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New Thoughts about an Old Result on Univalent Functions

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1. Prologue.

In the theory of functions of a complex variable there are certain areas which have attracted and challenged a large number of mathematicians. One such area is the theory of univalent functions (or "schlicht" functions, according to German tradition). Restricted to the most known case it deals with functions f, analytic and univalent (one to one) in the open unit disk U, and normalized by $f(0) = 0, f'(0) = 1$. The Taylor expansion around $z = 0$ thus has the form

$$w = f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots , \tag{1.1}$$

and converges in U. When Ludwig Bieberbach in 1916 proved that $|a_2| \leq 2$ and (in a foot-note) conjectured that generally $|a_n| \leq n$ [1] the theory really "took off", and it did not even stop when de Branges 68 years after, greatly helped by Richard Askey, Walter Gautschi and, above all, a Leningrad group of mathematicians, proved the truth of the Bieberbach conjecture [2].

Few things are more useful in mathematics than difficult conjectures (in particular if they have a simple appearence). Throughout the years a large and great theory has been developed, with growth and distortion theorems, coefficient results of different types, results for subfamilies, for generalizations etc. Most important perhaps, is that methods have been created, methods of value far beyond the original intention. Some results have become classical parts of the theory, such as for instance Löwner's parameter method, inspired by thoughts in hydrodynamics, and Schiffer's two variational methods (together with other variational methods). But most results have been small, modest contributions to the theory, individually meaning very little, but together, in virtue of multitude and wide scope, being indispensable in the theory.

The topic of the present paper is such a small result, being of little or no value in itself, but of some pleasure and joy for the one who created it, and hopefully also to the ones who have read it. The result was, at

least indirectly, inspired by Ernst Jacobsthal, who once told a young assistant about univalent functions and in particular about the Bieberbach conjecture. The assistant readily started off to prove it. After several attempts and a lot of reading in books and papers, he realized that it was not all that simple, and put it aside (for a while). He still, however, maintained his interest in the field. One day he asked himself the question: "What is the effect on the second coefficient a_2 to require a_3 to be 0?" The answer was written down, then supported by Jacobsthal, and (surprisingly) approved of by Bieberbach, and turned out to be the very first paper in this field by the young mathematician [10].

43 years later, when the young mathematician had turned into an old mathematician, he looked again at his paper. He got some new thoughts, where things that had happened in the field in the meantime were taken into account. The purpose of the present paper is to tell about these thoughts. But first we need to take a look at the old result.

2. The old result.

Let S denote the family of "normalized, univalent functions", i. e. the family of functions presented in the introduction, and where we have the expansion

$$w = f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots + a_n z^n + \cdots . \tag{2.1}$$

Let S' be the subfamily of S, defined by the additional requirement $a_3 = 0$, and hence with the expansion

$$w = f(z) = z + a_2 z^2 + a_4 z^4 + \cdots + a_n z^n + \cdots . \tag{2.1'}$$

Two questions were asked: a) What is the best upper bound for $|a_2|$ in S'? b) What is the best (largest) number d, such that the disk $|w| \leq d$ is contained in $f(U)$ for all $f \in S'$?

Remark.

If the question b) had been asked for the class S instead of S', the answer had been $1/4$, *the Koebe constant.* Paul Koebe proved, already in 1907, the existence of such a positive constant [5]. The actual value $1/4$ was proved by Bieberbach. The functions (the only ones) for which the nearest boundary point of $f(U)$ has the distance $1/4$ from the origin are the functions

$$w = k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots + nz^n + \cdots \tag{2.2}$$

and its rotations, i. e. the functions $k_\phi(z) := e^{-i\phi} k(e^{i\phi} z)$. These functions are also the only ones for which $|a_2| = 2$. The function (2.2) is

called *the Koebe function* and maps U onto the whole plane, minus a slit from $-1/4$ to ∞ along the negative real axis. A solution to our question b) could then be called a "Koebe result", and the best d-value could be called a "Koebe constant" for the family S'.

