choice and thus used throughout the work.

Appendix H

Alternative justification of the BGK-formula

This appendix is based on a personal note from professor Johan Høye to undersigned. The aim is to derive an approximative formula for the bulk viscosity of a two-component expanding gas.

Assuming adiabatic motion one finds

$$TV^{\gamma-1} = \text{const} \tag{H.1}$$

Forming a differential one thus has

$$V^{\gamma-1}dT + (\gamma-1)TV^{\gamma-2}dV = 0 \quad \rightarrow \quad \frac{dT_i}{dV} = -(\gamma_i - 1)\frac{T}{V} \tag{H.2}$$

Where $T_i \approx T$ has been used. For $i = 1$ and $i = 2$ this should mean that

$$\dot{T}_1 = -(\gamma_1 - 1)\frac{T}{V}\dot{V} \quad \text{and} \quad \dot{T}_2 = -(\gamma_2 - 1)\frac{T}{V}\dot{V} \tag{H.3}$$

which should give the rates of temperature change due to the volume expansion. If the temperature difference $\Delta T = T_2 - T_1$ is large, the change in the temperatures should happen faster. This suggests incorporating a term proportional to ΔT. The overall temperature change becomes

$$\frac{d\Delta T}{dt} = \dot{T}_2 - \dot{T}_1 \rightarrow d\Delta T = -(\gamma_2 - \gamma_1)\frac{T}{V}\dot{V} - \frac{1}{\tau}\Delta T \tag{H.4}$$

where the proportionality constant $1/\tau$ is to be determined. This must give

$$\Delta\dot{T} = -(\gamma_2 - \gamma_1)\frac{T}{V}\dot{V} - \frac{1}{\tau}\Delta T \tag{H.5}$$

Where $\gamma = c'_p/c'_V$ is the adiabatic constant and c'_i is (as before) defined to be heat capacity per particle when variable i is kept constant. For stationary conditions ($\Delta\dot{T} = 0$) this gives

$$\Delta T = -\tau(\gamma_2 - \gamma_1)\frac{T}{V}\dot{V} \tag{H.6}$$

Now assume that one has two interacting fluids in expansion. Considering the energy exchange between the two and requiring that energy be conserved one finds

$$\Delta E_2 - \Delta E_1 = c_2' N_2 (T_2 - T) c_1' N_1 (T_1 - T) = 0. \tag{H.7}$$

The mean free time for particle of e.g. type 1 to collide is

$$\tau_1 = \frac{\lambda_1}{v} = \frac{1}{\sigma v n_2} = \frac{V}{\sigma v N_2} \tag{H.8}$$

where σ is the collisional cross-section. Thus the number of collisions per time should be

$$R = \frac{N_1}{\tau_1} = \sigma v \frac{N_1 N_2}{V} \tag{H.9}$$

which is symmetric in 1 and 2, just like it should be. Let $\Delta K = k_B \Delta T$ be the amount of energy transferred in each collision. The number of collisions then required to obtain equilibrium is

$$M = \frac{\Delta E_2}{\Delta K} = \frac{c_2' N_2 (T_2 - T)}{\Delta K} = \frac{c_2'}{k_B} N_2 \frac{T_2 - T}{T_2 - T_1} \tag{H.10}$$

To go on, a relation between T, T_1 and T_2 is needed. The equilibrium temperature is defined such that

$$c_{1v}' N_1 T_1 + c_{2v}' N_2 T_2 \equiv (c_{1v}' N_1 + c_{2v}' N_2) T \tag{H.11}$$

and thus

$$T_1 - T = \frac{c_{2v}' N_2}{c_{1v}' N_1 + c_{2v}' N_2} \Delta T \quad \text{and} \quad T_2 - T = \frac{c_{1v}' N_1}{c_{1v}' N_1 + c_{2v}' N_2} \Delta T. \tag{H.12}$$

Since M is the number of collisions, and R is the collision rate, the relaxation time τ for the system as a whole should be given by

$$\tau = \frac{M}{R}. \tag{H.13}$$

Inserting this and also (H.12)into (H.10) one finds that

$$M = \frac{c_{2v}' c_{1v}' N_1 N_2}{k_B (c_{1v}' N_1 + c_{2v}' N_2)} = \tau R = \tau \sigma v \frac{N_1 N_2}{V}. \tag{H.14}$$

From this the relaxation time, or mean free time of the system as a whole is found to be

$$\tau = \frac{c_{1v} c_{2v}}{K} \frac{V}{c_{1v} N_1 + c_{2v} N_2} \frac{1}{\sigma v} \approx \frac{1}{n \sigma v}, \tag{H.15}$$

as in the usual one gas scenario. This was found by using that $c_{1v}' \sim c_{2v}' \sim k_B$ and by the definition

$$n \equiv n_1 + n_2. \tag{H.16}$$

Because of the temperature differences between the two gas components, the pressure will deviate from the equilibrium pressure P_e as the gas expands. From the ideal gas law one finds the difference to be

$$\Delta P = P - P_e = k_B N_1 (T_1 - T) \frac{1}{V} + k_B N_2 (T_2 - T) \frac{1}{V} \overset{(H.12)}{=} -\frac{k_B}{V} N_1 N_2 \frac{c_{2v}' - c_{1v}'}{c_{1v}' N_1 + c_{2v}' N2} \Delta T \tag{H.17}$$

With equation (H.6) for ΔT one finds

$$\Delta P = k\frac{1}{V}N_1N_2\frac{c'_{2v}-c'_{1v}}{c'_{1v}N_1+c'_{2v}N_2}\tau(\gamma_2-\gamma_1)\frac{T}{V}\dot{V} \approx -kT\tau\frac{N_1N_2}{c'_{1v}N_1+c'_{2v}N_2}\frac{(\gamma_2-\gamma_1)^2}{V}\frac{\dot{V}}{V} \tag{H.18}$$

where again $c'_{1v} \sim c'_{2v} \sim k_B$ was used, and also

$$c_{2v} - c_{1v} = k_B\frac{\gamma_1-\gamma_2}{(\gamma_2-1)(\gamma_1-1)} \approx -k_B(\gamma_2-\gamma_1).$$

One finds

$$\Delta P = -\tau kT(\gamma_2-\gamma_1)^2\frac{n_1n_2}{n_1+n_2}\frac{\dot{V}}{V} \tag{H.19}$$

Now employing that[1]

$$\Delta P = -\theta\zeta \tag{H.20}$$

and using that the volume expansion in co-moving coordinates must be

$$\frac{\dot{V}}{V} = \frac{d(a^3)/dt}{a^3} = 3\frac{\dot{a}}{a} \equiv \theta \tag{H.21}$$

one thus finally obtains

$$\boxed{\zeta = \frac{kT\tau}{3}(\gamma_2-\gamma_1)^2\frac{n_1n_2}{n_1+n_2}} \tag{H.22}$$

which indeed resembles (4.111) found from (Zimdahl, 1996):

$$\zeta = \tau Tk(\omega_2-\omega_1)^2\frac{n_1n_2}{n_1\omega_2+n_2\omega_1} \tag{H.23}$$

The pre-factors are not entirely the same, but they are of the same order of magnitude ($\gamma \sim \omega \sim 1$). Estimating the order of magnitude is precisely what is sought in the present work.

[1]The minus sign follows to make the sign convention used for ζ consistent with the rest of the work.

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