The results in the paper [10] are:

$$|a_2| \leq 1. \tag{2.3}$$

The disk

$$|w| \leq \frac{1}{3} \tag{2.4}$$

is the largest disk, centered at the origin, that is contained in all sets $f(U)$, $f \in S'$.

We shall not include the proof here, only mention that it was established by means of the Löwner method, where (among many other things) we have, that for a dense subfamily of S the following formulas hold for the first coefficients:

$$a_2 = -2 \int_0^\infty \kappa(\tau) e^{-\tau} d\tau, \tag{2.5}$$

$$a_3 = 4(\int_0^\infty \kappa(\tau) e^{-\tau} d\tau)^2 - 2 \int_0^\infty \kappa^2(\tau) e^{-2\tau} d\tau, \tag{2.6}$$

where $\kappa(\tau)$ is piecewise continuous and of absolute value 1, $|\kappa(\tau)| = 1$ [7].

The bounds (2.3) and (2.4) are best possible. The extremal functions (and the only ones) are

$$w = f(z) = \frac{z}{1 - z + z^2} \tag{2.7}$$

and its rotations. The function (2.7) maps the unit disk onto the whole plane minus two slits, one going from $-1/3$ along the negative real axis to $-\infty$, the other one going from 1 to ∞ along the positive real axis.

In the paper [10] are also studied k-symmetric functions in S with vanishing third coefficient, i. e. functions

$$w = g_k(z) = z + a_{k+1}^{(k)} z^{k+1} + a_{2k+1}^{(k)} z^{2k+1} + a_{3k+1}^{(k)} z^{3k+1} + \cdots, \tag{2.8}$$

where $a_{2k+1}^{(k)} = 0$. The following sharp bound was found:

$$|a_{k+1}^{(k)}| \leq \sqrt{\frac{2}{k(k+1)}} \tag{2.9}$$

Moreover, the "Koebe constant" for this family was proved to be

$$\left(2+\sqrt{\frac{2k}{k+1}}\right)^{-1/k}. \tag{2.10}$$

The only extremal functions are

$$w=g_k(z)=\frac{z}{\left(1-\sqrt{\frac{2k}{k+1}}z^k+z^{2k}\right)^{1/k}} \tag{2.11}$$

and its rotations.

So far the old result. In the following we shall restrict ourselves to the case $k=1$, and we shall not be aiming at any "Koebe result".

3. The new thoughts.

Let M be an arbitrary number in $[1,\infty]$. An interesting subfamily of S is the one obtained by the additional condition

$$|f(z)|<M \tag{3.1}$$

for all $z\in U$. This family is usually denoted S_M. Two trivial cases: For $M=1$ the family consists of merely one function, $f(z)=z$, whereas $S_\infty=S$. From now on we shall assume that $1<M<\infty$.

The different questions dealt with in the class S are in most cases substantially more difficult to handle in S_M. For the first two coefficients the sharp bounds in S_M are:

$$|a_2|\le 2(1-\frac{1}{M}), \quad [8] \tag{3.2}$$

$$|a_3|\le 1-\frac{1}{M^2} \quad \text{for} \quad M\le e, \tag{3.3a}$$

$$|a_3|\le 2e^{-2\nu}-\frac{4e^{-\nu}}{M}+1+\frac{1}{M^2} \quad \text{for} \quad M\ge e, \tag{3.3b}$$

where $\nu\in[0,1]$ is given by $M^{-1}=\nu e^{-\nu}$. See e. g. [9], and also [4, p. 54].

Apart from rotations there is only one function in S_M with equality in (3.2), namely the function

$$w=Mk^{-1}\left(\frac{k(z)}{M}\right)=z+2(1-\frac{1}{M})z^2+(1-\frac{1}{M})(3-\frac{5}{M})z^3+\cdots. \tag{3.4}$$

There are some scattered results under additional conditions for higher order coefficients, but a sharp upper bound for $|a_n|$ generally seems to be located way beyond mountains of difficulties.

In the present section we shall determine the sharp upper bound for $|a_2|$ in the family S'_M, consisting of all $f \in S_M$ for which the coefficient $a_3 = 0$. The non-triviality we will experience in this invesigation of the very first, presumeably simple coefficient, will throw some light on the impact of the two conditions $|f(z)| < M$ and $a_3 = 0$ on the complexity of problems in the family of functions.

As in the old problem we shall also here use the Löwner formulas, which, however, differ from the formulas (2.5) and (2.6) by having $\log M$ as the upper limit instead of ∞, log being the natural logarithm. We thus have

$$a_2 = -2 \int_0^{\log M} \kappa(\tau) e^{-\tau} d\tau \tag{3.5}$$

and

$$a_3 = 4 \left(\int_0^{\log M} \kappa(\tau) e^{-\tau} d\tau \right)^2 - 2 \int_0^{\log M} \kappa^2(\tau) e^{-2\tau} d\tau \quad . \tag{3.6}$$

To simplify writing we define

$$I_1 := \int_0^{\log M} \kappa(\tau) e^{-\tau} d\tau \tag{3.7}$$

and

$$I_2 := \int_0^{\log M} \kappa^2(\tau) e^{-2\tau} d\tau \quad . \tag{3.8}$$

Membership in S'_M is then equivalent to the relation

$$4I_1^2 = 2I_2 \quad . \tag{3.9}$$

(This form is convenient, since $a_2^2 = 4I_1^2$.)

The equation (3.9) produces right away a trivial upper bound for $|a_2|$, obtained by taking $\kappa(\tau)$ to be 1, which gives us $1 - 1/M^2$ as the sharp upper bound for the right-hand side of (3.9), from which we get

$$|a_2| \leq \sqrt{1 - \frac{1}{M^2}} \quad . \tag{3.10}$$

In spite of the fact that this bound comes from a sharp bound of the right-hand side of (3.9) it does not have to be sharp, the reason being that we

may not always have the possibility of the combination of equality in (3.9) and maximality on the right-hand side. We can immediately find a case when the bound (3.10) is not sharp, by looking at the sharp bound (3.2) for $|a_2|$ in S_M. Since $S'_M \subset S_M$, the sharp upper bound for $|a_2|$ in S'_M must be less than or equal to the bound in S_M. Hence, if

$$\sqrt{1-\frac{1}{M^2}} > 2(1-\frac{1}{M}),$$

which means that $M < 5/3$, the bound (3.10) is not sharp. This raises two questions: a) Is the bound sharp for $M \geq 5/3$? b) How can we find a sharp upper bound in the case $M < 5/3$? We shall see that the answer to the question a) is *yes*:

Proposition 1.
For $M \geq 5/3$ the following sharp bound holds in S'_M:

$$|a_2| \leq \sqrt{1-\frac{1}{M^2}} \quad . \tag{3.10}$$

Proof:

Take first $M = \frac{5}{3}$. In this case the extremal functions in S_M are in S'_M, as seen in (3.4), and the two $|a_2|$-bounds are equal. Hence the statement in Proposition 1 is established for $M = 5/3$.

Next take $M > \frac{5}{3}$. in this case

$$\sqrt{1-\frac{1}{M^2}} < 2(1-\frac{1}{M}). \tag{3.11}$$

In order to prove sharpness we need to find a $\kappa(\tau)$, such that $|2I_1| = \sqrt{1-1/M^2}$. For any $t_0 \in [0, \log M]$ take $\kappa(\tau)$ to be 1 for $\tau \in [0, t_0]$ and -1 for $\tau \in (t_0, \log M]$. Here we have $2I_2 = 1 - 1/M^2$, regardless of the t_0-value. When t_0 increases from 0 to $\log M$, the expression $-2I_1$ decreases from $2(1-1/M)$ to $-2(1-1/M)$. Since (3.11) holds, and since $-2I_1$ is a continuous function of t_0, there are two t_0- values for which

$$|2I_1| = \sqrt{1-\frac{1}{M^2}},$$

(one is such that $-2I_1 = \sqrt{1-1/M^2}$, the other one such that $-2I_1 = -\sqrt{1-1/M^2}$. This proves the existence of functions $\kappa(\tau)$, such that we have equality in (3.10), and the sharpness of the bound is thus established. This concludes the proof of Proposition 1.

The remaining case $M < 5/3$ is more difficult. We shall, as the main tool, use the theorem:

Theorem 2.

Let t be a fixed positive number, $\lambda(\tau)$ be a real, piecewise continuous function in $[0,t]$ and

$$|\lambda(\tau)| \leq e^{-\tau}. \tag{3.12}$$

Next, let K be a real number in $[0,(1-e^{-2t})/2]$. If

$$\int_0^t \lambda^2(\tau)d\tau = K, \tag{3.13}$$

then the following sharp bound holds:

$$|\int_0^t \lambda(\tau)d\tau| \leq V(K), \tag{3.14}$$

where $V(K)$ is defined as follows:

For $K \leq e^{-2t}t$ we have

$$V(K) = \sqrt{K \cdot t} \quad , \tag{3.15}$$

with equality only for the constant function

$$\lambda(\tau) = \sqrt{K/t} \quad \text{for} \quad 0 \leq \tau \leq t. \tag{3.15'}$$

For $K \geq e^{-2t}t$ we have

$$V(K) = e^{-\nu}(\nu+1) - e^{-t}, \tag{3.16}$$

where $\nu \in [0,t]$ is given by

$$K = e^{-2\nu}(\nu + \frac{1}{2}) - \frac{1}{2}e^{-2t}. \tag{3.17}$$

Equality holds for the function

$$\lambda(\tau) = e^{-\nu} \quad \text{for} \quad 0 \leq \tau \leq \nu \quad \text{and} = e^{-\tau} \quad \text{for} \quad \nu \leq \tau \leq t \tag{3.17'}$$

and no other function.

For the case $t = \infty$ (where the first subcase is empty) this theorem goes back to Valiron and Landau [6], and was used by Landau in the theory of univalent functions. The case presented above was established in the paper [11], and was there used in the theory of bounded univalent functions. See also [12].

For our purpose we choose

$$\lambda(\tau) = \cos\vartheta(\tau)e^{-\tau} \quad \text{and} \quad t = \log M \quad \text{, where} \quad M < 5/3. \tag{3.18}$$

We thus have $e^{-t} = 1/M$.

In (3.9) we may without loss of generality assume that $I_2 \geq 0$ and at the same time $I_1 \geq 0$. This may be seen as follows: If not, replace f by f_ϕ, then I_1 is replaced by $e^{-i\phi}I_1$ and I_2 by $e^{-2i\phi}I_2$. By a proper choice of ϕ both expressions are non- negative. With

$$\kappa(\tau) = \cos\vartheta(\tau) + i\sin\vartheta(\tau)$$

we have

$$I_1 = \int_0^{\log M} \cos\vartheta(\tau)e^{-\tau}d\tau \quad , \tag{3.19}$$

and

$$2I_2 = 2\int_0^{\log M} \cos 2\vartheta(\tau)e^{-2\tau}d\tau =$$

$$= 4\int_0^{\log M} \cos^2\vartheta(\tau)e^{-2\tau}d\tau - (1 - \frac{1}{M^2}). \tag{3.20}$$

Moreover

$$\int_0^{\log M} \sin\vartheta(\tau)e^{-\tau}d\tau = 0, \quad \text{and} \quad \int_0^{\log M} \sin 2\vartheta(\tau)e^{-2\tau}d\tau = 0. \tag{3.21.}$$

The relation (3.9) thus takes the form (when we write ϑ instead of $\vartheta(\tau)$):

$$\left(\int_0^{\log M} \cos\vartheta \cdot e^{-\tau}d\tau\right)^2 = \int_0^{\log M} \cos^2\vartheta \cdot e^{-2\tau}d\tau - \frac{1}{4}(1 - \frac{1}{M^2}). \tag{3.22}$$

Now let K be the value of the integral

$$\int_0^{\log M} \cos^2\vartheta e^{-2\tau}d\tau.$$

Then $V(K)$ is the sharp upper bound for the integral

$$\int_0^{\log M} \cos\vartheta e^{-\tau}d\tau.$$

The question now is: Given an M-value $< 5/3$. Is it possible to have (3.9) satisfied and at the same time have the maximal value of the second integral, i.e. is it possible to have

$$V(K)^2 = K - \frac{1}{4}(1 - \frac{1}{M^2}) \quad ? \tag{3.23}$$

It turns out that it is convenient to deal with the second case $K \geq$

$\log M/M^2$ first. In this case (3.23) takes the form

$$[e^{-\nu}(\nu+1)-\frac{1}{M}]^2 = e^{-2\nu}(\nu+\frac{1}{2})-\frac{1}{4}(1+\frac{1}{M^2}), \tag{3.24}$$

or

$$[e^{-\nu}(\nu+1)-\frac{1}{M}]^2 - e^{-2\nu}(\nu+\frac{1}{2})+\frac{1}{4}(1+\frac{1}{M^2}) = 0. \tag{3.24'}$$

Simple computation shows that the derivative with respect to ν of the lefthand expression in (3.24') is

$$2\nu e^{-\nu}(\frac{1}{M}-\nu e^{-\nu}),$$

which is positive for all $\nu \in (0, \log M]$. Hence the lefthand side of (3.24') is strictly increasing in the interval $(0, \log M]$, and the equation (3.24') has at most one solution in this interval. The values of the lefthand side of (3.24') in the endpoints are:

For $\nu = 0$:

$$(1-\frac{1}{M})^2 - \frac{1}{4}(1-\frac{1}{M^2}) = (1-\frac{1}{M})(\frac{3}{4}-\frac{5}{4M}),$$

which is < 0 for all $M \in (1, 5/3)$.

For $\nu = \log M$:

$$\frac{1}{M^2}((\log M)^2 - \log M + \frac{M^2}{4} - \frac{1}{4}). \tag{3.25}$$

Simple computation shows that there is a unique M-value $M_0 = 1.361...$, such that the expression (3.25) is negative for all $M \in (1, M_0)$ and positive for all $M \in (M_0, 5/3)$. Keeping in mind the monotonicity property established above we have: For any M in the interval $(M_0, 5/3)$ the equation (3.24') has exactly one root in the interval $(0, \log M)$. For any M in the interval $(1, M_0)$ the equation (3.24') has no such root.

The only remaining question, as far as the interval $(M_0, 5/3)$ is concerned, is the question about a possible $\kappa(\tau)$, such that, in addition to (3.22) also (3.21) is satisfied. The real parts $\cos\vartheta$ and $\cos 2\vartheta$ are already such that (3.22) is satisfied. Any change from ϑ to $-\vartheta$ does not violate the validity of (3.22). In order to find a change such that (3.21) is satisfied, we go back to Theorem 2, second case. From (3.17') with

$$\lambda(\tau) = \cos\vartheta(\tau)e^{-\tau}$$

we find

$$\sin\vartheta(\tau)e^{-\tau} = \pm\sqrt{e^{-2\tau} - e^{-2\nu}} \quad \text{for} \quad 0 \le \tau \le \nu \quad \text{and} \quad 0 \quad \text{for} \quad \nu \le \tau \le t$$

and

$$\sin 2\vartheta(\tau)e^{-2\tau} = 2e^{-\nu}e^{-\tau}\sin\vartheta(\tau) \quad \text{for} \quad 0 \le \tau \le \nu \quad \text{and} \quad 0 \quad \text{for} \quad \nu \le \tau \le t.$$

Hence

$$\int_0^t \sin 2\vartheta(\tau)e^{-2\tau}d\tau = 2e^{-\nu}\int_0^t \sin\vartheta(\tau)e^{-\tau}d\tau. \tag{3.26}$$

Take a $\nu_0 \in [0,\nu]$ and let ϑ be positive in $[0,\nu_0]$ and negative in $(\nu_0,\nu]$. By a proper choice of ν_0-value the "$\sin\vartheta$-integral" is 0, and, by (3.26) also the "$\sin 2\vartheta$- integral". Hence (3.21) is satisfied without violation of (3.22). We have thus proved the following:

Proposition 3.

Let M_0 be the unique M-value in $(1,5/3)$ for which (3.25) is 0. Then for $1.316... = M_0 < M < 5/3$ the following sharp bound holds in S'_M:

$$|a_2| \le 2[e^{-\nu}(\nu+1) - \frac{1}{M}], \tag{3.27}$$

where $\nu \in [0,\log M]$ is uniquely given by (3.24').

In the remaining case $1 < M < M_0$ we shall use the first case in Theorem 2, and (3.23) takes the form

$$(\sqrt{K\log M})^2 = K - \frac{1}{4}(1 - \frac{1}{M^2}), \tag{3.28}$$

and hence

$$K = \frac{1 - 1/M^2}{4(1 - \log M)}. \tag{3.28'}$$

From Theorem 2 we know that we have equality in (3.14) if and only if

$$\lambda(\tau) = \cos\vartheta(\tau)e^{-\tau} = \sqrt{\frac{K}{\log M}}.$$

From this we get

$$\sin 2\vartheta(\tau)e^{-2\tau} = 2\sqrt{\frac{K}{\log M}}\sin\vartheta(\tau)e^{-\tau},$$

showing that in this case $\sin 2\vartheta(\tau)e^{-2\tau}$ and $\sin\vartheta(\tau)e^{-\tau}$ only differ by a constant factor. This was, in the previous case the crucial point in proving that the two equalities in (3.21) can be satisfied simultaneously. This will, essentially in the same way, also hold in the present case. This shows that $2\sqrt{K\log M}$ is the sharp upper bound for $|a_2|$, and we have the following result:

Proposition 4.

Let M_0 be the unique M-value in $(1, 5/3)$ for which (3.25) is 0. Then for $1 < M < M_0 = 1.316\ldots$ the following sharp bound holds in S'_M:

$$|a_2| \leq \sqrt{\frac{(1 - 1/M^2)\log M}{1 - \log M}}. \tag{3.29}$$

We put the results in the Propositions 1,3 and 4 together in one theorem:

Theorem 5.

Let $A_2(M)$ be the maximum value of $|a_2|$ in S'_M. Then the following holds:

For $M \geq 5/3$ we have

$$A_2(M) = \sqrt{1 - \frac{1}{M^2}}.$$

For $M_0 \leq M \leq 5/3$, where $M_0 = 1.316\ldots$ is uniquely given in $(1, 5/3)$ by

$$(\log M_0)^2 - \log M_0 - \frac{1}{4} + \frac{M_0^2}{4} = 0,$$

we have

$$A_2(M) = 2[e^{-\nu}(\nu + 1) - \frac{1}{M}],$$

where $\nu \in [0, \log M]$ is given by

$$[e^{-\nu}(\nu + 1) - \frac{1}{M}]^2 = e^{-2\nu}(\nu + \frac{1}{2}) - \frac{1}{4}(1 + \frac{1}{M^2}).$$

For $1 \leq M \leq M_0$ we have

$$A_2(M) = \sqrt{\frac{(1 - 1/M^2)\log M}{1 - \log M}}.$$

Remarks.

a) It is a straightforward verification to prove that $A_2(M)$ is a continuous function of M in the interval $[1, \infty)$, the critical points being $1, M_0$ and $5/3$. This justifies the use of the equality signs in distinguishing the cases above.

b) It is not difficult to carry out the investigations above for k-symmetric functions in S_M with vanishing third coefficient. Such an investigation is presently in progress by the Master of Science- student Muhammad Mushtaq.

c) The condition $a_3 = 0$ is equivalent to the condition (3.9), i. e. to $4I_1^2 = 2I_2$. Here the number 4 could be replaced by a positive number q, and a similar investigation as the one in the present paper could be carried out, only with an additional discussion of which q-values are permitted.

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